ON ELIMINATION IN BOOLEAN ALGEBRA

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In their paper [6], K. Yamada and K. Yoshida have proven a theorem which can be stated as follows:

THEOREM. *Let f and g be two Boolean functions of n variables defined over an arbitrary Boolean algebra $\langle B, +, \cdot, ', 0, 1\rangle$.*
Then

$$f(x_1, \cdots, x_n)=0 \tag{1}$$

and

$$g(x_1, \cdots, x_n)>0 \tag{2}$$

are simultaneously solvable in $(x_1, \cdots, x_n)$ if and only if

$$\prod_{\varepsilon_1, \cdots, \varepsilon_n} f(\varepsilon_1, \cdots, \varepsilon_n)=0 \tag{3}$$

and

$$\sum_{\varepsilon_1, \cdots, \varepsilon_n} g(\varepsilon_1, \cdots, \varepsilon_n)f'(\varepsilon_1, \cdots, \varepsilon_n)>0 \tag{4}$$

hold, where each ε_i runs over 1 and 0.

This Theorem expresses an elimination property in Boolean algebra; it was proven in [6] by induction on n, using elementary Boolean properties. The aim of the present note is to point out another aspect of the Theorem, by proving its equivalence with a result due to Löwenheim. It can be stated as follows:

LÖWENHEIM'S LEMMA ([2]; see also [4]). *Assume the equation (1) is solvable. Then the following conditions are equivalent:*

$$f(x_1, \cdots, x_n)=0 \quad \text{implies} \quad g(x_1, \cdots, x_n)=0; \tag{5}$$

$$g(x_1, \cdots, x_n)\leqq f(x_1, \cdots, x_n) \quad \text{identically}; \tag{6}$$

$$\sum_{\varepsilon_1, \cdots, \varepsilon_n} g(\varepsilon_1, \cdots, \varepsilon_n)f'(\varepsilon_1, \cdots, \varepsilon_n)=0. \tag{7}$$

(As a matter of fact, the conditions (6) and (7) are equivalent even without the assumption that (1) is solvable. For (7) means that $g(\varepsilon_1, \cdots, \varepsilon_n)f'(\varepsilon_1, \cdots, \varepsilon_n)=0$ for all $\varepsilon_1, \cdots, \varepsilon_n=0$ and 1; in view of Löwenheim's Verification Theorem [2], this is equivalent with the identity $g(x_1, \cdots, x_n)f'(x_1, \cdots, x_n)=0$, which can also be expressed in the form (6)).

Proof of the equivalence between Löwenheim's Lemma and the Theorem. Assume first the validity of Löwenheim's Lemma.

If the system (1) and (2) is solvable, relation (3) holds, by a well-known necessary-and-sufficient solvability condition (generalizing Theorem 9 from [6]; see, for instance, [5]). On

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** This means that $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ are obtained from variables and possibly constants of B, by superpositions of the operations $+, \cdot, '$. We recall that if B has more than two elements, then there exist functions $f: B^n \to B$ which are not Boolean [3].

the other hand, the implication (5) is not valid; hence, by Löwenheim's Lemma, we deduce the negation of (7), i.e. (4).

Conversely, if the system (1) and (2) is not solvable, there are two possibilities: I) the equation (1) is inconsistent, whence the relation (3) is not fulfilled, by the above mentioned solvability condition; II) the equation (1) is solvable, but the implication (5) holds; hence we deduce (7) by Löwenheim's Lemma. Thus either (3) or (4) is not fulfilled. This completes the proof of the Theorem.

Assume now the validity of the Theorem. Assume further the solvability of equation (1). Now (5) implies the inconsistency of the system (1) and (2), hence the relations (3) and (4) are not both fulfilled, in view of the Theorem. But (3) is satisfied, by the solvability criterion; therefore we deduce that (4) is not fulfilled, so that (7) holds. We have already noticed that (7) is equivalent with (6). Clearly (6) implies (5), thus completing the proof of Löwenheim's Lemma.

Remark 1. The original proof of Löwenheim's Lemma is different from the proof of the Theorem given in [6].

Remark 2. The Lemma from [6], i.e. the Theorem for $n=1$, is already known; see, for instance, [1].

Remark 3. In the case of the two-element Boolean algebra, the Theorem reduces to the following (almost trivial) property: the equations (1) and

$$g(x_1, \cdots, x_n)=1 \tag{2'}$$

are simultaneously solvable if and only if the relation

$$g(\varepsilon_1, \cdots, \varepsilon_n)f'(\varepsilon_1, \cdots, \varepsilon_n)=1 \tag{4'}$$

holds.

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