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A MODEL OF THE COMPREHENSION AXIOM WITHOUT NEGATION

By Takashi Nagashima*

In [1], the author has proved that the comprehension axiom
\[ \exists y \forall x (x \in y \iff F(x)) \]
is valid in Skolem's model if \( F \) is a positive formula. In this paper, we construct Skolem's model within the theory of natural numbers and we prove that the predicate \( \in \) is elementary (in Kalmar's sense) in this model. In the language of first-order predicate calculus with logical constants \( \forall, \land, \neg, \lor, \rightarrow, \forall, \exists \), and non-logical constant \( \in \), a formula \( F \) is positive if neither \( \neg \) nor \( \rightarrow \) occurs in \( F \). \( F \rightarrow G \) is considered as an abbreviation of the formula \( (F \rightarrow G) \land (G \rightarrow F) \). For any formula \( F \), the universal closure of \( F \) is denoted \( \forall F \). Let \( I \) be the set of axioms of the form
\[ \forall \exists y \forall x (x \in y \iff F(x)) \]
where the formula \( F(a) \) is positive and let \( \Theta \) be the set of axioms of the form
\[ \forall \forall x \forall y (\forall z (x \in z \iff z \in y \rightarrow (G(x) \iff G(y))) \).

Let \( B \) be the set of free individual variables. For any structure \( M \) and any assignment \( m : B \rightarrow |M| \), we abbreviate
\[ M, m \models F(a_1, \ldots, a_n) \]
to
\[ M \models F(m(a_1), \ldots, m(a_n)) \).

We now define some number-theoretic functions and predicates. The elementary functions \( \rho, \sigma, \tau, \nu, \xi, \delta_0, \delta_1, \eta, \zeta \) and \( \theta \) are defined explicitly as follows.
\[
\begin{align*}
\rho(x) & = x + \text{sg}(x), \\
\sigma(x) & = \text{rm}(x + 1, 2) + \text{rm}(x - 1, 2), \\
\tau(x, y, z) & = \max((x - y)(1 - z), (y - x)z)\sigma(x), \\
\nu(x, y) & = (1 - (\sigma(x) - \sigma(y))) + (\sigma(y) - \sigma(x)), \\
\gamma(x, y) & = \text{sg}(\max(\sigma(x) - \sigma(y), \tau(x, y, \sigma(x))\nu(x, y))), \\
\xi(x, y) & = 1 - (\gamma(y, x) - \gamma(x, y)), \\
\delta_0(x, y) & = \max(x\xi(y, x), y\xi(x, y)), \\
\delta_1(x, y) & = \max(x\xi(x, y), y\xi(y, x)), \\
\eta(x, y) & = \gamma(\rho(x), y), \\
\zeta(x, y) & = (1 - \eta(x, y)) + 1, \\
\theta(x) & = \rho(\rho(x)).
\end{align*}
\]
The elementary predicates \( \subset \) and \( \in \) are defined by
\[ x \subset y \iff \gamma(x, y) = 0 \]

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and
\[ x \in y \iff \eta(x, y) = 0. \]
Then we have \( x \in y \iff \rho(x) \subseteq y \). The relation \( \subseteq \) linearly orders the set \( N \) as
\[ 1 \subseteq 3 \subseteq 5 \subseteq \ldots \subseteq 0 \subseteq \ldots \subseteq 6 \subseteq 4 \subseteq 2, \]
and hence
\[ x \subseteq y \iff \forall z (z \subseteq x \iff z \subseteq y). \]

The functions \( \zeta, \theta, \delta_0 \) and \( \delta_1 \) are monotone with respect to the order \( \subseteq \) for each argument.

**Lemma 1.** If a function \( \varphi \) is explicit in \( \zeta, \theta, \delta_0, \delta_1 \) and constants, then
\[ \forall z (z \in \varphi(1) \iff \forall x (z \in \varphi(x))) \]
and
\[ \forall z (z \in \varphi(2) \iff \exists x (z \in \{ \varphi(x) \})). \]

**Proof.** By assumption, \( \varphi \) is monotone with respect to \( \subseteq \). Hence
\[ \varphi(1) \subseteq \varphi(3) \subseteq \ldots \subseteq \varphi(0) \subseteq \ldots \subseteq \varphi(4) \subseteq \varphi(2), \]
thence follows the conclusion.

We define \( \mathcal{M} = (N, \subseteq) \). Let \( \mathcal{F} \) be the set of constants 0, 1, 2, and functions \( \zeta, \theta, \delta_0, \delta_1 \).

**Lemma 2.** For any positive formula \( F(a_1, a_2, \ldots, a_n) \) with no free variables but the indicated ones, there exists a function \( \varphi \) explicit in \( \mathcal{F} \) such that
\[ \mathcal{M} \models \forall x (x \in \varphi(a_1, a_2, \ldots, a_n) \iff \gamma(x, a_1, a_2, \ldots, a_n)). \]

**Proof.** By induction on the construction of \( F \). If \( F \) is a prime formula \( \gamma, \eta, \chi, \delta_1 \), let \( \gamma(a_1, a_2, \ldots, a_n) = 2 \), \( \eta(a_1, a_2, \ldots, a_n) = 1 \), \( \chi(a_1, a_2, \ldots, a_n) = 0 \), \( \delta_1(a_1, a_2, \ldots, a_n) = \delta_0(a_1, a_2, \ldots, a_n) \) respectively. Then
\[ \mathcal{M} \models \forall x (x \in \varphi(a_1, a_2, \ldots, a_n) \iff \gamma(x, a_1, a_2, \ldots, a_n)). \]

Suppose \( F \) is \( G_1 \wedge G_2 \). By induction hypothesis, there are functions \( \varphi_1 \) and \( \varphi_2 \) explicit in \( \mathcal{F} \) such that
\[ \mathcal{M} \models \forall x (x \in \varphi_k(a_1, a_2, \ldots, a_n) \iff G_k(x, a_1, a_2, \ldots, a_n)) \]
for \( k = 1, 2 \). Define \( \varphi(a_1, a_2, \ldots, a_n) = \delta_0(\varphi_1(a_1, a_2, \ldots, a_n), \varphi_2(a_1, a_2, \ldots, a_n)) \), then Lemma holds for \( F \). If \( F \) is \( G_1 \wedge G_2 \), proof is similar. Suppose \( F \) is \( \forall y G(a_1, a_2, \ldots, a_n, y) \). By induction hypothesis, there is a function \( \varphi \) explicit in \( \mathcal{F} \) such that
\[ \mathcal{M} \models \forall x (x \in \varphi(a_1, a_2, \ldots, a_n, b) \iff G(x, a_1, a_2, \ldots, a_n, b)). \]
Define \( \varphi(a_1, a_2, \ldots, a_n) = \varphi(a_1, a_2, \ldots, a_n, 1) \), then Lemma holds for \( F \) by Lemma 1. If \( F \) has \( \exists \) as its outermost symbol, proof is similar.

**Theorem 1.** \( \mathcal{M} \) is a model for the system of axioms \( \Gamma, \Theta \).

**Proof.** If \( A \) is an axiom in \( \Gamma \), then \( \mathcal{M} \models A \) by Lemma 2. If \( A \) is in \( \Theta \), \( \mathcal{M} \models A \) is evident.

The model \( \mathcal{M} \) is indeed isomorphic to Skolem's model in [2], §2. Thus we have shown that the proof of Theorem in [1] can be carried through within the theory of natural numbers by using elementary functions.

**Theorem 2.** In the model \( \mathcal{M} \), any positive formula is equivalent to an elementary predicate.

**Proof.** Suppose \( F(a_1, a_2, \ldots, a_n) \) is a positive formula with no free variables but the indicated ones. In \( \mathcal{M} \), \( F(a_1, a_2, \ldots, a_n) \) is equivalent to
\[ \eta(a_1, \varphi(a_2, \ldots, a_n)) = 0, \]
where \( \varphi \) is a function explicit in \( \mathcal{F} \), by Lemma 2. The left-hand side of this equation is an elementary function of \( a_1, a_2, \ldots, a_n \).

Construction of the other model in [2], §2 can also be carried through similarly by using elementary functions.
REFERENCES
