ON LINEAR FRACTIONAL PROGRAMMING

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Introduction

In this paper we shall deal with the problem to maximize a linear fraction of some variables under the condition that they satisfy some linear inequalities.

We shall use the following notations and definition:

\[ A \ (m \times n), \ B \ (m \times 1), \ C \ (1 \times n), \ D \ (1 \times n) \] given matrix and vectors,

\[ X \ (n \times 1), \ z \ (1 \times 1) \] variable vector and scalar,

\[
\begin{align*}
CX &= \begin{cases} 
\infty, & \text{if } CX > 0 \text{ and } DX = 0, \\
-\infty, & \text{if } CX < 0 \text{ and } DX = 0, \\
0/0, & \text{if } CX = DX = 0,
\end{cases}
\end{align*}
\]

where 0/0 is to be understood as merely a symbol between which and other symbols or numbers we do not define the order relation.

The object of this paper is to find an optimal solution of

**Problem 0:** Maximize \( CX/DX \) subject to \( AX \leq B, \ DX \geq 0, \ X \geq 0. \)

Isbell and Marlow [4] considered Problem 0 under the condition that the set of all feasible solutions of Problem 0 is a bounded polyhedron and that there does not exist a solution such that \( DX = 0. \) Yamada [5] assumed further that \( B > 0, \ C \geq 0, \ C \neq 0, \ D \geq 0, \ D \neq 0. \) I do not assume these conditions. Dorn [2] reports further results about the problem but I have not been able to know the details. 

I take up the condition \( DX \geq 0 \) explicitly because the problem to maximize \( CX/DX \) over both regions \( DX > 0 \) and \( DX < 0 \) is meaningless, as is the case to maximize \( 1/x \) over the region \(-1 \leq x \leq 1.\)

**Theorem 1:** The set \( S \) of all feasible solutions of Problem 0 is of the following form, if it is not empty,

\[
S = \{ X ; X = u_1P_1 + \ldots + u_kP_k + v_1Q_1 + \ldots + v_lQ_l \text{,} \quad u_1 \geq 0, \ldots, u_k \geq 0, \quad u_1 + \ldots + u_k = 1, \quad v_1 \geq 0, \ldots, \quad v_l \geq 0 \},
\]

where \( k \geq 1, \ l \geq 0. \) The set \( T \) of all solutions of

\[
AX \leq 0, \quad DX \geq 0, \quad X \geq 0
\]

is

\[
T = \{ X ; X = v_1Q_1 + \ldots + v_lQ_l, \quad v_1 \geq 0, \ldots, v_l \geq 0 \}.
\]

The proof of Theorem 1 is given e.g. in Goldman [3].

**Theorem 2:** If \( S \) is not empty, then we have

\[
\sup_{X \in S} \frac{CX}{DX} = \max \left( \frac{CP_1}{DP_1}, \ldots, \frac{CP_k}{DP_k}, \frac{CQ_1}{DQ_1}, \ldots, \frac{CQ_l}{DQ_l} \right).
\]

If the maximum in the right side is among first \( k \) members, say \( CP_i/DP_i, \) then we have
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\[
\max_{x \in S} \frac{CX}{DX} = \frac{CP}{DP},
\]

If the maximum in the right side of (2) is e.g. \( CQ_1/DQ_1 \) and if every \( CP_i/DP_i \), \( i \neq 0/0 \), is less than it, then we have

\[
\sup_{x \in S} \frac{CX}{DX} = \lim_{v \to \infty} \frac{C(P+vQ_1)}{D(P+vQ_1)} = CQ_1,
\]

where \( P \) is an arbitrary feasible solution of Problem 0. If further there exists a feasible solution \( P \) such that \( CP=DP=0 \), then the above supremum is attained, thus we have

\[
\max_{x \in S} \frac{CX}{DX} = \frac{C(P+vQ_1)}{D(P+vQ_1)} = CQ_1.
\]

If all members in the right side of (2) are 0/0, then \( CX/DX=0/0 \) for every feasible solution \( X \) of Problem 0.

The proof of Theorem 2 is evident.

We admit the values \( \infty \) and \( -\infty \) as maximum, minimum, supremum and infimum.

We understand that Problem 0 is solved when we can find

1) a vector \( P \) such that \( P \in S \) and

\[
\max_{x \in S} \frac{CX}{DX} = \frac{CP}{DP},
\]

if the maximum exists, or

2) a pair of vectors \((P, Q)\) such that \( P \in S, Q \in T \) and

\[
\sup_{x \in S} \frac{CX}{DX} = \lim_{v \to \infty} \frac{C(P+vQ)}{D(P+vQ)},
\]

if the maximum does not exist but the supremum has a sense, or

3) that \( CX/DX=0/0 \) for every \( X \in S \), though \( S \) is not empty, or

4) that \( S \) is empty, i.e. Problem 0 is non-feasible.

In case 1) or 2) we call \( P \) or \((P, Q)\) respectively an optimal solution of Problem 0.

**Simplex Criterion for Problem 0**

In this section we shall generalize and complete the simplex criterion for Problem 0 which was given by Yamada [5] under the restricted conditions.

At first we assume that

\( a) \ S \) is a bounded polyhedron,
\( b) \) there exists such an \( X \in S \) that \( DX>0 \),
\( c) \) the degeneracy does not occur.

Let

\[
A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},
\]

\[
C = (c_1, \ldots, c_n), \quad D = (d_1, \ldots, d_n),
\]

and put

\[
c_{n+1} = \ldots = c_{m+n+1} = d_{n+1} = \ldots = d_{m+n+1} = 0.
\]

Introducing \( m+1 \) slack variables \( x_{n+1}, \ldots, x_{m+n+1} \), Problem 0 can be stated as follows:

To maximize

\[
\sum_{i=1}^{m+n+1} c_i x_i / \sum_{i=1}^{m+n+1} d_i x_i
\]
subject to
\[ \sum_{i=1}^{m+n+1} A_i x_i = \begin{pmatrix} B \\ 0 \end{pmatrix} \quad \text{and} \quad x_1, \ldots, x_{m+n+1} \geq 0, \]
where
\[ A_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ -d_1 \end{pmatrix}, \ldots, A_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \\ -d_n \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_{m+n+1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \]

By the above assumptions (a) and (b), the function (3) attains the maximum \((-\infty, \infty)\) at some vertex of the solution set of (4). Any vertex can be obtained as follows. Let \( I \) be a set of \( m+1 \) indices taken from the set \( \{1, 2, \ldots, m+n+1\} \), and let \( m+1 \) vectors \( A_i \) (\( i \in I \)) be linearly independent. Solve \( \sum_{i \in I} A_i x_i = \begin{pmatrix} B \\ 0 \end{pmatrix} \) for \( x_i \) (\( i \in I \)) and put \( x_i = 0 \) (\( i \notin I \)). If all \( x_i \) (\( i \in I \)) are non-negative, then the values thus obtained are coordinates of a vertex. By the assumption (c) all \( x_i \) (\( i \in I \)) are positive in this case.

Let \( I \) determine a vertex \( (x_1, \ldots, x_{m+n+1}) \), and let the value of (3) be finite or \( \infty \) at it. One of such vertices can be found by solving a problem of linear programming.

We can find a set of coefficients \( \lambda_{ij} \) such that
\[ A_j = \sum_{i \in I} \lambda_{ij} A_i \quad (j = 1, \ldots, m+n+1). \]
Let \( j \notin I \) and \( \theta > 0 \), then we have
\[ \sum_{i \in I} A_i (x_i - \theta \lambda_{ij}) + A_j \theta = \begin{pmatrix} B \\ 0 \end{pmatrix}, \]
therefore
\[ x_i^* = x_i - \theta \lambda_{ij} \quad (i \notin I), \quad x_j^* = \theta, \quad x_k^* = 0 \quad (k \notin I, k \neq j) \]
is a feasible solution of (4), if
\[ x_i^* = x_i - \theta \lambda_{ij} \geq 0 \quad (i \in I). \]
Note that for every \( j \notin I \), there exists at least one \( i \in I \) such that \( \lambda_{ij} > 0 \).

We have
\[ \sum_{i \in I} c_i (x_i - \theta \lambda_{ij}) + c_j \theta = \sum_{i \in I} c_i x_i \]
according as
\[ (\sum_{i \in I} d_i x_i)(c_j - \sum_{i \in I} c_i \lambda_{ij}) \equiv (\sum_{i \in I} c_i x_i)(d_j - \sum_{i \in I} d_i \lambda_{ij}), \]
provided the right side of (6) is finite and the left side is not \( 0/0 \). If the right side of (6) is \( \infty \), then the inequality \( > \) does not hold in (7).

There occurs one of two exclusive cases.

1) there exists at least one \( j \) such that \( j \notin I \) and the inequality \( > \) holds in (7).

2) for all \( j \) inequalities or equalities \( \leq \) hold in (7).

In case 1) we have another vertex from (5), putting
\[ \theta = \min_{x_i^* \geq 0} \frac{x_i^*}{\lambda_{ij}}, \]
and then (3) will have a greater value. The case 1) does not hold indefinitely, so we can arrive at the case 2) in finite steps.

If the case 2) occurs, then for any feasible solution \( x_1^*, \ldots, x_{m+n+1}^* \) of (4), we have
\[ x_i^* = x_i - \sum_{j \notin I} \lambda_{ij} x_j^* \quad (i \in I), \]
and then (3) will have a greater value. The case 1) does not hold indefinitely, so we can arrive at the case 2) in finite steps.
and
\[(\sum_{i \in I} d_i x_i) \sum x_i^* (c_i - \sum_{i \in I} c_i \lambda_i) \leq (\sum_{i \in I} c_i x_i) \sum x_i^* (d_i - \sum_{i \in I} d_i \lambda_i)\].

From (8) we have
\[
\sum x_i^* (c_i - \sum_{i \in I} c_i \lambda_i) = \sum_{i=1}^{m+n+1} c_i x_i^* - \sum_{i \in I} c_i x_i,
\]
\[
\sum x_i^* (d_i - \sum_{i \in I} d_i \lambda_i) = \sum_{i=1}^{m+n+1} d_i x_i^* - \sum_{i \in I} d_i x_i.
\]
Inserting these two equalities into (9) and noting that
\[-\infty < \sum_{i \in I} c_i x_i / \sum_{i \in I} d_i x_i \leq \infty,
\]
we have
\[
\sum_{i=1}^{m+n+1} c_i x_i^* / \sum_{i \in I} d_i x_i^* \leq \sum_{i \in I} c_i x_i / \sum_{i \in I} d_i x_i,
\]
unless the left side is 0/0. Therefore I gives a vertex where (3) takes on maximum.

Thus we have proved that if the three conditions (a), (b), (c) are satisfied, Problem 0 can be solved by the simplex method based on (7). Of these three conditions (b) is essential. If it is not satisfied, \(CX/DX\) can take only \(-\infty, -\infty, \infty\) and 0/0, so the Problem 0 can be solved trivially, as seen in Case (5) of the next section. If only (b) is satisfied, (a) and (c) can be satisfied by the simple devices used in linear programming (see Charnes, Cooper and Henderson [1]).

We could give a method of solving Problem 0 in the most general situation in the sense given at the end of Introduction, by using the above simplex method along with some problems of linear programming.

**Reduction to Problems of Linear Programming**

In this section we shall give a method of solving Problem 0 in the most general case by using only problems of linear programming. It will be much simpler than that described in the previous section.

We shall use the following problems.

**Problem 1:** Maximize \(CX\) subject to \(AX-Bz \leq 0, DX=1, X \geq 0, z \geq 0\).

**Problem 2:** Maximize \(CX\) subject to \(AX \leq B, DX=0, X \geq 0\).

**Problem 3(r):** Maximize \((C-rD)X\) subject to \(AX \leq B, DX=0, X \geq 0\), where \(r\) is a parameter.

**Problem 4:** Minimize \(CX\) subject to \(AX \leq B, DX=0, X \geq 0\).

Note that if \(\left(\begin{array}{c} X \\ z \end{array}\right)\) is a feasible solution of Problem 1 and \(z > 0\), then \(\frac{X}{z}\) is a feasible solution of Problem 0 such that \(D(\frac{X}{z}) > 0\) and \(C(\frac{X}{z})/D(\frac{X}{z}) = CX\).

In the following we shall subdivide cases according to the decimal system. We shall denote by \(\max_i (\min_i)\) the maximum (minimum) given by solving Problem \(i\), if it exists. If it does not exist but the corresponding supremum (infimum) has a sense, then we shall denote it by \(\sup_i (\inf_i)\).

The method proceeds as follows.

Solve Problem 1.
Case (1): \( \sup_1 = \infty \) and \( z_{11} > 0 \). We denote by \( \begin{pmatrix} X_{10} \\ z_{10} \end{pmatrix} \) a feasible solution of Problem 1, and by \( \begin{pmatrix} X_{11} \\ z_{11} \end{pmatrix} \) a solution of

\[
C \mathbf{x} > 0, \quad A \mathbf{x} - B \mathbf{z} \leq 0, \quad D \mathbf{z} = 0, \quad \mathbf{x} \geq 0, \quad \mathbf{z} \geq 0.
\]

We can find these two vectors in the last step when we solve Problem 1 by the simplex method. In this case \( \frac{X_{11}}{z_{11}} \) is a feasible solution of Problem 0 and \( C \left( \frac{X_{11}}{z_{11}} \right) / D \left( \frac{X_{11}}{z_{11}} \right) = \infty \). Therefore \( \frac{X_{11}}{z_{11}} \) is an optimal solution of Problem 0 and we have \( \max_0 = \infty \).

Case (2): \( \sup_1 = \infty \) and \( z_{11} = 0 \). We use the symbols \( \begin{pmatrix} X_{10} \\ z_{10} \end{pmatrix} \) and \( \begin{pmatrix} X_{11} \\ z_{11} \end{pmatrix} \) in the same sense as in Case (1). \( X_{11} \) satisfies

\[
C X_{11} > 0, \quad A X_{11} \leq B, \quad D X_{11} = 0, \quad X_{11} \geq 0.
\]

Solve Problem 2. It can not give a finite \( \max_2 \). For, if we had a finite \( \max_2 \) and \( X_2 \) were an optimal solution of Problem 2, then \( X_2 + X_{11} \) would be a feasible solution of Problem 2 such that

\[
C(X_2 + X_{11}) > CX_2 = \max_2.
\]

Case (2.1): \( \sup_2 = \infty \). Let \( X_2 \) be a solution of

\[
C \mathbf{x} > 0, \quad A \mathbf{x} \leq B, \quad D \mathbf{x} = 0, \quad \mathbf{x} \geq 0,
\]

then \( X_2 \) is an optimal solution of Problem 0 and we have \( \max_0 = \infty \).

Case (2.2): Problem 2 is non-feasible and \( z_{10} > 0 \). In this case there does not exist a feasible solution \( X \) of Problem 0 such that \( CX/DX = \infty \). As \( \frac{X_{10}}{z_{10}} \) is a feasible solution of Problem 0 and \( X_{11} \) satisfies (10), \( \begin{pmatrix} X_{10} \\ z_{10} \\ X_{11} \end{pmatrix} \) is an optimal solution of Problem 0 and we have \( \max_0 = \infty \).

Case (2.3): Problem 2 is non-feasible and \( z_{10} = 0 \). Solve Problem 3(0).

Case (2.3.1): Problem 3(0) is feasible. Let \( X_3 \) be a feasible solution of Problem 3(0), then we can see by the same reason as in Case (2.2) that \( (X_3, X_{11}) \) is an optimal solution of Problem 0 and we have \( \max_0 = \infty \).

Case (2.3.2): Problem 3(0) is non-feasible. In this case Problem 0 is also non-feasible.

Case (3): \( \max_1 = s < \infty \) and \( z_1 > 0 \). We denote by \( \begin{pmatrix} X_1 \\ z_1 \end{pmatrix} \) an optimal solution of Problem 1.

In this case \( \frac{X_1}{z_1} \) is a feasible solution of Problem 0 and we have \( C \left( \frac{X_1}{z_1} \right) / D \left( \frac{X_1}{z_1} \right) = s \).

From \( \max_1 = s < \infty \) we can prove that we have no feasible solution \( X \) of Problem 0 such that \( s < CX/DX \leq \infty \). First, if we had a feasible solution \( X \) of Problem 0 such that \( CX/DX = \infty \), then it would satisfy (11), so \( \left( \frac{X_1 + X}{z_1 + 1} \right) \) would be a feasible solution of Problem 1 such that

\[
C(X_1 + X) > CX_1 = \max_1.
\]

Secondly, let \( X \) be an arbitrary feasible solution of Problem 0 such that \( -\infty < CX/DX < \infty \). Then we have \( DX > 0 \). Putting \( z = 1/DX > 0 \), we then have a feasible solution \( \begin{pmatrix} z X \\ z \end{pmatrix} \) of Problem 1, therefore

\[
CX/DX = C(zX) \leq \max_1 = s.
\]
Thus we have proved that \( \frac{X_i}{z_i} \) is an optimal solution of Problem 0 and that we have \( \max_s = s \).

Case (4): \( \max_t = s \) and \( z_i = 0 \). We use the symbol \( \left( \frac{X_i}{z_i} \right) \) in the same sense as in Case (3). \( X_i \) satisfies

\[
CX_i = s, \quad AX_i \leq 0, \quad DX_i = 1, \quad X_i \geq 0.
\]

Solve Problem 3(s). By the same reason as in Case (3) we have no feasible solution \( X \) of Problem 0 such that \( CX/DX > s \). Therefore we have \( \max_s \leq s \), if Problem 3(s) is feasible.

Case (4.1): \( \max_3 = 0 \). Let \( X_3 \) be an optimal solution of Problem 3(s), then \( X_i \) satisfies

\[
CX_i = 0, \quad AX_i \leq B, \quad DX_i \geq 0, \quad X_i \geq 0,
\]

therefore \( X_3 + X_1 \) is a feasible solution of Problem 0 and we have \( CX(X_3 + X_1)/DX(X_3 + X_1) = s \). Hence \( X_3 + X_1 \) is an optimal solution of Problem 0 and we have \( \max_s = s \).

Case (4.2): \( \max_3 < 0 \). In this case we have no feasible solution \( X \) of Problem 0 such that \( CX/DX < s \). Let \( X_3 \) be an optimal solution of Problem 3(s), then it is a feasible solution of Problem 0. (\( X_3, X_i \)) is an optimal solution of Problem 0 and we have \( \sup_3 = s \).

Case (4.3): Problem 3(s) is non-feasible. In this case Problem 0 is also non-feasible.

Case (5): Problem 1 is non-feasible. In this case we have no feasible solution \( X \) of Problem 0 such that \( DX > 0 \). Solve Problem 2.

Case (5.1): \( \sup_2 = \infty \) or \( \max_s > 0 \). Let \( X_2 \) be a solution of (11), then \( X_2 \) is an optimal solution of Problem 0 and we have \( \max_s = \infty \).

Case (5.2): \( \max_2 = 0 \). There remains only possibilities of \( CX/DX = \infty \) or \( 0 \) for every feasible solution \( X \) of Problem 0. Solve Problem 4.

Case (5.2.1): \( \inf_4 = 0 \) or \( \min_4 < 0 \). Let \( X_4 \) be a solution of

\[
CX < 0, \quad AX \leq B, \quad DX = 0, \quad X_i \geq 0,
\]

then \( X_4 \) is an optimal solution of Problem 4 and we have \( \max_s = \infty \).

Case (5.2.2): \( \min_4 = 0 \). \( CX/DX = 0/0 \) for every feasible solution \( X \) of Problem 0.

Case (5.3): \( \max_2 < 0 \). Let \( X_2 \) be an optimal solution of Problem 2, then \( X_2 \) is an optimal solution of Problem 0 and we have \( \max_s = \infty \).

Case (5.4): Problem 5 is non-feasible. In this case Problem 0 is also non-feasible.

According to the above method we can arrive at a complete solution of Problem 0 in any case by solving at most three problems of linear programming, and especially in case when \( \max_s \) or \( \sup_s \) is finite, we can arrive by solving at most two, occasionally one.

\textbf{Example 1.} Maximize \( \frac{x_1 + x_2}{2x_1 - x_2} \) subject to

\[
\begin{align*}
-2x_1 + x_2 & \leq -2, \\
x_1 - x_2 & \leq 2, \\
2x_1 - x_2 & \geq 0, \\
x_1, x_2 & \geq 0.
\end{align*}
\]

1) Solve Problem 1:

Maximize \( x_1 + x_2 \) subject to

\[
\begin{align*}
-2x_1 + x_2 + 2x & \leq 0, \\
x_1 - x_2 - 2x & \leq 0, \\
2x_1 - x_2 & = 1, \\
x_1, x_2, z & \geq 0.
\end{align*}
\]

We have

\[
\sup_t = \infty; \quad \left( \begin{array}{c} X_{10} \\ x_{10} \end{array} \right) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad \left( \begin{array}{c} X_{11} \\ x_{11} \end{array} \right) = \left( \begin{array}{c} 1 \\ 2 \end{array} \right),
\]

hence we are in Case (2).
2) Solve Problem 2:

Maximize $x_1 + x_2$ subject to

\[
\begin{align*}
-2x_1 + x_2 &\leq -2, \\
x_1 - x_2 &\leq 2, \\
2x_1 - x_2 &= 0, \\
x_1, x_2 &\geq 0.
\end{align*}
\]

This problem is non-feasible, hence we are in Case (2.3).

3) Solve Problem 3(0):

Maximize $x_1 + x_2$ subject to

\[
\begin{align*}
-2x_1 + x_2 &\leq -2, \\
x_1 - x_2 &\leq 2, \\
2x_1 - x_2 &\geq 0, \\
x_1, x_2 &\geq 0.
\end{align*}
\]

This problem is feasible and \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is a feasible solution.

Therefore we are in Case (2.3.1), hence \( \sup_0 = \infty \) and \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) is an optimal solution of this example.

**Example 2.** Maximize $\frac{x_1 + x_2}{3x_1 + 2x_2}$ subject to

\[
\begin{align*}
x_1 &\leq 2, \\
-x_1 + x_2 &\leq -1, \\
3x_1 + 2x_2 &\geq 0, \\
x_1, x_2 &\geq 0.
\end{align*}
\]

1) Solve Problem 1:

Maximize $x_1 + x_2$ subject to

\[
\begin{align*}
x_1 &\leq -2z \leq 0, \\
-x_1 + x_2 + z &\leq 0, \\
3x_1 + 2x_2 &= 1, \\
x_1, x_2 &\geq 0.
\end{align*}
\]

We have an optimal solution

\[
x_1 = \frac{1}{4}, \ x_2 = \frac{1}{8}, \ z = \frac{1}{8} > 0; \ \text{max} = \frac{3}{8}.
\]

Hence we are in Case (3), and \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) is an optimal solution of this example.

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**References**


