# ON STOCHASTIC PROGRAMMING IV

## ——A Note on a Generalized Stochastic Programming Model——

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This paper extends the stochastic programming model proposed in my previous papers (1) (2) (3) to a more general one, and discusses a possible device for computation by the cutting plane method.

### I *Assumptions and Formulation*

Suppose $x(x_1, x_2, \ldots\ldots, x_n)$ and $p(p_1, p_2, \ldots\ldots, p_n)$ be $n$-vectors, $a_{n+1}(a_{1,\,n+1}, a_{2,\,n+1}, \ldots\ldots, a_{m,n+1})$ an $m$-vector and $A(a_{ij})$ be an $m\times n$-matrix. Furthermore the components of the vectors $p, a_{n+1}$ and the matrix $A$ be random variables of known distributions.

Then we define the general stochastic programming as follows:

$$\max \ f, \tag{1.1}$$

subject to

$$\text{Prob}\,(p'x-f\geqq 0)\geqq\beta_0, \tag{1.2}$$

and

$$\text{Prob}\,(\sum_{j=1}^{n} a_{ij}x_j+a_{i,\,n+1}\geqq 0)\geqq\beta_i, \tag{1.3}$$

$$x_j\geqq 0, \tag{1.4}$$

$$i=1, 2, \ldots\ldots, m;\ \ j=1, 2, \ldots\ldots, n,$$

where $p'$ is a transposed vector of $p$, and $\beta_0$ and $\beta_i$ are lowest values of probabilities of the conditions being held.[1]

*Assumption* 1: The random variables $p_j$s, $a_{i,\,n+1}$s and $a_{ij}$s have multinormal[2] distributions of which means, variances and covariances are

$$E_jp=\bar{p}_j,\ E(p_k-\bar{p}_k)(p_l-\bar{p}_l)=v_{kl} \qquad k, l=1, 2, \ldots\ldots, n,$$

$$Ea_{ij}=\bar{a}_{ij},\ E(a_{ik}-\bar{a}_{ik})(a_{il}-\bar{a}_{il})=w_{ikl} \qquad k, l=1, 2, \ldots\ldots, n+1.$$

In order to transform the condition (1.3) into such a nonlinear constraint as used in (3) and to prove its convexity, a dummy variable $x_{n+1}$, identically equal to unity, is introduced

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[1] The physical meanings of these probabilistic conditions were discussed in the previous paper (3) pp. 44—45.

[2] In the references (4) and (5), Charnes and Cooper treated a normal distribution case.

for $\bar{a}_{i,n+1}$ $(i=1, 2, \ldots\ldots, m)$.

Since $a_{ij}$s are multinormal random variables, their linear combination

$$\sum_{j=1}^{n} a_{ij}x_j + a_{i,n+1}x_{n+1}$$

is also a normal random variable, of which mean and variance are

$$\sum_{j=1}^{n+1} \bar{a}_{ij}x_j$$

and

$$\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l \quad (x_{n+1}=1).$$

Using these values, we transform the condition (1.3) as follows:

$$\mathrm{Prob}\left(\frac{\sum_{j=1}^{n+1}(a_{ij}-\bar{a}_{ij})x_j}{\sqrt{\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l}} \geqq \frac{-\sum_{j=1}^{n+1}\bar{a}_{ij}x_j}{\sqrt{\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l}}\right) \geqq \beta_i,$$

$$(x_{n+1}=1). \tag{1.5}$$

Because the left-hand side in the argument is a normalized random variable, we obtain the following equivalent inequality,

$$\frac{-\sum_{j=1}^{n+1}\bar{a}_{ij}x_j}{\sqrt{\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l}} \leqq G^{-1}(\beta_i)$$

or

$$\sum_{j=1}^{n+1}\bar{a}_{ij}x_j + G^{-1}(\beta_i)\sqrt{\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l} \geqq 0, \tag{1.6}$$

where

$$G(y) = \frac{1}{\sqrt{2\pi}}\int_y^{\infty} e^{-\frac{z^2}{2}}dz.$$

As in the appendix of (3), the convexity of the function

$$\sqrt{\sum_{k,l} v_{kl}x_kx_l}$$

is proved for the variance matrix $V(v_{kl})$, it is easily concluded that a function

$$\sqrt{w_{i,n+1,n+1} + 2\sum_{j=1}^{n} w_{i,n+1,j}x_j + \sum_{k,l=1}^{n} w_{ikl}x_kx_l}$$

is also convex, putting $x_{n+1}=1$ in the convex function,

$$\sqrt{\sum_{k,l=1}^{n+1} w_{ikl}x_kx_l}.$$

Furthermore we assume a plausible assumption as follows:
*Assumption* 2:

$$\beta_0, \beta_i \geqq 0.5,$$

which provide us a negative value for $G^{-1}(\beta_i)$,

$$G^{-1}(\beta_i) = -q_i, \qquad q_i > 0.$$

From (1. 6) and the *Assumption* 2, we have a convex domain of $x$ for the probabilistic condition (1. 3),

$$g_i(x) = \sum_{j=1}^{n} \bar{a}_{ij} x_j + \bar{a}_{i,\,n+1} - q_i \sqrt{\sum_{k,l=1}^{n} w_{ikl} x_k x_l + 2 \sum_{j=1}^{n} w_{i,\,n+1,\,j} x_j + w_{i,\,n+1,\,n+1}} \geqq 0. \quad (1.7)$$

In the similar way, the constraint (1. 2) is also transformed into a convex domain,

$$g_0(x) = \sum_{j=1}^{n} \bar{p}_j x_j - q_0 \sqrt{\sum_{k,l=1}^{n} v_{kl} x_k x_l} - f \geqq 0, \quad (1.8)$$

where

$$q_0 = -G^{-1}(\beta_0) > 0.$$

For the sake of brevity, we combine (1. 7) and (1. 8) together in the following way:

$$g_i(x) = \sum_{j=0}^{n} \bar{a}_{ij} x_j + \bar{a}_{i,\,n+1} - q_i \sqrt{\sum_{k,l=0}^{n} w_{ikl} x_k x_l + 2 \sum_{j=0}^{n} w_{i,\,n+1,\,j} x_j + w_{i,\,n+1,\,n+1}} \geqq 0, \quad (1.9)$$

$$i = 0, 1, \ldots\ldots, m$$

where

$$\bar{a}_{0,\,j} = \bar{p}_j, \quad \bar{a}_{0,\,n+1} = 0, \quad x_0 = f, \quad \bar{a}_{00} = -1,$$

$$w_{0kl} = \begin{cases} v_{kl} & \text{if } k, l = 1, 2, \ldots\ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{a}_{i0} = 0, \quad w_{i,\,k,\,0} = w_{i,\,0,\,l} = 0 \quad \text{for} \quad \begin{array}{l} i = 1, 2, \ldots\ldots, m, \\ k, l = 0, 1, 2, \ldots\ldots, n+1. \end{array} \quad (1.10)$$

Consequently the stochastic programming (1. 1), (1. 2), (1. 3) and (1. 4) would be solved as a maximization problem in a convex domain.

## II *Cutting Plane Method*

The idea of the cutting plane method was originally conceived by Dantzig in his famous "Traveling Salesman Problem" and promoted the later investigation on integer programming by Gomory. The same idea was also applied to convex nonlinear programming by Kelley.[6]

Before mentioning the computational procedure of the convex nonlinear programming, let us take a note on some properties of a convex domain.

*Property* 1: If a function of a vector $x, g(x)$, is concave and differentiable then the domain of $x$

$$g(x)\geqq 0$$

is convex and is included in a half space

$$\sum_j (x_j-x_j^0)\cdot\frac{\partial}{\partial x_j}g(x^0)\geqq 0 \tag{2.1}$$

or simply

$$(x-x^0)\cdot\frac{\partial}{\partial x}g(x^0)\geqq 0, \tag{2.2}$$

where $x^0$ is an $m$-vector satisfying

$$g(x^0)=0.$$

*Proof.* The convexity of the domain $g(x)\geqq 0$ was proved in (2). Since $g(x)$ is concave, we have

$$g(x^0)+\lambda(g(x)-g(x^0))\leqq g(x^0+\lambda(x-x^0)). \tag{2.3}$$

Expanding the right-hand side of (2.3) for a sufficiently small $\lambda$, we obtein

$$g(x^0+\lambda(x-x^0))=g(x^0)+\lambda(x-x^0)\cdot\frac{\partial}{\partial x}g(x^0)+0(\lambda^2). \qquad (2.4)^3$$

From (2.3) and (2.4), it follows that

$$0\leqq g(x)\leqq(x-x^0)\cdot\frac{\partial}{\partial x}g(x_0) \tag{2.5}$$

in the limiting case, $\lambda\to 0$. Q.E.D.

Geometrically, a point such that $g(x)\geqq 0$ is always found in the inner side of the tangential plane at the point $x^0$.

Let us return to the main problem. From (1.9) and (1.10), we have the following nonlinear problem for our stochastic programming:

$$\max\quad x_0 \tag{2.6}$$

subject to

$$g_i(x)=\sum_{j=0}^{n}\bar{a}_{ij}x_j+\bar{a}_{i,\,n+1}-q_i\sqrt{\sum_{k,l=0}^{n}w_{ikl}x_kx_l+2\sum_{j=0}^{n}w_{i,\,n+1,\,j}x_j+w_{i,\,n+1,\,n+1}}\geqq 0, \tag{2.7}$$

$$x_j\geqq 0,\qquad j=1,\,\ldots\ldots,n;$$

$$i=0,1,\,\ldots\ldots,m, \tag{2.8}$$

where $g_i(x)$ is a concave function of $x$.

In order for the origin $x=0$ to be a feasible point of the convex domain (2.7), we assume

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[3] $\frac{\partial}{\partial x}g(x)$ means a gradient vector,

$$\left(\frac{\partial g(x)}{\partial x_1},\ \frac{\partial g(x)}{\partial x_2},\ \ldots\ldots,\ \frac{\partial g(x)}{\partial x_n}\right).$$

*Assumption* 3:

$$\bar{a}_{i,\,n+1}-q_i\sqrt{w_{i,\,n+1,\,n+1}}\geqq 0.$$

This assumption is plausible, since the standard deviation should be sufficiently small comparing with the mean value in the practical case.

Using *Property* 1 and (2.6), (2.7) and (2.8), the following computational procedure would be conceived.

*Step* 1: Let $C$ be a set of linear constraints,

$$\sum_{j=0}^{n}\bar{a}_{ij}x_j+\bar{a}_{i,\,n+1}\geqq 0,\qquad x_j\geqq 0,\ i=0,1,\ldots\ldots,m. \tag{2.9}$$

*Step* 2: (i) Solve a linear problem

$$\max\ x^0, \tag{2.10}$$

subject to $C$.

(ii) Rejecting slack variables, if any, from the bases, form a reduced set of constraints $C'$ and a solution $x$.

*Step* 3: If $x$ satisfies (2.7), $x$ is a required solution and the computation is terminated. If $x$ does not satisfy (2.7), proceed to the next step.

*Step* 4: Let $I$ be a set of constraints of (2.7),

(i) Compute $\mu_i$ such that

$$g_i(\mu_i S_j)=0\quad\text{for } i\in I, \tag{2.11}$$

where

$$x_j=\mu_i S_j,$$

and $S_i$ is the $i$-th direction cosine of the vector $x$.

(ii) Find minimum (positive) $\mu_\alpha$,

$$\mu_\alpha=\min_{i\in I}\mu_i. \tag{2.12}$$

(iii) Add a cutting plane

$$(x-x^0)\cdot\frac{\partial}{\partial x}g(x_0)\geqq 0, \tag{2.13}$$

where

$$x^0=\mu_\alpha S$$

to the set of constraints $C'$ and form $C$.

(iv) Return to the *Step* 2.

## III *Discussions*

Finally we give some explanations for this cutting plane method of the stochastic programming.

(1) The *Step* 2 (ii) keeps the number of constraints from increasing infinitely: the maximum number never exceeds $n+1$.

(2) *Step* 4 rejects an unsatisfactory solution by adding a tangential plane to the innermost furface of the convex domain. Consequently the maximal value of $x_0(=f)$ will be decreased, and continuing this process, we will attain the optimal solution at last.

(3) However, the sequence of the linear programming (2.10) is generally infinite. Therefore, in practice, it should be truncated after a finite number of steps at a point when the desired degree of approximation is attained.

(4) Another device for computation of convex programming is the gradient method which was described in (2). A general discussion will be made later.

## REFERENCES

(1) Kataoka S., "On Stochastic Programming I—Stochastic Programming and its Application to Production Horizon Problem—", The Hitotsubashi Journal of Arts and Sciences Vol. 2, No. 1, 1962 pp. 23—36.

(2) Kataoka S., "On Stochastic Programming II—A Preliminary Study of a Stochastic Programming Model—", Hitotsubashi Journal of Arts and Sciences Vol. 2, No. 1, 1962 pp. 36—44.

(3) Kataoka S., "On Stochastic Programming III—A Stochastic Programming Model—", The Hitotsubashi Journal of Arts and Sciences Vol. 2, No. 1, 1962 pp. 44—55.

(4) Charnes A., W.W. Cooper, "Chance Constrained Programming," Management Science, Vol. 4, 1958, pp. 235—263.

(5) Charnes A, W.W. Cooper, "Chance Constrains and Normal Deviates", Journal of American Statistical Association, March 1962, pp. 134—148.

(6) Kelley J.E. Jr., "The Cutting Plane Method for Solving Convex Programs," Industrial and Applied Mathematics, Vol. 8, 1960, pp. 703—712.