ON DIOPHANTINE EQUATION OF 1st DEGREE

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Throughout this paper N and Z are the set of all natural numbers and rational integers respectively. We use the symbols of interval [,], [,) etc. as the symbols of intervals defined on linearly ordered set Z.

For n $(n\geq 2)$ elements $a_j\in N$ $(1\leq j\leq n)$ such that $(a_1, a_2, \ldots, a_n)=1$ we consider the set

$$S(a_1, a_2, \ldots, a_n)=\{\sum_{j=1}^{n} a_j x_j\in N;\ x_j\in N,\ (1\leq j\leq n)\}.$$

Obviously $b\in S(a_1, a_2, \ldots, a_n)$ is eqivalent to the fact that Diophantine equation of 1st degree $\sum_{j=1}^{n} a_j X_j=b$ has at least one solution in N, and it is obvious

$$\exists j\ (1\leq j\leq n);\ a_j=1\Rightarrow S(a_1, a_2, \ldots, a_n)=[\sum_{j=1}^{n} a_j, \infty).$$

So throughout this paper we assume $a_j\geq 2$ for all j, $(1\leq j\leq n)$.

1. We put

$$\begin{array}{lll}
d_1=(a_2, a_3, \ldots, a_n) & a_j=d_1 a'_j & 2\leq j\leq n \\
d_2=(a'_3, a'_4, \ldots, a'_n) & a'_j=d_2 a''_j & 3\leq j\leq n \\
\vdots & \vdots & \vdots \\
d_r=(a^{(r-1)}_{r+1}, a^{(r-1)}_{r+2}, \ldots, a^{(r-1)}_{n}) & a^{(r-1)}_j=d_r a^{(r)}_j & r+1\leq j\leq n \\
\vdots & \vdots & \vdots \\
d_{n-2}=(a^{(n-3)}_{n-1}, a^{(n-3)}_{n}) & a^{(n-3)}_j=d_{n-2} a^{(n-2)}_j & n-1\leq j\leq n \\
d_{n-1}=(a^{(n-2)}_{n}) & a^{(n-2)}_n=d_{n-1} a^{(n-1)}_n &
\end{array}$$

It is obvious that $d_{n-1}=a^{(n-2)}_n$, $a^{(n-1)}_n=1$. Now we can prove

$$(\sum_{j=1}^{n-1} a_j d_j, \infty)\subseteq S(a_1, a_2, \ldots, a_n)$$

by induction on n.

If $n=2$, then $d_1=a_2$. Let us prove that the equation

$$a_1X_1+a_2X_2=b$$

has at least one solution in N for all $b\in N$ such that $a_1a_2<b$. By the assumption $(a_1, a_2)=1$ the eqation has at least one rational integral solution, which we denote $X_1=x^{(0)}_1$, $X_2=x^{(0)}_2$. For all $t\in Z$, $X_1=x^{(0)}_1-a_2t$, $X_2=x^{(0)}_2+a_1t$ are also rational integral solution of $a_1X_1+a_2X_2=b$. So the fact to be proved is

$$\{t\in Z;\ x^{(0)}_1-a_2t>0, x^{(0)}_2+a_1t>0\}\neq\phi \qquad \text{i.e.}\ \left(-\frac{x^{(0)}_2}{a_1}, \frac{x^{(0)}_1}{a_2}\right)\neq\phi.$$

But this is obvious by the relation

$$\frac{x^{(0)}_1}{a_2}-\left(-\frac{x^{(0)}_1}{a_1}\right)=\frac{b}{a_1a_2}>1.$$

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Next let us assume

$$b\in(\sum_{j=2}^{n-1}a_jd_j,\ \infty),$$

and let us adopt as the assumption of induction

$$(\sum_{j=2}^{n-1}a'_jd_j,\ \infty)\subseteq S(a'_2,\ a'_3,\ \ldots,\ a'_n)$$

where a'_j $(2\leq j\leq n)$ have been defined $a_j=d_1a'_j$ $(2\leq j\leq n)$. Thus the equation $\sum_{j=2}^{n}a'_jX_j=b_0$ has at least one solution in N for all $b_0\in(\sum_{j=2}^{n-1}a'_jd_j,\ \infty)$.

If we can prove the fact that the equation $a_1X+d_1Y=b$ has at least one solutiou $X=x^{(0)}$, $Y=y^{(0)}$ such that $x^{(0)}\in N,\ y^{(0)}\in(\sum_{j=2}^{n-1}a'_jd_j,\ \infty)$, then we finish the proof. But it is equivalent to the fact that $a_1X_1+d_1(Y_1+\sum_{j=2}^{n-1}a'_jd_j)=b$, i.e. $a_1X_1+d_1Y_1=b-\sum_{j=2}^{n-1}a_jd_j$ has at least one solution $X_1=x_1^{(0)}\in N,\ Y_1=y_1^{(0)}\in N$. But this is guaranteed by the assumption $b\in(\sum_{j=1}^{n-1}a_jd_j,\ \infty)$.

We can improve on this result by changing the order of a_j $(1\leq j\leq n)$ suitably. Namely let us $\mathfrak{S}_n$ be symmetric group of degree n. For any $\sigma\in\mathfrak{S}_n$ we put

$$\begin{array}{lll}
d_1(\sigma)=(a_{\sigma(2)},\ a_{\sigma(3)},\ \ldots,\ a_{\sigma(n)}) & a_{\sigma(j)}=d_1(\sigma)a'_{\sigma(j)} & 2\leq j\leq n\\
d_2(\sigma)=(a'_{\sigma(3)},\ a'_{\sigma(4)},\ \ldots,\ a'_{\sigma(n)}) & a'_{\sigma(j)}=d_2(\sigma)a''_{\sigma(j)} & 3\leq j\leq n\\
\vdots & \vdots & \vdots\\
d_r(\sigma)=(a^{(r-1)}_{\sigma(r+1)},\ a^{(r-1)}_{\sigma(r+2)},\ \ldots,\ a^{(r-1)}_{\sigma(n)}) & a^{(r-1)}_{\sigma(j)}=d_r(\sigma)a^{(r)}_{\sigma(j)} & r+1\leq j\leq n\\
\vdots & \vdots & \vdots\\
d_{n-2}(\sigma)=(a^{(n-3)}_{\sigma(n-1)},\ a^{(n-3)}_{\sigma(n)}) & a^{(n-3)}_{\sigma(j)}=d_{n-2}(\sigma)a^{(n-2)}_{\sigma(j)} & r-1\leq j\leq n\\
d_{n-1}(\sigma)=(a^{(n-2)}_{\sigma(n)}) & a^{(n-2)}_{\sigma(n)}=d_{n-1}(\sigma)a^{(n-1)}_{\sigma(n)}. &
\end{array}$$

It is obvious $d_{n-1}(\sigma)=a^{(n-2)}_{\sigma(n)},\ a^{(n-1)}_{\sigma(n)}=1$. We put

$$M=\{\sigma_0\in\mathfrak{S}_{n-1};\ \sum_{j=1}^{n-1}a_{\sigma_0(j)}d_j(\sigma_0)=\min_{\sigma\in\mathfrak{S}_n}\sum_{j=1}^{n-1}a_{\sigma(j)}d_j(\sigma)\}$$

Following the above proof, we have a result,

$$(\sum_{j=1}^{n-1}a_{\sigma_0(j)}d_j(\sigma_0),\ \infty)\subseteq S(a_{\sigma_0(1)},\ a_{\sigma_0(2)},\ \ldots,\ a_{\sigma_0(n)})$$

for any $\sigma_0\in M$, and this is better than the above result.

With respect to $\sigma\in\mathfrak{S}_n$, the fact

$$a_{\sigma(1)}\leq a_{\sigma(2)}\leq\ \ldots\ \leq a_{\sigma(n)}\Rightarrow\sigma\in M$$

is not always correct and there are two cases where

$$\sum_{j=1}^{n-1}a_{\sigma_0(j)}d_j(\sigma_0)\in S(a_{\sigma_0(1)},\ a_{\sigma_0(2)},\ \ldots,\ a_{\sigma_0(n)})$$

holds and does not hold.

Example 1. $a_1=2,\ a_2=3,\ a_3=4$.

$$\sum_{j=1}^{2}a_{\sigma(j)}d_j(\sigma)=\begin{cases}14 & \text{for}\quad \sigma\in\{\iota=\text{identity of }\mathfrak{S}_3,\ (23)\}\\ 10 & \text{for}\quad \sigma\in\{(12),(13),(123),(132)\}\end{cases}$$

So $M=\{(12),(13),(123),(132)\}$. But $10\notin S(a_1,\ a_2,\ a_3)$, because if $10\in S(a_1,\ a_2,\ a_3)$, the equation $a_1X_1+a_2X_2+a_3X_3=10$ has at least one solution $X_j=x_j^{(0)}\in N$ $(1\leq j\leq 3)$ and $x_2^{(0)}\equiv 0 \pmod 2$. Accordingly $3x_2^{(0)}\geq 6$, then $a_1x_1^{(0)}+a_2x_2^{(0)}+a_3x_3^{(0)}\geq 12$. This is a contradiction.

Example 2. $a_1=3,\ a_2=4,\ a_3=5$.

$$\sum_{j=1}^{2}a_{\sigma(j)}d_j(\sigma)=\begin{cases}23 & \text{for}\quad \sigma\in\{\iota=\text{identity of }\mathfrak{S}_3,\ (23)\}\\ 19 & \text{for}\quad \sigma\in\{(12),(132)\}\\ 17 & \text{for}\quad \sigma\in\{(123),(13)\}\end{cases}$$

So $M=\{(123),(13)\}$. But $17\in S(a_1,\ a_2,\ a_3)$, because the equation $a_1X_1+a_2X_2+a_3X_3=17$ has a solution $X_1=2$, $X_2=X_3=1$.

2. It is obvious that

$$[1, \sum_{j=1}^{n} a_j) \cap S(a_1, a_2, \dots, a_n)=\phi$$

$$\sum_{j=1}^{n} a_j \in S(a_1, a_2, \dots, a_n).$$

So we are interested in the following finite set

$$[\sum_{j=1}^{n} a_j, \sum_{j=1}^{n-1} a_j d_j) \cap S(a_1, a_2, \dots, a_n).$$

Let us assume

$$S(a_2', a_3', \dots, a_n')=\{c_1, c_2, \dots, c_l\} \cup (b_0', \infty)$$

where $c_1<c_2< \cdots <c_l$, $c_1=\sum_{j=2}^{n} a_j'$, and

$$b_0'=\mathrm{Max}\{x \in N;\ x \notin S(a_2', a_3', \dots, a_n')\} \leq \sum_{j=2}^{n-1} a_j' d_j,$$

and a_j $(2 \leq j \leq n)$ have been defined as $a_j=d_1 a_j'$ $(2 \leq j \leq n)$. By the relation

$$\sum_{j=1}^{n} a_j x_j = a_1 x_1 + d_1 \sum_{j=2}^{n} a_j' x_j$$

we have

$$\begin{aligned} S(a_1, a_2, \dots, a_n) &= a_1 N + d_1 S(a_2', a_3', \dots, a_n') \\ &= (a_1 N + \{d_1 c_1, d_1 c_2, \dots, d_1 c_l\}) \cup (a_1 N + d_1(b_0' + N)) \\ &= (a_1 N + \{d_1 c_1, d_1 c_2, \dots, d_1 c_l\}) \cup (d_1 b_0' + S(a_1, d_1)). \end{aligned}$$

Accordingly the problem is generally reduced to consider the set $S(a_1, a_2)$.

3. Let us consider the special case $n=2$. a_1, a_2 are two elements of N such that $(a_1, a_2)=1$ and $a_1<a_2$. (If $a_1=a_2$, then $a_1=a_2=1$ by the assumption $(a_1, a_2)=1$).

At first $a_1 a_2 \notin S(a_1, a_2)$, because by $(a_1, a_2)=1$

$$\left\{(x_1, x_2) \in N^2;\ x_2=\frac{a_1}{a_2} x_1,\ (0<) x_1<a_2\right\}=\phi,$$

then
$$\left\{(x_1, x_2) \in N^2;\ \frac{x_1}{a_2}+\frac{x_2}{a_1}=1\right\}=\phi.$$

Next $\varphi : (x_1^{(0)}, x_2^{(0)}) \to a_1 x_1^{(0)} + a_2 x_2^{(0)}$ is a bijection from $\{(x_1, x_2) \in N^2;\ a_1 x_1 + a_2 x_2 < a_1 a_2\}$ onto $[a_1+a_2, a_1 a_2) \cap S(a_1, a_2)$.—This result was suggested by Mr. T. Nagashima, who is a lecturer at Hitotsubashi University—. The reason is

$$\begin{aligned} & b \in [a_1+a_2, a_1 a_2) \cap S(a_1, a_2) \\ & \Rightarrow \exists (x_1^{(0)}, x_2^{(0)}) \in N^2,\ b=a_1 x_1^{(0)} + a_2 x_2^{(0)} < a_1 a_2 \\ & \Rightarrow \varphi(x_1^{(0)}, x_2^{(0)})=b, \\ & \varphi(x_1^{(0)}, x_2^{(0)})=\varphi(x_1^{(1)}, x_2^{(1)}) \\ & \Rightarrow a_1 x_1^{(0)} + a_2 x_2^{(0)} = a_1 x_1^{(1)} + a_2 x_2^{(1)} \\ & \Rightarrow a_1(x_1^{(0)} - x_2^{(1)}) = a_2(x_2^{(1)} - x_2^{(0)}) \\ & \Rightarrow x_1^{(0)} \equiv x_1^{(1)} \pmod{a_2} \qquad (\text{by } (a_1, a_2)=1) \end{aligned}$$

but $1 \leq x_1^{(0)} \leq a_2-1$, $1 \leq x_1^{(1)} \leq a_2-1$, then $x_1^{(0)}=x_1^{(1)}$, $x_2^{(0)}=x_2^{(1)}$.

Next we have

$$\begin{aligned} & \text{number of the elements in } \{(x_1, x_2) \in N^2;\ a_1 x_1 + a_2 x_2 < a_1 a_2\} \\ & = \frac{1}{2}(\text{number of the elements in } \{(x_1, x_2) \in N^2;\ 0<x_1<a_2,\ 0<x_2<a_1\}). \end{aligned}$$

So we have

$$\text{number of the elements in } [a_1+a_2,\, a_1a_2)\cap S(a_1a_2)$$
$$=\frac{1}{2}(a_1-1)(a_2-1)$$
$$=\frac{1}{2}(\text{number of the elements in } [a_1+a_2,\, a_1a_2)).$$

Let us consider $S(a_1, a_2)$ more precisely.

i) When $a_1=2$, there exists c in N such that $a_2=2c+1$ by $(a_1, a_2)=1$.

i)-1 When $c=1$ i.e. $a_2=3$, it is obvious that

$$(\text{number of the elements in } [a_1+a_2,\, a_1a_2)\cap S(a_1, a_2))=1,$$
$$S(a_1, a_2)=\{5\}\cup(6, \infty).$$

i)-2 When $c\geq 2$ i.e. $a_2\geq 5$, it is obvious that

$$\text{number of the elements in } [a_1+a_2,\, a_1a_2)\cap S(a_1, a_2)=\frac{a_2-1}{2},$$
$$S(a_1, a_2)=\left\{2s+a_2;\ s=1, 2, \ldots, \frac{a_2-1}{2}\right\}\cup(a_1a_2, \infty).$$

ii) When $a_1\geq 3$, we put

$$a_2=a_1q+r,\ 0\leq r<a_1.$$

Then $q\geq 1$ and $1\leq r<a_1$, and

$$a_1a_2-(a_1+a_2)=a_2(a_1-2)+(a_2-a_1)\geq 5$$

So $\quad$ number of the elements in $[a_1+a_2, a_1a_2)\geq 5$.

Now we put for any k such that $1\leq k\leq a_1-1$

$$V_k=\left\{a_1x_1+a_2x_2\in N;\ (k-1)\left(q+\frac{r}{a_1}\right)<x_1<k\left(q+\frac{r}{a_1}\right),\ 1\leq x_2\leq a_1-k\right\}.$$

Then $\quad [a_1+a_2, a_1a_2)\cap S(a_1, a_2)=\bigcup_{k=1}^{a_1-1} V_k.$

This is obvious by the bijection $\varphi: (x_1^{(0)}, x_2^{(0)})\to a_1x_1^{(0)}+a_2x_2^{(0)}$ from $\{(x_1, x_2)\in N^2;\ a_1x_1+a_2x_2< a_1a_2\}$ onto $[a_1+a_1, a_1a_2)\cap S(a_1, a_2)$.

Example 3. $a_1=5$, $a_2=6$, $a_3=8$.

As a preparation

$$S(2, 5)=\{2s+5;\ s=1, 2\}\cup(10, \infty)$$
$$=\{7, 9\}\cup(10, \infty),$$
$$S(3, 4)=\{7, 10, 11\}\cup(12, \infty),$$

because by $4=3\cdot 1+1$,

$$V_1=\left\{3x_1+4x_2;\ 0<x_1\leq 1+\frac{1}{3},\ 1\leq x_2\leq 3-1\right\}$$
$$V_2=\left\{3x_1+4x_2;\ 1+\frac{1}{3}<x_1\leq 2\left(1+\frac{1}{3}\right),\ 1\leq x_2\leq 1\right\}.$$

Now $d_1=2$, $d_2=4$, then

$$\sum_{j=1}^{2} a_jd_j=34, \qquad \sum_{j=1}^{2} a_j=19.$$

Accordingly

$$(34, \infty)\subseteq S(a_1, a_2, a_3), \qquad (0, 19)\cap S(a_1, a_2, a_3)=\phi.$$

By $\quad 5x_1+6x_2+8x_3=5x_1+2(3x_2+4x_3)$

we have $\quad [19, 34]\cap S(a_1, a_2, a_3)$

$$=[19, 34]\cap(5N+2S(3, 4))$$

$$=[19,34]\cap((5N+\{14,20,22\})\cup(24+S(2,5)))$$
$$=\{19,24,25,27,29,30,32,34\}\cup\{31,33\}$$
$$=\{19,24,25,27,29,30,31,32,33,34\}.$$

Then $\quad S(a_1,a_2,a_3)=\{19,24,25,27\}\cup(28,\infty)$

and $\quad \mathrm{Max}\{x\in N;\ x\notin S(a_1,a_2,a_3)\}=28.$

4. Finally I state formulae which give us general solution of Diophantine equation of 1st degree.

i) For two rational integers a_1, a_2 such that $(a_1,a_2)=1$ and $a_1<a_2$, let us put

$$r_{j-2}=r_{j-1}q_j+r_j \qquad (1\leq j\leq m)$$
$$r_{m-1}=r_mq_{m+1}$$

where $a_1=r_0$, $a_2=r_{-1}$. Then we have

$$r_0>r_1>r_2>\cdots>r_{m-1}>r_m>0$$

and $\quad r_m=(a_1,a_2)=1.$

For arbitrary rational integer b, let us put

$$S_j=\left\{\begin{pmatrix}x_1^{(j)}\\x_2^{(j)}\end{pmatrix}\in Z^2;\ r_jx_1^{(j)}+r_{j-1}x_2^{(j)}=b\right\} \qquad (0\leq j\leq m),$$

then we have

$$S_0=\left\{\begin{pmatrix}x_1^{(0)}\\x_2^{(0)}\end{pmatrix}\in Z^2;\ a_1x_2^{(0)}+a_2x_2^{(0)}=b\right\},$$

$$S_m=\left\{\begin{pmatrix}b & -r_{m-1}\\0 & 1\end{pmatrix}\begin{pmatrix}1\\t\end{pmatrix}\in Z^2;\ t\in Z\right\},$$

and
$$\begin{pmatrix}x_1^{(j)}\\x_2^{(j)}\end{pmatrix}\to Q_j\begin{pmatrix}x_1^{(j)}\\x_2^{(j)}\end{pmatrix},\ Q_j=\begin{pmatrix}-q_j & 1\\1 & 0\end{pmatrix}, \qquad (1\leq j\leq m)$$

are the bijection from S_j onto S_{j-1} $(1\leq j\leq m)$. Accordingly the general solution $X_1=x_1^{(0)}$, $X_2=x_2^{(0)}$ of the equation $a_1X_1+a_2X_2=b$ are given by the following formula.

$$\begin{pmatrix}x_1^{(0)}\\x_2^{(0)}\end{pmatrix}=Q_1, Q_2, \ldots, Q_m\begin{pmatrix}b & -r_{m-1}\\0 & 1\end{pmatrix}\begin{pmatrix}1\\t\end{pmatrix}, \qquad t\in Z.$$

ii) Now let us consider n dimensional case. For n $(n\geq 2)$ rational integers a_j $(1\leq j\leq n)$ such that $(a_1,a_2,\ldots,a_n)=1$ and all of them are not negative, we put

$$\mathfrak{a}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}, \qquad a_{m(\mathfrak{a})}=\mathrm{Min}\{a_j\in Z;\ 1\leq j\leq n,\ a_j>0\}$$

and

$$\mathfrak{a}'=\begin{pmatrix}a_1'\\a_2'\\\vdots\\a_n'\end{pmatrix}, \quad \text{where } a_j'=a_j-(1-\delta_{j,m(\mathfrak{a})})a_{m(\mathfrak{a})}\left[\frac{a_j}{a_{m(\mathfrak{a})}}\right]$$

$$\mathfrak{a}^{(k+1)}=\mathfrak{a}^{(k)\prime} \qquad k=1,2,3,\ldots.$$

Then we have the following result which is easily proved by induction on n,

$$\exists k_0\in N;\ \mathfrak{a}^{(k_0)}=\begin{pmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix} m(\mathfrak{a}^{(k_0)})$$

Now we put for any fixed $b \in Z$

$$S_k=\left\{\begin{pmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{pmatrix} \in Z^n;\ \sum_{j=1}^{n} a_j^{(k)} x_{j,k}=b\right\} \qquad (0 \leq k \leq k_0)$$

Then S_0=the set of all solutions in Z of $\sum_{j=1}^{n} a_j X_j=b$

$$S_{k_0}=\left\{\begin{pmatrix} t_1 \\ \vdots \\ t_{\nu-1} \\ b \\ t_{\nu+1} \\ \vdots \\ t_n \end{pmatrix} \in Z^n;\ \nu=m(\mathfrak{a}^{(k_0)}),\ t_l=\text{arbitrary element in } Z \text{ for } 1 \leq l \leq n,\ l \neq m(\mathfrak{a}^{(k_0)})\right\}$$

and

$$\begin{pmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{pmatrix} \to Q_k \begin{pmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{pmatrix},\quad Q_k=\begin{array}{r} \\ \\ \\ \\ \nu) \\ \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ -q_1^{(k-1)} & -q_2^{(k-1)} & \cdots & -q_{\nu-1}^{(k-1)} & 1 & -q_{\nu+1}^{(k-1)} & \cdots & -q_n^{(k-1)} \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

where $\nu=m(\mathfrak{a}^{(k-1)})$, $q_j^{(k-1)}=\left[\dfrac{a_j^{(k-1)}}{a_{m(\mathfrak{a}^{(k-1)})}}\right]$, $1 \leq j \leq n$, $j \neq m(\mathfrak{a}^{(k-1)})$, is a bijection from S_k onto S_{k-1}.

Accordingly the general solution $X_j=x_{j,0}$ $(1 \leq j \leq n)$ of equation $\sum_{j=1}^{n} a_j X_j=b$ is given by the following formula,

$$\begin{pmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{n,0} \end{pmatrix}=Q_1 \cdot Q_2 \cdot \ldots Q_{k_0} \begin{pmatrix} t_1 \\ \vdots \\ t_{\nu-1} \\ b \\ t_{\nu+1} \\ \vdots \\ t_n \end{pmatrix} (\nu \qquad \begin{array}{l} \nu=m(\mathfrak{a}^{(k_0)}) \\ t_l \in Z,\ 1 \leq l \leq n,\ l \neq m(\mathfrak{a}^{(k_0)}). \end{array}$$