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AN APPLICATION OF BOOLEAN ALGEBRA IN PRACTICAL SITUATIONS

By

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In this paper, the authors will establish a theorem on elimination in Boolean algebra, apply it in some practical situations and propose a switching network thereof.

§ 0 Introduction

Much has been proposed and studied on the theoretic applications of symbolic logic, but very little on its pragmatic applications. By theoretic, we mean scientific: it has been applied in elaborating definitions, in arranging assumptions and in systematizing theories of sciences other than logic. By pragmatic, we mean practical.

In a previous paper by Yamada¹, three kinds of practical applications of Boolean algebra were alluded to, which is essentially a logic of propositions. They are arrangement of given rules, abstraction of mutual relations of assigned objects and construction of collectives with desired structure.

§ 1 Problem

The abstraction of relations is a procedure to find mutual relations of the assigned objects out of the given mutual relations of given objects.

Various inferences in classical logic are ascertaining rather than discovering. The classical inferences are quite powerful in ascertaining the presupposed conclusions, but not very in discovering the unknown. They can make it certain that all A is B and no C is B, therefore no C is A, but not very quick at answering such kind of question as “all A is B and no C is B. What is the relation between A and C?” The abstraction procedure can easily answer the latter type of question.

As this procedure stands in the previous paper, however, it can manipulate solely those problems which involve nothing but universal propositions.

In this paper, firstly a theorem shall be established on a new elimination method, which enables us to manage some types of problems involving not only universal propositions but also particular ones.

Secondly, a switching network shall be designed, which will operate to solve these kinds of problems.
§ 2 Definitions and Theorems Reviewed

For the sake of completeness, in this section, fundamental definitions and theorems shall be summarized without either explanations or proofs.

Definition I: Boolean algebra is a set $B$ of elements with one unary operation $'$ and two binary $\cdot$ and $+$ defined to satisfy the following conditions:

1. **Closure**
   - To every pair of elements $a$, $b$ of $B$, there belongs a unique element of $B$, called the product of $a$ and $b$ which is written $p = ab$.

2. **Commutativity**
   - If $a$ and $b$ are any elements of $B$, then $ab = ba$.

3. **Distributivity**
   - If $a$, $b$ and $c$ are any elements of $B$, then $a(b+c) = ab + ac$.

4. **Absorptivity**
   - If $a$ and $b$ are any elements of $B$, then $a(a+b) = a + ab = a$.

5. **Identity**
   - $B$ contains elements 1 and 0 such that for every element $a$ of $B$ respectively $a1 = a$ and $a + 0 = a$.

6. **Complementarity**
   - Corresponding to each element $a$ of $B$, there exists an element $a'$ of $B$ such that $a = a' = 0 + a + a' = 1$.

The following lemma and theorems are of fundamental importance.

**Theorem 1** Associativity: For any three elements $a$, $b$ and $c$ of $B$, $ab \cdot c = a \cdot bc$.

**Theorem 2** Idempotence: For any element $a$ of $B$, $aa = a$.

**Theorem 3** Uniqueness of Identity: 1 and 0 defined in (5) are unique respectively.

(Lemma) If $xz = yz$ and $x + z = y + z$, then $x = y$, where $x$, $y$ and $z$ are elements of $B$.

**Theorem 4** Uniqueness of Complement: For each element $a$ of $B$, $a'$ defined in (6) is unique.

(Corollary) $a'' = a$.

**Theorem 5** Annihilation: For every element $a$ of $B$, $a0 = 0$ and $a1 = 1$.

**Theorem 6** Dualisation (Law of De Morgan): For any two elements $a$ and $b$ of $B$, $(ab)' = a' + b'$ and $(a + b)' = a'b'$.

**Theorem 7** Symmetric Difference: $a = b$ and $ab' + a'b = 0$ are equivalent with each other for any two elements $a$ and $b$ of $B$.

Definition II: An expression which is constructed by finite number of applications of $'$,
+ and upon given element or elements and one unknown \( x \) of \( B \) is called a function of \( x \) in \( B \) and written \( f(x) \).

**Theorem 8** Expansion: \( f(x)=f(1)x+f(0)x' \)

Definition III: An equation \( f(x)=0 \)
in \( B \) is said to be solvable in \( B \), when there exists such an element \( \alpha \) of \( B \) as \( f(\alpha)=0 \)

**Theorem 9** Solvability and Root: A necessary and sufficient condition that \( f(x)=0 \) is solvable in \( B \) is that \( f(1)f(0)=0 \).

And \( x=f(0)+f'(1)u \) \( u \) being an arbitrary element of \( B \), provided that \( f(x)=0 \) is solvable in \( B \).

Definition IV: An element \( a \) of \( B \) is said to be not larger than an element \( b \) of \( B \), if \( ab'=0 \),

and written \( a \leq b \).

**Theorem 10** Any two of the following equalities and inequalities are equivalent with each other: \( ab'=0, a=ab, b'=b'a', a'+b=1, a+b=b, a\leq b, b'\leq a' \).

**Theorem 11** Extrema:

\[ f(1)f(0)\leq f(x)\leq f(1)+f(0) \]

Definition V: An element \( a \) of \( B \) is said to be smaller than an element \( b \) of \( B \), if \( ab'=0 \) and \( a\neq b \),

and written \( a<b \).

**Theorem 12** For any three elements \( a, b \) and \( c \) of \( B \):

If \( a\leq b \) and \( b\leq c \), then \( a\leq c \)

If \( a\leq b \) and \( b<c \), then \( a<c \)

If \( a<b \) and \( b\leq c \), then \( a<c \)

If \( a<c \) and \( b<c \), then \( a<c \)

For the sake of simplicity, let us introduce some nomenclatures.

Given and unanalysed letter is called an atom. By the universe of discourse, we mean the Boolean algebra generated by a given set of atoms. The degree of a universe is defined, after Yule, as the number of its generating atoms. In a universe of degree \( n \), any product of \( n \) atomic letters or their complements is called a fundamental constituent, unless it is 0.

§ 3 Lemma and Theorem

In order to establish the principal theorem, the following lemma shall be proved.

**Lemma** A necessary and sufficient condition that

(1) \( f(x)=0 \)

and

(2) \( g(x)>0 \)

are simultaneously solvable is that both

(3) \( f(1)f(0)=0 \)

and
Proof: Let (1) and (2) be simultaneously solvable, then the single equation (1) is solvable. We have, therefore, (3) and
\[ x = f(0) + f'(1)u \]
by means of Theorem 9.

Due to the simultaneous solvability of (1) and (2), there exists \( u \) such that
\[ g(f(0) + f'(1)u) > 0. \]

Theorem 11 yields
\[ g(f(0) + f'(1)) + g(f(0)) \geq g(f(0)) + f'(1)u \]
and these inequalities are combined to show that
\[ g(f(0) + f'(1)) + g(f(0)) > 0. \]

Expansion of the left-hand side by Theorem 8 and simplification by Theorem 10 conclude the inequality
\[ g(f'(1)) + g(f(0)) > 0. \]

Thus the condition is necessary.

Conversely, let (3) and (4) hold. Then either
\[ (5) \quad g(f'(1)) > 0 \]
or
\[ (6) \quad g(f(0)) > 0 \]
holds.

In the case when (5):
By (3) and Theorem 9 with \( u = 1 \), we have
\[ f(f'(1)) = 0. \]

In general, the inequality
\[ g(f'(1)) = g(f'(1)) + g(f(1)) \geq g(1)f'(1) \]
holds, from which
\[ (7) \quad g(f'(1)) > 0 \]
follows by (5).

In the case when (6):
By (3) and Theorem 9 with \( u = 0 \), we have
\[ f(f(0)) = 0. \]

The general inequality
\[ g(f(0)) = g(f(0)) + g(0)f'(0) \geq g(0)f'(0), \]
combined with (6), yields
\[ (8) \quad g(f(0)) > 0. \]

Thus the condition is sufficient due to either (7), (8) or (9), (10). And the proof is completed.

Now, by finite induction, this lemma shall be generalized into the

[Principal Theorem] A necessary and sufficient condition that
\[ f(x_1, \ldots, x_n) = 0 \]
and
\[ g(x_1, \ldots, x_n) > 0 \]
are simultaneously solvable in \( (x_1, \ldots, x_n) \) is that both
(13) \[ \Pi_{\epsilon_1, \ldots, \epsilon_n} f(\epsilon_1, \ldots, \epsilon_n) = 0 \]

and

(14) \[ \sum_{\epsilon_1, \ldots, \epsilon_n} g(\epsilon_1, \ldots, \epsilon_n) f(\epsilon_1, \ldots, \epsilon_n) > 0 \]

hold, where each \( \epsilon_i \) runs over 1 and 0.

**Proof:** This theorem is true for \( n=1 \) by the lemma. Suppose this theorem to be true for \( n=m \).

And let

(15) \[ f(x_1, \ldots, x_m, x_{m+1}) = 0 \]

and

(16) \[ g(x_1, \ldots, x_m, x_{m+1}) > 0 \]

be simultaneously solvable in \((x_1, \ldots, x_m, x_{m+1})\).

Then there exists such a vector \((\alpha_1, \ldots, \alpha_m, \alpha_{m+1})\) as

(17) \[ f(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}) = 0 \]

and

(18) \[ g(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}) > 0 \]

This means that

(19) \[ f(\alpha_1, \ldots, \alpha_m, x_{m+1}) = 0 \]

and

(20) \[ g(\alpha_1, \ldots, \alpha_m, x_{m+1}) > 0 \]

are simultaneously solvable in \(x_{m+1}\).

Therefore, by the lemma, it is necessary that both

(21) \[ f(\alpha_1, \ldots, \alpha_m, 1) f(\alpha_1, \ldots, \alpha_m, 0) = 0 \]

and

(22) \[ g(\alpha_1, \ldots, \alpha_m, 1) f(\alpha_1, \ldots, \alpha_m, 1) + g(\alpha_1, \ldots, \alpha_m, 0) f(\alpha_1, \ldots, \alpha_m, 0) > 0 \]

hold.

From this necessity, it follows that

(23) \[ f(x_1, \ldots, x_m, 1) f(x_1, \ldots, x_m, 0) = 0 \]

and

(24) \[ g(x_1, \ldots, x_m, 1) f(x_1, \ldots, x_m, 1) + g(x_1, \ldots, x_m, 0) f(x_1, \ldots, x_m, 0) > 0 \]

are solvable in \((x_1, \ldots, x_m)\).

On the inductive assumption for \( n=m \), we have

(25) \[ \Pi_{\epsilon_1, \ldots, \epsilon_m} f(\epsilon_1, \ldots, \epsilon_m, 1) f(\epsilon_1, \ldots, \epsilon_m, 0) = 0 \]

and

(26) \[ \sum_{\epsilon_1, \ldots, \epsilon_m} (g(\epsilon_1, \ldots, \epsilon_m, 1) f(\epsilon_1, \ldots, \epsilon_m, 1) + g(\epsilon_1, \ldots, \epsilon_m, 0) f(\epsilon_1, \ldots, \epsilon_m, 0)) (f(\epsilon_1, \ldots, \epsilon_m, 1) f(\epsilon_1, \ldots, \epsilon_m, 0))^2 > 0 \]

(25) can easily be rewritten in the form

(27) \[ \Pi_{\epsilon_1, \ldots, \epsilon_{m+1}} f(\epsilon_1, \ldots, \epsilon_{m+1}) = 0. \]

The law of dualisation and that of absorption simplify (26) into
Thus, (27) and (28) are the necessary conditions for \( n = m + 1 \).

Conversely, let (27) and (28) hold simultaneously. They can be easily reread as (25) and (26) respectively. And on the inductive assumption for \( n = m \), (23) and (24) are simultaneously solvable in \((x_1, \ldots, x_m)\).

This means that there exists such a vector

\[(a_1, \ldots, a_m)\]

as (21) and (22).

By the lemma, however, they are the sufficient conditions for simultaneous solvability of (19) and (20) in \( x_{m+1} \). Therefore, we have \( \alpha_{m+1} \) together with \( \alpha_1, \ldots, \alpha_{m-1} \) and \( \alpha_m \) such that both (17) and (18) hold.

Thus, (27) and (28) are the sufficient conditions for \( n = m + 1 \).

And the theorem is true for \( n = m + 1 \), provided that such is the case for \( n = m \).

This completes the proof by induction.

This theorem provides us with a new elimination method in Boolean algebra, by means of which such abstraction problems will be mechanically manipulated, as contain particular proposition besides universal ones.

We might justly say that it is a generalized procedure of inferences in classical logic.

§ 4 Applications and Examples

The four kinds of categorical propositions in classical logic might be realized in Boolean algebra as follows:

- all A is B \( ab' = 0 \)
- no A is B \( ab = 0 \)
- some A is B \( ab > 0 \)
- some A is not B \( ab' > 0 \).

Thus, we can manage many kinds of inferences in classical logic as problems in Boolean algebra.

And we can manage them not only from systematic point of view but also heuristic.

For instance, by means of the lemma in § 3 we can answer the question “some A is B and no C is B. What is the relation between A and C?”, instead of proving that some A is B and no C is B, therefore some A is not C.

The procedure is as follows:

“no C is B” is realized in \( f(b) = cb = 0 \)

“some A is B” is realized in \( g(b) = ab > 0 \).

Thus, we have

\[
\begin{align*}
  f(1) &= c & f(0) &= 0 \\
  g(1) &= a & g(0) &= 0,
\end{align*}
\]

and due to the lemma

\[
\begin{align*}
  c0 &= 0 & ac' + 00' &= 0 \\
  0 &= 0 & ac' &= 0,
\end{align*}
\]

i.e.

The first equality tells nothing. The second inequality, however, yields a new knowledge
on A and C: some A is not C.

A somewhat complicated and rather practical problem shall be solved as an example.

Example. One evening, Mr. E received independently following four informations about behaviours of four students A, B, C and D on that day:
1. Whenever C remained at hostel, B did also and A was absent therefrom.
2.Whenever B remained, either D or C did.
3. When both B and C were out, A did not remain.
4. There were some hours, when B was in and both A and C were out or when C in and both B and D out, or when D in and both B and C out.

What can Mr. E infer on the relation between A and D?

Solution: Let a designate that A remained at hostel. Then a' designates that A was out. And similarly with other letters and students.

The above mentioned informations are realized in Boolean algebra as follows:

(1) \( \neg B' + A \) = 0
(2) \( BD'C' = 0 \)
(3) \( B'C'A = 0 \)
(4) \( BA'C' + CB'D' + DB'C' > 0 \)

(1), (2) and (3) are combined into a single equation

(5) \( \neg B' + A + BD' + B'C'A = 0 \).

Since B and C are to be eliminated, the left-hand side of (5) and (4) shall be designated by \( f(b, c) \) and \( g(b, c) \) respectively, i.e.

(6) \( f(b, c) = \neg (a + b') + BD' + B'C'A \)
(7) \( g(b, c) = BA'C' + CB'D' + DB'C' \).

Thus, we have

\( f(1, 1) = a, f(1, 0) = d', f(0, 1) = 1, f(0, 0) = a \)

and

\( g(1, 1) = 0, g(1, 0) = a', g(0, 1) = d', g(0, 0) = d. \)

The principal theorem yields

\( ad = 0 \) and \( a'd > 0, \)

by means of which Mr. E is able to conclude that whenever A remained, D did also, and there were some hours when A was out and D was in.

§ 5 A Switching Network

In this section, a switching network shall be proposed to realize the algebraic procedures explained in the preceding sections.

It was initiated by K. Yamada in principle and by K. Yoshida in synthesis. And it has been improved by Yamada to be workable with those problems which involve particular proposition besides universal ones.

This device consists of six major parts: three switch boards, one selector and two eliminators as shown in Fig. 1.

For the sake of concreteness, hereafter, explanations and figures shall be of degree 4.

The selection board (SE. B.) contains eight on-off switches, each labeled with one letter unprimed or primed. They are called selection switches. The selector (S.) contains thirty normally closed contacts, each for one letter unprimed or primed, and sixteen signals, each
for one fundamental constituent, numbered 1 to 16.

These two parts are connected in a way such that for instance when the switch $a$ is set, each contact for $a'$ will be opened and when the switch $a'$ is set, each contact for $a$ will be opened.

After some applications of selection switches, there will remain one signal or more glowing to tell us which of the sixteen constituents should be stored as informations on the
second step.

The storage board (ST. B.) contains sixteen on-off switches, each for one fundamental constituents labeled with the corresponding number. The eliminators are of two kinds. The STORAGE BOARD

![Diagram of Storage Board]

first is that of pure eliminator (P. E.) which consists of ten switch circuits and will work to yield (13) of the principal theorem. Four of the circuits are the monomial eliminators, each

**PURE ELIMINATOR (a)**

![Diagram of Pure Eliminator (a)]

**PURE ELIMINATOR (b)**

![Diagram of Pure Eliminator (b)]

**PURE ELIMINATOR (c)**

![Diagram of Pure Eliminator (c)]

**PURE ELIMINATOR (d)**

![Diagram of Pure Eliminator (d)]
for one letter to be eliminated, containing sixteen normally opened contacts and eight conclusion signals. Six of them are the binomial eliminators, each for one set of two letters to be eliminated, containing sixteen normally opened contacts and eight conclusion signals. The second is that of mixed eliminator (M. E.) which will work to yield (14) of the principal theorem. It consists of two arrays A and B of sixteen contacts and sixteen conclusion signals. Each contact of A is for one fundamental constituent and normally opened. Each contact of B is for one fundamental constituent and normally opened.

The storage board is connected with the eliminators in the following way. For instance, when the switch 1 is set, each contact for 1 of the pure eliminators will be closed and simultaneously that of A of the mixed eliminator will be opened. B of the mixed eliminator is connected with the storage board in a way such that it will work only when the informations of (12) of the principal theorem are being put in, and for instance when switch 1 is set, each contact for 1 will be closed.

After some applications of switches by means of glowing signals of the selector, both of the eliminators will be ready to yield conclusions desired on the third step.

The elimination board (E. B.) contains two dial switches. One is the monomial elimination switch with four points, each for one letter to be eliminated. The other is the binomial elimination switch with six points, each for one set of two letters to be eliminated.
The elimination board and the eliminators are connected in a way such that when the dial switch is set at one of the ten points, the current will flow in the corresponding pure eliminator and in those signal lamps of the mixed eliminator with other letters from the letter or letters labeling the points.

Thus, the glowing conclusion signal lamp or lamps will abstract the mutual relations of the assigned letters: those of the pure eliminator shall read universally and conjunctively and those of the mixed particularly and disjunctively.

§ 6 An Example

Let us solve the problem of § 4 by means of the switching network to illustrate the
procedure.
Let us store the first half
of the information (1).

The selection switch $c$ is set and the selector will be in the state shown in Fig. 17. Again
the selection switch $b'$ is set and the selector will be in the state shown in Fig. 18, where
signal lamps 5, 6, 13 and 15 are glowing.

Now the storage switches 5, 6, 13 and 14 shall be set and the eliminators will be in the
states shown in Figs. 19 to 29.

P. E. a

P. E. b
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P. E. c

Fig. 21

P. E. d

Fig. 22

P. E. (ab)

Fig. 23

P. E. (ac)

Fig. 24

P. E. (ad)

Fig. 25

P. E. (bc)

Fig. 26

P. E. (bd)

Fig. 27

P. E. (cd)

Fig. 28
After all the informations (1) to (4) are stored, the eliminators will be in the states shown in Figs. 30 to 40 to be ready for conclusion.
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P. E. c

Fig. 32

P. E. d

Fig. 33

P. E. (ab)

Fig. 34

P. E. (ac)

Fig. 35

P. E. (ad)

Fig. 36

P. E. (be)

Fig. 37

P. E. (bd)

Fig. 38

P. E. (cd)

Fig. 39
Note, however, that no conclusion lamp is yet glowing.

Now we are able to have the conclusions only by pointing the binomial elimination switch to $(b \land c)$. Thus current will flow through the pure eliminator $(b \land c)$ and through $a$ and $d$ lamps of the mixed eliminator. And the $ad'$ signal will glow in the pure eliminator and two $a'ds$ in the mixed eliminator as shown in Figs. 41 and 42.

They should read respectively, "Whenever $A$ remained, $D$ did also", and "There were some hours when $A$ was out and $D$ was in".

§ 7 Concluding Remarks

The principal theorem is not very general in that it yields a simultaneous solvability condition for $f(x_1, \ldots, x_n)=0$ and $g(x_1, \ldots, x_n)>0$, but not for $f(x_1, \ldots, x_n)=0$, $g_1(x_1, \ldots, x_n)>0$, $\ldots$, $g_{n-1}(x_1, \ldots, x_n)>0$ and $g_n(x_1, \ldots, x_n)>0$.

It might be, however, of high significance in that it enables us to work with problems involving one particular proposition algebraically and mechanically, while it has been hitherto rather difficult to manage particular propositions algebraically or mechanically.

Generalisation of the principal theorem is a most interesting and significant problem not only from mathematical point of view but also logical.

References


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