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AN APPLICATION OF NEW BARRIER OPTIONS (EDOKKO OPTIONS) FOR PRICING BONDS WITH CREDIT RISK

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Abstract

In order to price bonds with credit risk, we can consider structural models. Basically, default occurs if the value of the firm hits some pre-specified barrier in these models. We extend traditional structural models to put the additional default condition such that the value of the firm remains under some pre-specified level for a long period of time until the maturity after the first time hitting this level. A new framework of barrier options (Edokko Options) allows us to extend default condition. In our approach, the way to describe default time can be applied more precisely to the real world.

Keywords: Credit Risk, Structural Model, First-Passage-Time Model, Edokko Options, Cumulative Parisian Edokko Option, Simple Parisian Like Edokko Option

I. *Introduction*

The models for pricing bonds with credit risk are usually classified into two categories. One group is based on the evolution of the firm value to determine default endogenously, called structural models. On the other hand, another group specifies default process exogenously, called reduced-form models.[1]

One of the first (structural) models for pricing credit-risky bonds was developed by Merton[12] using the option pricing theory developed by Black and Scholes[5] and Merton [11]. In Merton's model[12], because the claims of bondholders are senior to those of equityholders, the payoff to the bondholders at the debt's maturity is regarded as the face value minus a put option with the face value as the strike price written on the value of the firm's assets. So, Black and Scholes formula is applicable in this model. But, it is assumed that default can occur only at the maturity, which is unrealistic.

To correct this deficiency, Black and Cox[4] extended Merton's model to allow premature default. In their model, it is assumed that default occurs when the value of the firm's assets crosses some pre-specified barrier. That is, default time is defined by the first passage time of the value of the firm's assets to some barrier. So, the models that belong to this approach are

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[1] Structural models are also called firm value models, and reduced-form models called intensity models.

called First-Passage-Time models. First-Passage-Time model has a natural interpretation as the safety covenant. It is a contractual mechanism that gives bondholders the right to bankrupt or force a reorganization of the firm. Black and Cox[4] assumed a time-dependent deterministic barrier and a constant risk free rate. Longstaff and Schwartz[10] and Cathcart and El-Jahel[6] allowed interest rates to be stochastic with dynamics proposed by Vasicek[13] and C.I.R.[8] respectively, while barriers were constants. But typically, with a constant barrier and stochastic interest rate, no closed-form solution for the bond pricing is derived.

As mentioned above, in structural models it is typically assumed that default occurs when the value of the firm's assets crosses some barrier. But, in the real markets it seems to be unrealistic that default time is the first passage time. Then, we extend these structural models to a more realistic model by applying the framework of new barrier options (Edokko Options). In the framework of Edokko Options, explained in detail in the next section, there are two random times: one is called Caution time, the other is K.O. time. In this paper we denote K.O. time as a default time and derive the prices of bonds with credit risk as closed-form.

The rest of this paper is organized as follows: Section 2 describes an outline of Edokko Options, and Section 3 presents the basic model throughout this paper. In Section 4, we apply Edokko Options to value corporate bonds. Section 5 summarizes this paper.

II. *New Barrier Options (Edokko Options)*

Edokko Options, which were developed by Fujita and Miura[9], are generalized barrier options. They denote the first hitting time of the underlying asset at a constant level A by a stopping time τ_A, and call it Caution time, $R_S=\{t\,|\,0\leq t<\tau_A\}$ Safety Region, and $R_C=\{t\,|\,t\geq\tau_A\}$ Caution Region. In Caution Region, the option vanishes when $g(\tau_A)$ called K.O. time occurs. $R_{K.O}=\{t\,|\,g(\tau_A)<t<T\}$ is called Knock Out Region, and if $R_{K.O.}=\emptyset$ the option should not be knocked out. This framework is called Edokko framework. It can be combined with other barrier options and make the design of barrier options more flexible. Fujita and Miura[9] proposed Cumulative Parisian Edokko Option, Simple Parisian Like Edokko Option, and various barrier options that belong to Edokko framework.

In this paper, we apply Cumulative Parisian Edokko Option and Simple Parisian Like Edokko Option. Cumulative Parisian Edokko Option is generalized Cumulative Parisian Option developed by Chesney, Jeanblanc-Picque and Yor[7]. This option is a down-and-out call option that is knocked out if the occupation time of the underlying asset below the level A exceeds a given fraction $\alpha(0<\alpha<1)$ of $T-\tau_A$. Let us denote the underlying asset of this option by S_t and the maturity by T, then Knock Out Region of this option is as follows:

$$R_{K.O.}=\{t\,|\int_{\tau_A}^{t}1_{(-\infty,A)}(S_u)du\geq\alpha(T-\tau_A)\}$$

Simple Parisian Like Edokko Option is a generalized Simple Parisian Like Option, and is a down-and-knock-out option that is knocked out if in Caution Region it takes more than $\alpha(T-\tau_A)$ for the underlying asset to return to another bar $B(>A)$. In other words Knock Out Region of this option is as follows:

$$R_{K.O.}=\{t\geq\tau_B'\,|\,\tau_B'\geq(1-\alpha)\tau_A+\alpha T\}$$

where, $\tau_B' = \inf\{t > \tau_A \mid S_t = B\}$.

In order to apply this framework to structural models, we take $g(\tau_A)$ to be default time and $R_{K.O.}$ as the default region. A caution is given if the value of the firm crosses some lower level. And after this, default occurs if a pre-specified condition is satisfied before the bond's maturity.

III. *The Basic Model*

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F})$. We assume trading occurs continuously, in a frictionless market with no taxes and transaction costs, and risk free rate is a constant r.

Consider a firm that issues a single zero-coupon bond with face value L and maturity T. We take the value of the firm as the value of the firm's assets, and assume that the value of the firm's assets V satisfies the following S.D.E.:

$$dV_t = rV_t dt + \sigma V_t dW_t, \ V_0 = v \tag{1}$$

where, σ and v are positive constants and W is a Brownian motion. (1) implies that P is a risk neutral measure.

Let A be a positive constant $(A < L < v)$, τ_A a stopping time as follows:

$$\tau_A = \inf\{t \geq 0 \mid V_t = A\}$$

and denote default time by $g(\tau_A)$ depending on $\tau_A (g(\tau_A) \geq \tau_A)$.

We take the payoff to the bondholder at the maturity as follows:

1. The face value L, if default does not occur before the maturity and the value of the firm's assets at maturity V_T is and over L.
2. The constant fraction $\beta_1 \, (0 < \beta_1 < 1)$ of V_T, if default doesn't occur before the maturity and V_T is less than L. This case is also essentially default.
3. The constant fraction $\beta_2 \, (0 < \beta_2 < 1)$ of the level A if default occurs before the maturity.

Then, the payoff to the bondholder at the maturity is:

$$X_T = L 1_{\{g(\tau) > T, \, V_T \geq L\}} + \beta_1 V_T 1_{\{g(\tau) > T, \, V_T < L\}} + \beta_2 A 1_{\{0 < g(\tau) \leq T\}} \tag{2}$$

where, $1_{\{\cdot\}}$ is a indicator function.

Under this setting, the value of the corporate zero-coupon bond at time 0 is derived as the expectation of the discounted payoff under the risk neutral measure. (See Baxter and Rennie [2], for example) That is,

$$D(0, T) = E[e^{-rT} X_T] \tag{3}$$

We begin with traditional structural models in which default can occur if the value of the firm's assets cross a pre-specified barrier, that is $g(\tau_A) = \tau_A$ in (2), the payoff to the bondholder at the maturity is described as follows:

$$X_T^{(1)} = L 1_{\{\tau_A > T, \, V_T \geq L\}} + \beta_1 V_T 1_{\{\tau_A > T, \, V_T < L\}} + \beta_2 A 1_{\{0 < \tau_A \leq T\}} \tag{4}$$

So, the value of the corporate bond at time 0 is:

$$\begin{aligned} D^{(1)}(0,T) &= E[e^{-rT}L1_{\{\tau_A>T,\ V_T\geq L\}}]+E[e^{-rT}\beta_1 V_T 1_{\{\tau_A>T,\ V_T<L\}}] \\ &\quad +E[e^{-rT}\beta_2 A1_{\{0<\tau_A\leq T\}}] \\ &=: I_1^{(1)}+I_2^{(1)}+I_3^{(1)} \end{aligned} \tag{4}$$

where,

$$\begin{aligned} I_1^{(1)} &= e^{-rT}L\left\{\Phi\left(\frac{\log\frac{L}{v}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{\log\frac{A}{v}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right. \\ &\quad \left.-\left(\frac{A}{v}\right)^{\frac{2r}{\sigma^2}-1}\left\{\Phi\left(\frac{\log\frac{vL}{A^2}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{\log\frac{v}{A}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right\}\right\} \end{aligned} \tag{6}$$

$$I_2^{(1)}=\beta_1 v\left\{\Phi\left(\frac{\log\frac{v}{L}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-\left(\frac{A}{v}\right)^{\frac{2r}{\sigma^2}+1}\Phi\left(\frac{\log\frac{A^2}{vL}+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right\} \tag{7}$$

$$I_3^{(1)}=e^{-rT}\beta_2 A\left\{\Phi\left(\frac{\log\frac{A}{v}-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)+\left(\frac{A}{v}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{\log\frac{A}{v}+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right\} \tag{8}$$

$\Phi(\cdot)$ is a normal distribution function.

IV. *Application and Valuation*

As described in Section 2, default time is regarded as τ_A in traditional structural models. In this section, we regard default time as $g(\tau_A)$, and derive the price of corporate bonds as closed-form using Edokko framework: Cumulative Parisian Edokko Option and Simple Parisian Like Edokko Option.

1. An Application of Cumulative Parisian Edokko Option

We assume that default occurs, if the value of the firm's assets V crosses a lower level A and after this the occupation time of V below A exceeds a given fraction α $(0<\alpha<1)$ of $T-\tau_A$. Whether default occurs depends on the occupation time of the financial distress. Under this assumption, the framework of Cumulative Parisian Edokko Option is applicable.

In this framework, the payoff to the bondholder at the maturity is described as follows:

$$\begin{aligned} X_T^{(2)} &= L1_{\{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du<\alpha(T-\tau_A),\ V_T\geq L\}\cup\{\tau_A\geq T,\ V_T\geq L\}\}} \\ &\quad +\beta_1 V_T 1_{\{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du<\alpha(T-\tau_A),\ V_T<L\}\cup\{\tau_A\geq T,\ V_T<L\}\}} \\ &\quad +\beta_2 A1_{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du\geq\alpha(T-\tau_A),\}} \end{aligned} \tag{9}$$

The value of the corporate bond at time 0 is:

$$\begin{aligned}D^{(2)}(0,T)=&E[e^{-rT}L1_{\{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du<\alpha(T-\tau_A),\ V_T\geq L\}\cup\{\tau_A\geq T,\ V_T\geq L\}\}}]\\&+E[e^{-rT}\beta_1V_T1_{\{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du<\alpha(T-\tau_A),\ V_T<L\}\cup\{\tau_A\geq T,\ V_T<L\}\}}]\\&+E[e^{-rT}\beta_2A1_{\{0<\tau_A<T,\ \int_{\tau_A}^{t}1_{(-\infty,A)}(V_u)du\geq\alpha(T-\tau_A)\}}]\\=&:I_1^{(2)}+I_2^{(2)}+I_3^{(2)}\end{aligned}\tag{10}$$

where,

$$I_1^{(2)}=e^{-rT}L\int_0^T du\int_{\log\frac{L}{A}}^{+\infty}da\int_0^{\alpha(T-u)}db\ h_{(W_{T-u}^{r-\frac{1}{2}\sigma^2,\sigma},\ \int_0^{T-u}1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)}(a,b)f_{\tau_A}(u)$$
$$+I_1^{(1)}$$

$$I_2^{(2)}=e^{-rT}\beta_1\int_0^T du\int_{-\infty}^{\log\frac{L}{A}}da\int_0^{\alpha(T-u)}db\ Ae^a\ h_{(W_{T-u}^{r-\frac{1}{2}\sigma^2,\sigma},\ \int_0^{T-u}1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)}(a,b)f_{\tau_A}(u)$$
$$+I_2^{(1)}$$

$$I_3^{(2)}=e^{-rT}\beta_2A\int_0^T du\int_{-\infty}^{+\infty}da\int_{\alpha(T-u)}^{T-u}db\ h_{(W_{T-u}^{r-\frac{1}{2}\sigma^2,\sigma},\ \int_0^{T-u}1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)}(a,b)f_{\tau_A}(u)$$

$$f_{\tau_A}(u)=\frac{\log\frac{v}{A}}{\sigma\sqrt{2\pi u^3}}e^{-\frac{\left(\log\frac{A}{v}-\left(r-\frac{1}{2}\sigma^2\right)u\right)^2}{2\sigma^2u}}$$

$I_1^{(1)}$ is given in (6) and $I_2^{(1)}$ in (7). $f_{\tau_A}(u)$ is the density function of τ_A. Let us denote Brownian motion with drift $(r-\frac{1}{2}\sigma^2)$ and volatility σ by $W_t^{r-\frac{1}{2}\sigma^2,\sigma}$.

$$h'_{(W_t,\ \int_0^t1_{(-\infty,0)}(W_s)ds)}(a,b)=\begin{cases}\displaystyle\int_b^t\frac{a}{2\pi\sqrt{s^3(t-s)^3}}e^{\frac{-a^2}{2(t-s)}}ds & \cdots\quad a>0\\ \displaystyle\int_0^b\frac{-a}{2\pi\sqrt{s^3(t-s)^3}}e^{\frac{-a^2}{2s}}ds & \cdots\quad a<0\end{cases}$$

$$h_{(W_t^{r-\frac{1}{2}\sigma^2,\sigma},\ \int_0^t1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)}(a,b)=e^{-\frac{\left(r-\frac{1}{2}\sigma^2\right)^2t}{2\sigma^2}+\frac{\left(r-\frac{1}{2}\sigma^2\right)a}{\sigma^2}}\frac{1}{\sigma}h'_{(W_t,\ \int_0^t1_{(-\infty,0)}(W_s)ds)}\left(\frac{a}{\sigma},b\right)$$

$h'_{(W_t,\int_0^t1_{(-\infty,0)}(W_s)ds)}(a,b)$ is the joint density function of $(W_t,\int_0^t1_{(-\infty,0)}(W_s)ds)$, and by applying Girsanov's theorem we get the joint density function of $(W_t^{r-\frac{1}{2}\sigma^2,\sigma},\int_0^t1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)$, $h_{(W_t^{r-\frac{1}{2}\sigma^2,\sigma},\int_0^t1_{(-\infty,0)}(W_s^{r-\frac{1}{2}\sigma^2,\sigma})ds)}(a,b)$. See Fujita and Miura[9] for this proof.

2. An Application of Simple Parisian Like Edokko Option

Here we assume that default occrus, if the value of the firm's assets V crosses a pre-specified level A and after this it takes more than $\alpha(T-\tau_A)$ for the underlying asset to return to another bar $B(A<L<B)$. In other words, default can occur if the value of the firm's assets continues to stay below a given level for a long period of time. Even though the firm gets into financial distress, default does not occur if shortly afterward the financial distress is settled. Under this assumption, the framework of Simple Parisian Like Edokko Option is applicable in this case.

Let τ_B' be the first hitting time of V_t at B after τ_A,

$$\tau_B'=\inf\{t>\tau_A \mid V_t=B\}$$

In this framework, the payoff to the bondholder at the maturity is described as follows:

$$\begin{aligned} X_T^{(3)}=&L1_{\{\{0<\tau_A<T,\ \tau_B'<(1-\alpha)\tau_A+\alpha T,\ V_T\geq L\}\cup\{\tau_A\geq T,\ V_T\geq L\}\}} \\ &+\beta_1 V_T 1_{\{\{0<\tau_A<T,\ \tau_B'<(1-\alpha)\tau_A+\alpha T,\ V_T<L\}\cup\{\tau_A\geq T,\ V_T<L\}\}} \\ &+\beta_2 A 1_{\{0<\tau_A<T,\ \tau_B'\geq(1-\alpha)\tau_A+\alpha T\}} \end{aligned} \tag{11}$$

The value of the corporate bond at time 0 is:

$$\begin{aligned} D^{(3)}(0,T)=&E[e^{-rT}L1_{\{\{0<\tau_A<T,\ \tau_B'<(1-\alpha)\tau_A+\alpha T,\ V_T\geq L\}\cup\{\tau_A\geq T,\ V_T\geq L\}\}}] \\ &+E[e^{-rT}\beta_1 V_T 1_{\{\{0<\tau_A<T,\ \tau_B'<(1-\alpha)\tau_A+\alpha T,\ V_T<L\}\cup\{\tau_A\geq T,\ V_T<L\}\}}] \\ &+E[e^{-rT}\beta_2 A 1_{\{0<\tau_A<T,\ \tau_B'\geq(1-\alpha)\tau_A+\alpha T\}}] \\ =:&\, I_1^{(3)}+I_2^{(3)}+I_3^{(3)} \end{aligned} \tag{12}$$

where,

$$I_1^{(3)}=e^{-rT}L\int_0^T du\int_u^{(1-\alpha)u+\alpha T} ds\int_{\log\frac{L}{B}}^{+\infty} dx\, l(x)\, m_{\tau_B'}(s) f_{\tau_A}(u)+I_1^{(1)}$$

$$I_2^{(3)}=e^{-rT}\beta_1\int_0^T du\int_u^{(1-\alpha)u+\alpha T} ds\int_{\log\frac{L}{B}}^{+\infty} dx\, Be^x l(x)\, m_{\tau_B'}(s) f_{\tau_A}(u)+I_2^{(1)}$$

$$I_3^{(3)}=e^{-rT}\beta_2 A\int_0^T du\int_{(1-\alpha)u+\alpha T}^{+\infty} ds\, m_{\tau_B'}(s) f_{\tau_A}(u)$$

$$m_{\tau_B'}(s)=\frac{\log\frac{B}{A}}{\sigma\sqrt{2\pi(s-u)^3}}\, e^{-\frac{\left(\log\frac{B}{A}-\left(r-\frac{1}{2}\sigma^2\right)(s-u)\right)^2}{2\sigma^2(s-u)}}$$

$$l_{W_{T-s}^{r-\frac{1}{2}\sigma^2,\sigma}}(x)=\frac{1}{\sigma\sqrt{2\pi(T-s)}}\, e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-s)\right)^2}{2\sigma^2(T-s)}}$$

$m_{\tau_B'}(s)$ is the density function of τ_B' conditional on $\tau_A=u$, and $l_{W_{T-s}^{r-\frac{1}{2}\sigma^2,\sigma}}(x)$ is the density function of $W_{T-s}^{r-\frac{1}{2}\sigma^2,\sigma}$.

V. *Conclusion*

The purpose of this paper was to describe default realistically compared with traditional structural models and to value corporate bonds as closed-form. We applied Edokko Options to structural models and derived the prices of bonds with credit risk. Our approach is more realistic to describe default, unlike traditional structural models in which default time is specified as the first hitting time at some barrier (the Caution time τ_A in terms of Edokko framework), we extend the conditions of default to depend on K.O. time $g(\tau_A)$. We adopted Cumulative Parisian Edokko Option and Simple Parisian Like Edokko Option. In the former, we assumed that whether default occurs depends on the occupation time of the financial distress until the debt's maturity, and in the latter assumed that default occurs if the value of the firm continues to stay below a given level for a long period of time. In both cases the prices of corporate bonds were derived as closed-form.

In this paper, we only derived the price of corporate bonds, and as assignments, numerical examples and empirical studies remain to compare with existing models. And we assumed that the recovery, risk free rate, and barriers were constants for simplicity, but it might be practical to the real world that the recovery depends on the value of the firm upon default, barriers depend on some factors, and interest rates are stochastic, which is left for future studies.

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