A MODEL OF THE COMPREHENSION AXIOM WITHOUT NEGATION

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In [1], the author has proved that the comprehension axiom

$$\exists y \forall x(x \in y \leftrightarrow F(x))$$

is valid in Skolem's model if F is a positive formula. In this paper, we construct Skolem's model within the theory of natural numbers and we prove that the predicate $\in$ is elementary (in Kalmár's sense) in this model. In the language of first-order predicate calculus with logical constants $\curlyvee$, $\curlywedge$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, and non-logical constant $\in$, a formula F is *positive* if neither $\neg$ nor $\rightarrow$ occurs in F. $F \leftrightarrow G$ is considered as an abbreviation of the formula $(F \rightarrow G) \wedge (G \rightarrow F)$. For any formula F, the universal closure of F is denoted $\dot{\forall} F$. Let Γ be the set of axioms of the form

$$\dot{\forall} \exists y \forall x(x \in y \leftrightarrow F(x))$$

where the formula $F(a)$ is positive and let Θ be the set of axioms of the form

$$\dot{\forall} \forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow (G(x) \leftrightarrow G(y))).$$

Let $\mathfrak{B}$ be the set of free individual variables. For any structure $\mathfrak{M}$ and any assignment $\mathfrak{m}: \mathfrak{B} \rightarrow |\mathfrak{M}|$, we abbreviate

$$\mathfrak{M}, \mathfrak{m} \models F(a_1, \ldots, a_n)$$

to

$$\mathfrak{M} \models F(\mathfrak{m}(a_1), \ldots, \mathfrak{m}(a_n)).$$

We now define some number-theoretic functions and predicates. The elementary functions ρ, σ, τ, υ, γ, ξ, δ_0, δ_1, η, ζ and θ are defined explicitly as follows.

$$\begin{aligned}
\rho(x) &= x + \mathrm{sg}(x), \\
\sigma(x) &= \mathrm{rm}(x+1, 2) + \mathrm{rm}(x \dot{-} 1, 2), \\
\tau(x, y, z) &= \max((x \dot{-} y)(1 \dot{-} z), (y \dot{-} x)z)\sigma(z), \\
\upsilon(x, y) &= (1 \dot{-} (\sigma(x) \dot{-} \sigma(y))) \dot{-} (\sigma(y) \dot{-} \sigma(x)), \\
\gamma(x, y) &= \mathrm{sg}(\max(\sigma(x) \dot{-} \sigma(y), \tau(x, y, \sigma(x))\upsilon(x, y))), \\
\xi(x, y) &= 1 \dot{-} (\gamma(y, x) \dot{-} \gamma(x, y)), \\
\delta_0(x, y) &= \max(x\xi(y, x), y\xi(x, y)), \\
\delta_1(x, y) &= \max(x\xi(x, y), y\xi(y, x)), \\
\eta(x, y) &= \gamma(\rho(x), y), \\
\zeta(x, y) &= (1 \dot{-} \eta(x, y)) + 1, \\
\theta(x) &= \rho(\rho(x)).
\end{aligned}$$

The elementary predicates $\subset$ and $\in$ are defined by

$$x \subset y \leftrightarrow \gamma(x, y) = 0$$

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and

$$x \in y \leftrightarrow \eta(x, y) = 0.$$

Then we have $x \in y \leftrightarrow \rho(x) \subset y$. The relation $\subset$ linearly orders the set N as

$$1 \subset 3 \subset 5 \subset \dots \subset 0 \subset \dots \subset 6 \subset 4 \subset 2,$$

and hence

$$x \subset y \leftrightarrow \forall z(z \in x \rightarrow z \in y).$$

The functions ζ, θ, δ_0 and δ_1 are monotone with respect to the order $\subset$ for each argument.

LEMMA 1. If a function φ is explicit in ζ, θ, δ_0, δ_1 and constants, then

$$\forall z(z \in \varphi(1) \leftrightarrow \forall x(z \in \varphi(x))$$

and

$$\forall z(z \in \varphi(2) \leftrightarrow \exists x(z \in \varphi(x))).$$

Proof. By assumption, φ is monotone with respect to $\subset$. Hence

$$\varphi(1) \subset \varphi(3) \subset \dots \subset \varphi(0) \subset \dots \subset \varphi(4) \subset \varphi(2),$$

thence follows the conclusion.

We define $\mathfrak{M} = \langle N, \in \rangle$. Let $\mathscr{F}$ be the set of constants 0, 1, 2, and functions ζ, θ, δ_0, δ_1.

LEMMA 2. For any positive formula $F(a, a_1, \dots, a_n)$ with no free variables but the indicated ones, there exists a function φ explicit in $\mathscr{F}$ such that

$$\mathfrak{M} \models \forall x(x \in \varphi(a_1, \dots, a_n) \leftrightarrow F(x, a_1, \dots, a_n)).$$

Proof, by induction on the construction of F. If F is a prime formula $\curlyvee$, $\curlywedge$, $x \in x$, $x \in a_i$, $a_i \in x$ or $a_i \in a_j$, let $\varphi(a_1, \dots, a_n)$ be 2, 1, 0, a_i, $\theta(a_i)$ or $\zeta(a_i, a_j)$ respectively. Then

$$\mathfrak{M} \models \forall x(x \in \varphi(a_1, \dots, a_n) \leftrightarrow F(a_1, \dots, a_n)).$$

Suppose F is $G_1 \wedge G_2$. By induction hypothesis, there are functions ψ_1 and ψ_2 explicit in $\mathscr{F}$ such that

$$\mathfrak{M} \models \forall x(x \in \psi_k(a_1, \dots, a_n) \leftrightarrow G_k(a_1, \dots, a_n))$$

for $k=1, 2$. Define $\varphi(a_1, \dots, a_n) = \delta_0(\psi_1(a_1, \dots, a_n), \psi_2(a_1, \dots, a_n))$, then Lemma holds for F. If F is $G_1 \wedge G_2$, proof is similar. Suppose F is $\forall y G(a, a_1, \dots, a_n, y)$. By induction hypothesis, there is a function ψ explicit in $\mathscr{F}$ such that

$$\mathfrak{M} \models \forall x(x \in \psi(a_1, \dots, a_n, b) \leftrightarrow G(x, a_1, \dots, a_n, b)).$$

Define $\varphi(a_1, \dots, a_n) = \psi(a_1, \dots, a_n, 1)$, then Lemma holds for F by Lemma 1. If F has $\exists$ as its outermost symbol, proof is similar.

THEOREM 1. $\mathfrak{M}$ is a model for the system of axioms Γ, Θ.

Proof. If A is an axiom in Γ, then $\mathfrak{M} \models A$ by Lemma 2. If A is in Θ, $\mathfrak{M} \models A$ is evident.

The model $\mathfrak{M}$ is indeed isomorphic to Skolem's model in [2], §2. Thus we have shown that the proof of Theorem in [1] can be carried through within the theory of natural numbers by using elementary functions.

THEOREM 2. In the model $\mathfrak{M}$, any positive formula is equivalent to an elementary predicate.

Proof. Suppose $F(a_1, a_2, \dots, a_n)$ is a positive formula with no free variables but the indicated ones. In $\mathfrak{M}$, $F(a_1, a_2, \dots, a_n)$ is equivalent to

$$\eta(a_1, \varphi(a_2, \dots, a_n)) = 0,$$

where φ is a function explicit in $\mathscr{F}$, by Lemma 2. The left-hand side of this equation is an elementary function of $a_1, a_2, \dots, a_n$.

Construction of the other model in [2], §2 can also be carried through similarly by using elementary functions.

REFERENCES

[1] T. Nagashima: *A remark on a comprehension axiom without negation*. Proc. Japan Acad. **42**, No. 5 (1966), 425–426.

[2] Th. Skolem: *Investigations on a comprehension axiom without negation in the defining propositional functions*. Notre Dame J. Formal Logic, **1** (1960), 13–22.