

ON ELEMENTARY FUNCTIONS OF NATURAL NUMBERS

By TAKASHI NAGASHIMA*

§ 0. In this paper, we shall deal with Kalmár's elementary functions. A number-theoretic function is called elementary, if it is explicitly definable by means of addition, subtraction and the operations Σ (finite sum) and Π (finite product). For the sake of completeness, some fundamental theorems will be shown in § 1. Most of the results in § 1, excepting Theorems 5, 6 and 12 which are obtained by the author, are due to Kalmár, Bereczki and Kleene (cf. Example 1, § 57 of [4]). In § 2, we shall consider Takahashi's 'bounded predicates' [6] and apply his result to the theory of elementary functions. In § 3, we consider axiomatizability of formal theories. Following Kalmár's idea, an arithmetization (cf. Chapter X of [4]) of a formal theory can be carried through by using elementary functions. Then it will be shown that a semi-decidable theory is elementarily axiomatizable (this theorem is published also in [5]). This is an extension of Craig's axiomatizability theorem [2]. In § 4, we consider Diophantine predicates (cf. Chapter 7 of [3]). We shall prove that Diophantine predicates of a certain form (viz. a polynomial predicate with an existential quantifier prefixed to it) are elementary. Moreover, we shall construct some functions and predicates concerning polynomial representation of finite sequences. Instead of the well-known prime factor representation, the polynomial representation may also be used for arithmetizations of formal systems. Finite sequences can be represented by elementary functions whose representing predicates are Diophantine.

§ 1. A *function* shall mean a number-theoretic function, i.e. a function from $N \times \dots \times N$ into N where N is the set of all non-negative integers. A *predicate* shall mean a number-theoretic predicate, i.e. a function from $N \times \dots \times N$ into the set $\{true, false\}$.

A function is called *elementary* (in $\varphi_1, \dots, \varphi_i$) if it can be expressed explicitly by means of variables, the constant 1, the functions $+$ and \div (and $\varphi_1, \dots, \varphi_i$), and the operations $\Sigma_{x < a}$ and $\Pi_{x < a}$. A predicate or a set is *elementary* if its representing function is elementary.

THEOREM 1. *The constant functions C_q^n (cf. [4]), the identity functions U_i^n , and the functions a' , ab , $\overline{sg}(a)$, $sg(a)$, $\max(a, b)$, $\min(a, b)$, $|a - b|$, $[a/b]$, $rm(a, b)$, $a \uparrow b (=a^b)$ and $a!$ are elementary. The predicates $a = b$, $a \neq b$, $a \leq b$ and $a \mid b$ are elementary.*

PROOF. The functions prove elementary successively by the following equalities.

$$(1) \quad C_q^n(a_1, \dots, a_n) = q = \begin{cases} 1 \div 1 & \text{if } q = 0, \\ 1 + \dots + 1 & \text{otherwise.} \end{cases}$$

* Lecturer (Kōshi) in Mathematics.

- (2) $U_i^n(a_1, \dots, a_n) = a_i.$
- (3) $a' = a + 1.$
- (4) $ab = \sum_{x < b} a.$
- (5) $\overline{\text{sg}}(a) = 1 \div a.$
- (6) $\text{sg}(a) = 1 \div (1 \div a).$
- (7) $\max(a, b) = (a \div b) + b.$
- (8) $\min(a, b) = a \div (a \div b).$
- (9) $|a - b| = (a \div b) + (b \div a).$
- (10) $[a/b] = \sum_{x < a} \text{sg}(b(a' \div x'b)).$
- (11) $\text{rm}(a, b) = a \div b[a/b].$
- (12) $a \uparrow b = \prod_{x < a} a.$
- (13) $a ! = \prod_{x < a} x'.$

The representing functions of the predicates $a=b$, $a \neq b$, $a \leq b$, $a < b$ and $a \mid b$ are $\text{sg}|a-b|$, $\overline{\text{sg}}|a-b|$, $\text{sg}(a-b)$, $\overline{\text{sg}}(b \div a)$ and $\text{sg}(\text{rm}(b, a))$ respectively, which are elementary.

THEOREM 2. *The predicate $\neg P(a_1, \dots, a_n)$ is elementary in (the representing function of) the predicate P . The predicates $P(a_1, \dots, a_n) \& Q(a_1, \dots, a_n)$, $P(a_1, \dots, a_n) \vee Q(a_1, \dots, a_n)$, $P(a_1, \dots, a_n) \rightarrow Q(a_1, \dots, a_n)$ and $P(a_1, \dots, a_n) \equiv Q(a_1, \dots, a_n)$ are elementary in P and Q .*

PROOF. Let the representing functions of $P(a_1, \dots, a_n)$ and $Q(a_1, \dots, a_n)$ be $\varphi(a_1, \dots, a_n)$ and $\psi(a_1, \dots, a_n)$ respectively. Then the representing function of $\neg P$ is $1 \div \varphi(a_1, \dots, a_n)$, which is elementary in φ . The representing functions of $P \& Q$, $P \vee Q$, $P \rightarrow Q$ and $P \equiv Q$ are $\max(\varphi(a_1, \dots, a_n), \psi(a_1, \dots, a_n))$, $\varphi(a_1, \dots, a_n) \vee \psi(a_1, \dots, a_n)$, $\varphi(a_1, \dots, a_n) \div \psi(a_1, \dots, a_n)$ and $|\varphi(a_1, \dots, a_n) - \psi(a_1, \dots, a_n)|$ respectively, which are elementary in φ and ψ .

THEOREM 3. *The predicates $(\forall x)_{x < a} P(a_1, \dots, a_n, x)$ and $(\exists x)_{x < a} P(a_1, \dots, a_n, x)$ and the function $\mu x_{x < a} P(a_1, \dots, a_n, x)$ are elementary in the predicate P .*

PROOF. Let $\varphi(a_1, \dots, a_n, b)$ be the representing function of $P(a_1, \dots, a_n, b)$. Then the representing functions of $(\forall x)_{x < a} P(a_1, \dots, a_n, x)$ and $(\exists x)_{x < a} P(a_1, \dots, a_n, x)$ are $\text{sg}(\sum_{x < a} \varphi(a_1, \dots, a_n, x))$ and $\prod_{x < a} \varphi(a_1, \dots, a_n, x)$ respectively, which are elementary in φ . Now we have

$$\mu x_{x < a} P(a_1, \dots, a_n, x) = \sum_{x < a} \prod_{y \leq x} \varphi(a_1, \dots, a_n, y),$$

and this is elementary in φ since $\prod_{y \leq x}$ is $\prod_{y < x+1}$.

THEOREM 4. *The function φ defined by*

$$\varphi(a_1, \dots, a_n) = \begin{cases} \varphi_1(a_1, \dots, a_n) & \text{if } P_1(a_1, \dots, a_n), \\ \dots & \dots \\ \varphi_m(a_1, \dots, a_n) & \text{if } P_m(a_1, \dots, a_n), \\ \varphi_{m+1}(a_1, \dots, a_n) & \text{otherwise,} \end{cases}$$

where P_1, \dots, P_m are mutually exclusive predicates is elementary in $\varphi_1, \dots, \varphi_m, \varphi_{m+1}, P_1, \dots, P_m$.

PROOF. Let ϕ_1, \dots, ϕ_m be the representing functions of P_1, \dots, P_m . Then φ is expressed explicitly in $+$, \div , \cdot , $\phi_1, \dots, \phi_m, \varphi_1, \dots, \varphi_{m+1}$.

THEOREM 5. *If a set A is enumerated by a strictly increasing function φ , then A is elementary in φ .*

PROOF. $(\forall x)(x \leq \varphi(x))$ by hypothesis, hence

$$y \in A \equiv (\exists x)_{x \leq y} (y = \varphi(x)) .$$

THEOREM 6. *If a strictly increasing function φ enumerates a set A and if $(\forall x)(\varphi(x) \leq \eta(x))$, then φ is elementary in A and η .*

PROOF. Let ψ be the representing function of A . Then the set $\{y | y < x \text{ \& } y \in A\}$ has $\sum_{y < x} (1 - \psi(y))$ elements, therefore we have

$$\varphi(a) = \mu x_{x \leq \eta(a)} (\sum_{y < x} (1 - \psi(y)) \geq a \text{ \& } \psi(x) = 0) ,$$

hence φ is elementary in ψ, η .

Now we introduce some predicates and functions concerning prime factor representation of finite sequences.

$$\begin{aligned} \text{Pr}(a) &\equiv a > 1 \text{ \& } (\forall x)_{x < a} (x | a \rightarrow x = 1) , \\ p_i &= \text{the } i+1\text{-st prime number,} \\ (a)_i &= \mu x_{x < a} (\neg p_i^{x+1} | a) , \\ \text{lh}(a) &= \sum_{i < a} \text{sg}((a)_i) , \\ m(a) &= \mu x_{x < a} (\forall i)_{i < a} ((a)_i > 0 \rightarrow i \leq x) , \\ a * b &= a \text{ sg}(b) \prod_{i < b} (p_{m(a)+i} \uparrow (b)_i) , \\ \text{Sqn}(a) &\equiv (\forall i)_{i < \text{lh}(a)} ((a)_i > 0) . \end{aligned}$$

$\text{Pr}(a)$, and hence the set of all prime numbers, are elementary. By Theorem 6, p_i is an elementary function of i , since $(\forall x)(p_x \leq 2 \uparrow (2 \uparrow x))$. Then $(a)_i$ is an elementary function of a and i , thence follows that $\text{lh}(a)$, $m(a)$, $a * b$ and $\text{Sqn}(a)$ are elementary.

The *course-of-values function* for a function φ is the function $\varphi^*(a; a_2, \dots, a_n)$ defined by

$$\varphi^*(a; a_2, \dots, a_n) = \prod_{i < a} (p_i \uparrow \varphi(i, a_2, \dots, a_n)) .$$

THEOREM 7. *φ^* is elementary in φ , and conversely, φ is elementary in φ^* .*

A function defined by a primitive recursion ([4], Schema V) is not always elementary in the given functions. However, we have the following

THEOREM 8. *If a function $\varphi(a, a_2, \dots, a_n)$ is defined from χ (from ψ, χ) by Schema Va (Schema Vb) of primitive recursion, and if*

$$(\forall x)(\forall x_2) \dots (\forall x_n) (\varphi(x, x_2, \dots, x_n) \leq \eta(x, x_2, \dots, x_n)) ,$$

then φ is elementary in χ, η (in ψ, χ, η).

PROOF, for the case $n=1$. $\varphi(a)=b$ if and only if there exists a finite sequence u_0, \dots, u_a such that

$$u_0 = q , \quad u_{i+1} = \chi(i, u_i) \quad (0 \leq i < a) , \quad u_a = b .$$

So we consider the prime factor representation of such a sequence. Let $F(a, u, b)$ be

$$(u)_0 = q \text{ \& } (\forall i)_{i < a} ((u)_{i+1} = \chi(i, (u)_i)) \text{ \& } (u)_a = b .$$

Then $F(a, u, b)$ is elementary in χ . If $\varphi(a)=b$, then $F(a, u, b)$ is satisfied by $u = \varphi^*(a+1)$. Conversely, $F(a, u, b)$ implies $\varphi(a)=b$. Since

$$(\forall x)(\varphi^*(x) \leq \eta^*(x))$$

follows from the hypothesis, we have

$$\varphi(a) = b \equiv (\exists u)_{u \leq \eta^*(a+1)} F(a, u, b) .$$

Therefore

$$\varphi(a) = \mu y_{y \leq \eta(a)} (\exists u)_{u \leq \eta^*(a+1)} F(a, u, y) ,$$

hence φ is elementary in χ, η . Proof for the case $n > 1$ is similar.

THEOREM 9. *If a function $\varphi(a, a_2, \dots, a_n)$ is defined from χ by a course-of-values recursion*

$$\varphi(a, a_2, \dots, a_n) = \chi(a, \varphi^*(a; a_2, \dots, a_n), a_2, \dots, a_n),$$

and if

$$(\forall x)(\forall x_2) \dots (\forall x_n)(\varphi(x, x_2, \dots, x_n) \leq \eta(x, x_2, \dots, x_n)),$$

then φ is elementary in χ, η .

PROOF. for the case $n=1$. The function φ^* satisfies the recursion equation

$$\begin{cases} \varphi^*(0) = 1, \\ \varphi^*(a+1) = \varphi^*(a) \cdot p_a \uparrow \chi(a, \varphi^*(a)), \end{cases}$$

and

$$(\forall x)(\varphi^*(x) \leq \eta^*(x)),$$

therefore φ^* is elementary in χ, η , by Theorem 8. Hence φ is elementary in χ, η . Similarly for the case $n > 1$.

COROLLARY. *If a bounded function is defined by either a primitive or a course-of-values recursion, then it is elementary in the given functions.*

Now we consider the functions and predicates for the arithmetization of the formalism of recursive functions in Kleene [4], § 56. The predicates $N, V, FL, Tm, Eq, SE, Sb, Ct$ and Cn are elementary. The function Nu^{-1} satisfies the equation

$$Nu^{-1}(a) = \begin{cases} Nu^{-1}((a)_1) + 1 & \text{if } N(a), \\ a & \text{otherwise,} \end{cases}$$

and since $(\forall x)(Nu^{-1}(x) \leq x)$, Nu^{-1} is elementary by Theorems 4 and 9. Since $Nu(a, b)$ is equivalent to $N(a) \& b = Nu^{-1}(a)$, Nu is elementary. Hence S_n, T_n and U are elementary.

THEOREM 10. *If a predicate $P(a_1, \dots, a_n)$ is expressible in the form*

$$(\exists x)R(a_1, \dots, a_n, x)$$

with a general recursive R , then P is expressible in the form

$$(\exists x)Q(a_1, \dots, a_n, x)$$

where Q is elementary.

PROOF. This follows from the elementariness of T_n by Kleene's Enumeration Theorem.

THEOREM 11. *Every general recursive predicate $R(a_1, \dots, a_n)$ is expressible in both of the forms*

$$(\forall x)P(a_1, \dots, a_n, x)$$

and

$$(\exists x)Q(a_1, \dots, a_n, x)$$

where P and Q are elementary.

PROOF. This follows immediately from Theorem 10. Another proof, however, will be given in the next section.

THEOREM 12. *A recursively enumerable set can be enumerated (allowing repetitions) by an elementary function.*

PROOF. Let A be a set enumerated by a recursive function ψ . By Theorem 10, there

is an elementary predicate P such that

$$a \in A \equiv (\exists x)P(a, x).$$

Define a function φ by

$$\varphi(a) = \begin{cases} K(a) & \text{if } P(K(a), L(a)), \\ \psi(0) & \text{otherwise,} \end{cases}$$

where K and L are the inverses of the pairing function

$$J(a, b) = \frac{1}{2}((a+b)^2 + 3a + b).$$

Then φ enumerates A and since J , K and L are elementary, φ is elementary.

§ 2. In this section, we shall consider an application of set theory to the theory of elementary functions by use of Ackermann's model [1]. Ackermann's model of the general set theory is a structure $\langle N, \in \rangle$ where

$$a \in b \equiv \neg 2 \mid [b/2^a].$$

By use of this model, Takahashi [6] has applied Lévy's hierarchy theory to the recursive function theory.

Now let E be a two-place predicate. Generalizing Takahashi's notion of bounded predicates, we define as follows. An *E*-bounded quantifier is a quantifier of the form $(\forall x)(E(x, a) \rightarrow \dots)$ or $(\exists x)(E(x, a) \& \dots)$. A predicate is *E*-bounded if it can be expressed explicitly by means of variables, the predicate E , propositional connectives and *E*-bounded quantifiers. Then Takahashi's bounded predicate is an \in -bounded predicate.

THEOREM 13. *If $(\forall x)(\forall y)(E(x, y) \rightarrow x \leq y)$, then every *E*-bounded predicate is elementary in E .*

This follows immediately from Theorems 2 and 3. Since Ackermann's predicate \in is elementary, we have:

COROLLARY. *Every bounded predicate is elementary.*

As an immediate consequence of Takahashi's Main Theorem [6], we have:

THEOREM 14. *There is an elementary predicate E such that for any recursive predicate R there exist *E*-bounded predicates P and Q satisfying*

$$R(a_1, \dots, a_n) \equiv (\forall x)P(a_1, \dots, a_n, x)$$

and

$$R(a_1, \dots, a_n) \equiv (\exists x)Q(a_1, \dots, a_n, x).$$

In fact, this Theorem is true for the case that E is Ackermann's \in , and hence this implies Theorem 11.

§ 3. In [2], Craig has proved that recursively enumerable formal theories are primitive recursively axiomatizable within the system. Modifying his method, we shall show the elementary axiomatizability of such theories.

Let R be a predicate of two variables. A set A is *closed with respect to R* if

$$(\forall x)(\forall y)(x \in A \& R(x, y) \rightarrow y \in A).$$

The *R*-closure $[A]_R$ of a given set A is the smallest set including A and which is closed with

respect to R .

THEOREM 15. *For any predicates P , Q , R and a set B satisfying*

$$(1) \quad x \in B \equiv (\exists y)P(x, y),$$

$$(2) \quad Q(x, y) \rightarrow R(x, y) \& R(y, x)$$

and

$$(3) \quad P(x, y) \rightarrow (\exists z)(z \geq x \& z \geq y \& Q(x, z)),$$

there exists a set A such that $[A]_R = [B]_R$ and A is elementary in P , Q .

PROOF. Define A by

$$z \in A \equiv (\exists x)_{x \leq z} (\exists y)_{y \leq z} (P(x, y) \& Q(x, z)),$$

then A is elementary in P , Q . Suppose $c \in A$, then there are a and b such that $P(a, b) \& Q(a, c)$, therefore $a \in B \& R(a, c)$ and hence $c \in [B]_R$. Conversely, suppose $a \in B$. Then there are b and c such that $a \leq c \& b \leq c \& Q(a, c)$ by (1) and (3), therefore $c \in A \& R(c, a)$ and hence $a \in [A]_R$. Thus we have $A \subseteq [B]_R$ and $B \subseteq [A]_R$, hence $[A]_R = [B]_R$.

Now consider a formal theory which is semi-decidable, i.e. the set of (the Gödel numbers of) theorems is recursively enumerable. Let R be the number-theoretic predicate corresponding to the deducibility relation and B the set of theorems. Then there is an elementary predicate satisfying (1). Similarly as in [2], an elementary predicate Q satisfying (2) and (3) can be found, hence by Theorem 15, there exists an elementary set A which is an axiom system of the given theory. Thus a semi-decidable theory is elementarily axiomatizable. It should be remarked that such an axiom system (as given above) is not satisfactorily simple, cf. Footnote 6 in [2].

§ 4. A *polynomial* (with natural number coefficients) is a function which is explicit in the functions $a+b$ and ab . A predicate $P(a_1, \dots, a_n)$ is called a *polynomial predicate* if it is expressible in the form

$$\varphi(a_1, \dots, a_n) = \psi(a_1, \dots, a_n)$$

where φ and ψ are polynomials. $P(a_1, \dots, a_n)$ is called a *Diophantine predicate* if it is expressible in the form

$$(\exists x_1) \dots (\exists x_m) Q(a_1, \dots, a_n, x_1, \dots, x_m)$$

where Q is a polynomial predicate. A function will be called a *Diophantine function* if its representing predicate is Diophantine.

THEOREM 16. *A Diophantine predicate is elementary if it is expressible in the form*

$$(\exists x)P(a_1, \dots, a_n, x)$$

with a polynomial predicate P .

PROOF. It suffices to find an elementary function η such that

$$(1) \quad (\exists x)P(a_1, \dots, a_n, x) \equiv (\exists x)(x < \eta(a_1, \dots, a_n) \& P(a_1, \dots, a_n, x)).$$

In fact, (1) will be satisfied by a polynomial η . Since P is a polynomial predicate, there is a polynomial φ with integral coefficients such that

$$P(a_1, \dots, a_n, b) \equiv \varphi(a_1, \dots, a_n, b) = 0.$$

Clearly $\varphi(a_1, \dots, a_n, b)$ can be expressed in the form

$$\sum_{i \leq r} \gamma_i(a_1, \dots, a_n) b^i$$

where $\gamma_0, \dots, \gamma_q$ are polynomials with integral coefficients. Let

$$\eta(a_1, \dots, a_n) = \sum_{i \leq q} \gamma_i(a_1, \dots, a_n)^2,$$

then η is a polynomial with natural number coefficients. Thus the proof is reduced to the following

LEMMA. Let $\phi(x) = \sum_{i \leq m} c_i x^i$ be a polynomial with integral coefficients c_0, \dots, c_m . Then $\phi(0) \neq 0$ and $\phi(a) = 0$ imply $a < \sum_{i \leq m} c_i^2$.

This Lemma is easily proved by induction on m , because

$$|\sum_{i < m} c_i a^i| \leq a^{m-1} \sum_{i < m} |c_i| \leq a^{m-1} \sum_{i < m} c_i^2.$$

In the rest of this section, we shall introduce some functions and predicates concerning the polynomial representation of finite sequences. Such functions and predicates might be useful for arithmetizing the formal theory of Diophantine predicates. Now the pairing function

$$J(a, b) = \frac{1}{2}((a+b)^2 + 3a + b)$$

and its inverses $K(a)$, $L(a)$ are Diophantine. J is a 1-1 function from $N \times N$ onto N , and it is strictly increasing with respect to each variable. We have $a + b \leq J(a, b)$ (= only for $a = 0$ & $b \leq 1$), $K(a) \leq a$ (= only for $a = 0$) and $L(a) \leq a$ (= only for $a \leq 1$). J , K and L are elementary because

$$K(a) = \mu x_{x \leq a} (\exists y)_{y \leq a} (J(x, y) = a)$$

and

$$L(a) = \mu y_{y \leq a} (\exists x)_{x \leq a} (J(x, y) = a).$$

Now we define $J_1(a) = a$ and

$$J_{n+1}(a_0, a_1, \dots, a_n) = J(a_0, J_n(a_1, \dots, a_n))$$

successively for $n = 1, 2, 3, \dots$. Then for each n , J_n is a Diophantine, elementary, and 1-1 function from N^n onto N . $L^n(a)$ is defined by the primitive recursion

$$\begin{cases} L^0(a) = a, \\ L^{n+1}(a) = L(L^n(a)). \end{cases}$$

Since $L^n(a) \leq a$, $L^n(a)$ is elementary (as a function of n and a). For each fixed n , $L^n(a)$ is a Diophantine function of a . We define

$$\langle a_0, \dots, a_{n-1} \rangle = J_{n+1}(a_0, \dots, a_{n-1}, 0)$$

for $n = 0, 1, 2, \dots$. Then for each n , $\langle a_0, \dots, a_{n-1} \rangle$ is elementary and also Diophantine. In fact, it is a polynomial of a_0, \dots, a_{n-1} with nonnegative rational coefficients. Moreover, $\langle a_0, \dots, a_{n-1} \rangle$ is a 1-1 function from N^n into N . We can represent the finite sequences a_0, \dots, a_{n-1} of positive integers by the numbers $a = \langle a_0, \dots, a_{n-1} \rangle$. Let $[a]_i = K(L^i(a))$, then $[a]_i$ is an elementary function of a and i , and it is a Diophantine function of a for each fixed i . $[a]_i$ is the $i+1$ -st member of a sequence represented by a . We have

$$[\langle a_0, \dots, a_{n-1} \rangle]_i = \begin{cases} a_i & \text{if } 0 \leq i < n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[a]_i > 0 \rightarrow 2^i < a,$$

hence

$$a \leq i + 1 \rightarrow [a]_i = 0.$$

The function le defined by $le(a) = \sum_{i < a} \text{sg}([a]_i)$ is elementary. If a represents a finite sequence of positive numbers, then $le(a)$ is the length of that sequence. We define

$$\text{Seq}(a) \equiv (\forall i)_{i < le(a)} ([a]_i > 0) \ \& \ L^{le(a)}(a) = 0$$

and

$$\text{Seq}(n, a) \equiv \text{Seq}(a) \& \text{le}(a) \leq n,$$

then $\text{Seq}(a)$ and $\text{Seq}(n, a)$ are elementary predicates. $\text{Seq}(a)$ means that a represents a finite sequence (the empty sequence $0 = \langle \rangle$ inclusive) of positive integers, or a is a *sequence number*. We have

$$\text{Seq}(0, a) \equiv a = 0,$$

$$\text{Seq}(n+1, a) \equiv \text{Seq}(n, a) \vee (K(a) > 0 \& \text{Seq}(n, L(a))),$$

and since $\text{le}(a) \leq a$,

$$\text{Seq}(a) \equiv (\exists n)_{n \leq a} \text{Seq}(n, a).$$

By induction on n , we can prove that $\text{Seq}(n, a)$ is a Diophantine predicate of a for each fixed n .

The functions $\text{jx}(n, a, b)$ and $a \# b$ are defined as follows:

$$\begin{cases} \text{jx}(0, a, b) = b, \\ \text{jx}(n+1, a, b) = J(K(a), \text{jx}(n, L(a), b)), \end{cases}$$

and

$$a \# b = \text{jx}(\text{le}(a), a, b).$$

For each fixed n , $\text{jx}(n, a, b)$ is a Diophantine function of a and b . Now we show that $\text{jx}(n, a, b)$ and $a \# b$ are elementary. We consider a function φ defined by

$$\varphi(a, b) = \begin{cases} b & \text{if } K(a) = 0, \\ J(K(L(a)), \varphi(J(K(a)-1, L^2(a)), b)) & \text{otherwise.} \end{cases}$$

Then we have

$$\varphi(a, b) \leq (\max(L(a), b) + 4) \uparrow (3 \uparrow K(a))$$

because

$$J_{n+1}(x, \dots, x) \leq (x+4) \uparrow (3 \uparrow n),$$

hence φ is elementary. Therefore jx is elementary since $\text{jx}(n, a, b) = \varphi(J(n, a), b)$, and hence $a \# b$ is elementary. Now we have

$$\text{jx}(n, a, b) = J_{n+1}([a]_0, \dots, [a]_{n-1}, b),$$

hence we obtain the following proposition.

Let $a = \langle a_0, \dots, a_{m-1} \rangle$ and $b = \langle b_0, \dots, b_{n-1} \rangle$ where $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1}$ are positive integers. Then

$$a \# b = \langle a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1} \rangle.$$

Thus the *juxtapositions* of finite sequences can be represented by the function $a \# b$.

Let $\text{le}(n, a) = \sum_{i < n} \text{sg}([a]_i)$, then $\text{le}(n, a)$ is elementary. For each fixed n , it is a Diophantine function of a . We have

$$a \leq n \rightarrow \text{le}(n, a) = \text{le}(a).$$

$\#(n, a, b)$ is defined by

$$\#(n, a, b) = \text{jx}(\text{le}(n, a), a, \text{jx}(n, b, 0)).$$

Then $\#(n, a, b)$ is elementary. For each n , it is a Diophantine function of a and b . We have

$$\text{Seq}(n, a) \& \text{Seq}(n, b) \rightarrow \#(n, a, b) = a \# b.$$

According to the above considerations, arithmetizations of formal systems can be carried through by the polynomial representations of finite sequences. In such arithmetizations, the number-theoretic predicates which correspond to metamathematical predicates will generally have the following properties: Let $P(a_1, \dots, a_k)$ be such a number-theoretic predicate. Then P is elementary and there corresponds a predicate $P'(n, a_1, \dots, a_k)$ such that

- (1) $P'(n, a_1, \dots, a_k)$ is elementary,
- (2) $P'(n, a_1, \dots, a_k)$ is Diophantine for each fixed n ,

$$(3) \quad n \leq m \ \& \ P'(n, a_1, \dots, a_k) \rightarrow P'(m, a_1, \dots, a_k),$$

$$(4) \quad (\exists n)(\forall m)(n \leq m \rightarrow (P'(m, a_1, \dots, a_k) \equiv P(a_1, \dots, a_k)))$$

and hence

$$(5) \quad P(a_1, \dots, a_k) \equiv (\exists n)P'(n, a_1, \dots, a_k).$$

(1968. IV. 27.)

REFERENCES

- [1] W. Ackermann: *Die Widerspruchsfreiheit der allgemeinen Mengenlehre*. Math. Ann. **114** (1937), 305-315.
- [2] W. Craig: *On axiomatizability within a system*. J. Symbolic Logic **18** (1953), 30-32.
- [3] M. Davis: *Computability and unsolvability*. New York. 1958.
- [4] S.C. Kleene: *Introduction to metamathematics*. Amsterdam, Groningen, New York and Toronto. 1952.
- [5] T. Nagashima: *On elementary functions in the theory of natural numbers* (in Japanese). Hitotsubashi Daigaku Kenkyū Nenpō, Shizen Kagaku Kenkyū **10** (1968), 21-37.
- [6] M. Takahashi: *Ackermann's model and recursive predicates*. Proc. Japan Acad. **44**, No. 2 (1968), 41-42.

(Added in proof, 1968. VII. 1.) Most of the theorems in §1 are also obtained by A. Grzegorzczuk: *Some classes of recursive functions*. *Rozprawy Matematyczne IV*. Warszawa, 1953.