Hitotsubashi Journal of Commerce and Management 34 (1999), pp.15-49.

A MATHEMATICAL STRUCTURE OF THE FIRM VALUE WHEN STOCK OPTIONS ARE ISSUED

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Abstract

In this paper, we mathematically formalize the structure of the firm value when stock options are issued. Using the result, we show that the wealth of the stock holders do not transfer to the stock option holders at the maturity of the stock option ,and that the stock option gives the stock option holders the incentive for better firm's performance which brings the stock holders the more profit.

Keywords: Stock option, Firm value, Option pricing theory, No Arbitrage, Balance sheet,

JEL classification: G12;G32

Introduction

Black and Scholes (1973) and Merton (1973) made the foundation of the option pricing theory. Applying the framework of standard option pricing theory founded by them, we mathematically formalize the structure of the firm value when stock options (what is called executive or employee stock options) are issued.

Briefly speaking, a stock option contract is that the firm gives its executives or employees the right to buy the firm's stocks by a certain (prefixed) date for a certain (prefixed) price. The firm has three choices of schemes to prepare for the exercise. First, the firm repurchases the treasury stocks before exercise, and receives the exercise price in cash upon the exercise (This scheme is called treasury stock type). Second, the firm issues new stock upon exercise and receives the exercise price in cash in exchange to the stock (This scheme is called warrant type). Third, the firm pays the subtraction of exercise price from the stock price at the exercise day (This scheme is called phantom stock). In this paper we restrict the discussion to the treasury stock type.

There have been many research works on stock option pricing. Among them, we review Smith and Zimmerman (1976), Lambert, Larcker and Verrecchia (1991), Kulatilaka and Marcus (1994), Cuny and Jorion (1995), Rubinstein (1995), and Carpenter (1997).

In Smith and Zimmerman (1976), assuming that stock option is a derivative defined on a stock price process and that stock options are tradable in the market, they provide the lower bound on stock option price. They applied their results to the problem of valuing an employee stock option plan from an accounting view point.

Lambert, Larcker and Verrecchia (1991) developed a model for assessing the value of stock option from the perspective of the holder (executive), incorporating that stock options are not tradable. Their framework is based on the executive compensation and his utility.

Kulatilaka and Marcus (1994) incorporate, into their model, that the stock option holders are unable to sell their options. Assuming that stock price process is binomial process and that stock option holders maximize their expected utility, they show that stock option holders have propensity to exercise their options earlier than the American options on the same stock and that the value of stock option is worth less than it.

In Cuny and Jorion (1995), they assume that stock option is a derivative defined on a process of stock price and that stock options are tradable in the market, They incorporate, into their pricing model, that the stock option holder (executive) leaves the firm because of their bad results.

Rubinstein (1995) develops a binomial valuation model that takes into consideration that the most significant differences between standard call options and stock options: long maturity, delayed vesting, forfeiture, non-tradability, increase in shares of the stock, and taxes.

Carpenter (1997) incorporate, into her pricing model, that stock option holder has a propensity to exercise an option early ,and that the stock option holder (executive) leaves the firm because of bad results.

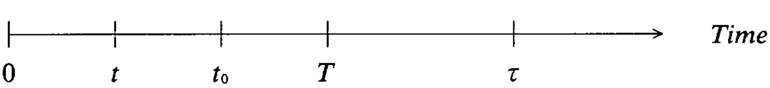
In those models they did not take in the value of the firm explicitly . A stochastic process representing a stock price is as underlying variable and the stock option is viewed as a derivative defined on the stochastic process. In this paper a stochastic process representing the firm value is as underlying variable , and the stock, the zero coupon bond and the stock option are viewed as derivatives defined on the stochastic process. Prices of three derivatives will be derived as a function of a firm value. We investigate the effect of introducing the stock option on both the behavior of the firm value and the stock price.

The framework which is based on the firm value has been applied to analyzing the problems in finance, as references, Merton (1977,1978) developed a model which is based on a stochastic process representing the firm value ,to examine financial problems.

This paper is organized as follows. In section 1, we describe assumptions and notations used in this paper. In section 2, we show the basic pricing formulae which will be used in later sections 3, 4 and 5. In sections 3,4, and 5, we show the pricing formulae for the stock, the zero coupon bond and the stock option in the general form. In section 6, we investigate problems of corporate finance. In section 7, we show the specific pricing formulae for the stock and the stock option in the case that the firm value follows a log-normal process.

1. Assumptions and Notations

In this section, we describe assumptions and notations used in this paper.



(1) No arbitrage

We assume that there are no arbitrage opportunities in the market.

(2) Frictionless market

We assume frictionless market, that is, there are no transactions costs, no taxes, all securities are infinitely divisible, and borrowing and short selling are allowed without restriction.

(3) Interest rate r

For any $t\in[0,+\infty)$, let $r(t)$ denote an instantaneous spot rate at time t. We assume that $r(t)$ is the stochastic process which satisfies stochastic differential equation;

$$\begin{aligned} dr(t) &= a(t, r(t))\, dt + b(t, r(t))\, dW_1(t) \\ r(0) &= r_0 \end{aligned} \tag{1.1}$$

where a and b are functions which satisfy certain conditions, and $W_1(t)$ is a standard Brownian motion.

(4) The finance of the firm

We set several assumptions for the firm and the necessary notations.

① The firm has a maturity. Let $\tau\in(0,+\infty)$ denote the maturity date. This assumption is to simplify the argument and the model.

② We assume that the decision makings of the firm are on raising fund and on investment. Raising funds is that the firm issues some securities to collect money and to employ executives. Investment is that the firm allocates its money or valuables among competing interests to increase the market value that the firm owns.

At time 0, the firm raises funds by issuing the stocks and the zero coupon bond, the firm issues stock options which is used to provide executives as the part of compensation, and decides its investment portfolio. In our model, assets, contracts and projects are the part of the firm's investment portfolio. For example, not only cash, account receivable, securities, goods, materials, equipment, buildings, and land but also contracts (futures or swaps), projects are elements of the firm's investment portfolio.

For any $t\in[0,\tau]$, let $V(t)$ denote the time t value of the firm's investment portfolio, which we describe as the value of the firm at time t hereafter.

③ At time 0, the firm issues N_1 (a positive integer) stocks.

For any $t\in[0,\tau]$, let $S\ (t, r(t), V(t))$ denote the price of one stock at time t.

④ At time 0, the firm issues zero coupon bond whose maturity date is τ and the face value is K(a positive constant).

At time τ, if $V(\tau)\geq K$, the bond holders receive K and the stockholders receive $V(\tau)-K$. However, if $V(\tau)<K$, the bond holders receive $V(\tau)$ and the stockholders receive nothing.

For any $t\in[0,\tau]$, let $L(t, r(t), V(t))$ denote the price of zero coupon bond at time t.

⑤ At time 0, the firm issues N_2 (a positive constant and less than N_1) stock options in the part of the executive compensation. The expiration date of the stock option is T for any fixed

$T\in(0,\tau)$, and the exercise price is set to be M(a positive constant). And the stock option holders have the right to buy one unit of the stock per one unit of the stock option for this exercise price.

We assume that the stock options are treasury stock type. It means that the firm repurchase N_2 treasury stocks from the stock market at time t_0 for any fixed $t_0\in(0,T)$ to prepare for the exercise of the stock option at time T. If stock option holders do not exercise stock options at time T, the firm sell those N_2 stocks in the market.

For any $t\in[0,T]$, let $C(t, r(T), V(t))$ denote the price of one stock option at time t.

⑥ From the assumptions②, ③, ④, and ⑤,

$$v_0=L(0,r_0,v_0)+N_1\cdot S(0,r_0,v_0)+N_2\cdot C(0,r_0,v_0)$$

where v_0 is the initial firm value (i.e. at time 0).

(5) The firm value

Following the above assumptions, we define three stochastic processes V_1, V_2 and V_3;

$$\begin{aligned}&dV_1(t)=\alpha_1(t,V_1(t))dt+\beta_1(t,V_1(t))dW_2(t) \quad \text{for } t\in[0,t_0]\\ &V_1(0)=v_0\end{aligned} \qquad (1.2a)$$

$$\begin{aligned}&dV_2(t)=\alpha_2(t,V_2(t))dt+\beta_2(t,V_2(t))dW_2(t) \quad \text{for } t\in[t_0,T]\\ &V_2(t_0)=V_1(t_0)-P_{t_0}\end{aligned} \qquad (1.2b)$$

$$\begin{aligned}&dV_3(t)=\alpha_3(t,V_3(t))dt+\beta_3(t,V_3(t))dW_2(t) \quad \text{for } t\in[T,\tau]\\ &V_3(T)=V_2(T)+Q_T\cdot(1-\mathrm{I_A})+R_T\cdot \mathrm{I_A}\end{aligned} \qquad (1.2c)$$

where A is the event that stock options are exercised at time T,
α_i and β_i are functions which satisfy certain conditions such that $V_i\geq 0$ for $i=1, 2, 3$,
$W_2(t)$ is a standard Brownian motion dependent on $W_1(t)$ such that $dW_1(t)\cdot dW_2(t)=\rho\cdot dt$, $-1<\rho<1$,
P_{t_0} is a random variable which denotes the decrease of the firm value caused by repurchasing N_2 units of stocks at time t_0,
Q_T is a random variable which denotes the increments of the firm value when stock options are not exercised at time T, and
R_T is a random variable which denotes the increments of the firm value when stock options are exercised at time T.

Then we define $V(t)$ as follows,

$$V(t)=\begin{cases}V_1(t) & \text{for } t\in[0,t_0)\\ V_2(t) & \text{for } t\in[t_0,T)\\ V_3(t) & \text{for } t\in[T,\tau]\end{cases} \qquad (1.2d)$$

We introduce the stochastic process V to represent the stochastic fluctuations of the firm value by using Ito process, the decrease of the firm value caused by repurchasing N_2 units of stocks at time t_0 by using a random variable P_{t_0}, and the increments of the firm value caused by not-exercising or exercising stock option at time T by using random variables Q_T and R_T. The functions α_t and β_t reflect the managers abilities.

(6) Stock Price S

The stock price function S is the function of the firm value , which is different in each of different three time interval. We introduce S_1, S_2 and S_3 as defined in the following;

$$S(t,r(t),\ V(t)) = \begin{cases} S_1(t,r(t),\ V_1(t)) & \text{for } t\in[0,t_0) \\ S_2(t,r(t),\ V_2(t)) & \text{for } t\in[t_0,T) \\ S_3(t,r(t),\ V_3(t)) & \text{for } t\in[T,\tau] \end{cases} \qquad (1.3a)$$

where,

S_1 : $[0,t_0]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$

S_2 : $[t_0,T]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$

S_3 : $[T,\tau]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$

S_i s are twice differentiable on its domain for $i=1$, 2, 3.

From time T to time τ, stocks are the security whose payoff at time τ is

$$S(\tau,r(\tau),V(\tau))\equiv S_3(\tau,r(\tau),V_3(\tau))=\frac{1}{N_1}\max\ (V_3(\tau)-K,0) \qquad (1.3b)$$

For the argument in section 3, (1.3b) will be the boundary condition to solve the partial differential equation which S_3 satisfy.

The value of one unit of stock at time T is

$$S_3(T,r(T),\ V_2(T)+Q_T)$$

if stock options are not exercised at time T, and if otherwise

$$S_3(T,r(T),\ V_2(T)+R_T).$$

Then from time t_0 to T, stocks are regarded as the security whose payoff at time T is

$$S(T,r(T),V(T))\equiv S_2(T,r(T)V_2(T))=S_3(T,r(T),V_2(T)+Q_T\cdot(1-\mathrm{I}_A)+R_T\cdot\mathrm{I}_A) \quad (1.3c)$$

In section 4, (1.3c) will be the boundary condition to solve the partial differential equation which S_2 satisfy.

The value of one unit of stock at time t_0 is

$$S_1(t_0,r(t_0),V_1(t_0)-P_{t_0}).$$

Then until time t_0, stocks are regarded as the security whose payoff at time t_0 is

$$S(t_0,r(t_0),V(t_0))\equiv S_1(t_0,r(t_0),V_1(t_0))=S_2(t_0,r(t_0),V_1(t_0)-P_{t_0}). \qquad (1.3d)$$

In section 5, (1.3d) will be the boundary condition to solve the partial differential equation which S_i satisfy.

(7) Zero coupon bond price L

Similar to the stock price S, the zero coupon bond price L is the function of the firm value ,then, we introduce L_1, L_2 and L_3 in the following;

$$L(t,r(t),V(t))=\begin{cases} L_1(t,r(t),V_1(t)) & \text{for } t\in[0,T_0) \\ L_2(t,r(t),V_2(t)) & \text{for } t\in[t_0,T) \\ L_3(t,r(t),V_3(t)) & \text{for } t\in[T,\tau] \end{cases} \qquad (1.4a)$$

where,
$L_1: [0,t_0]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$
$L_2: [t_0,T]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$
$L_3: [T,\tau]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$
L_i s are twice differentiable on its domain for $i=1, 2, 3$.

The boundary conditions at time τ, T and t_0 are as follows;

$$L(\tau,r(\tau),V(\tau))\equiv L_3(\tau,r(\tau),V_3(\tau))=\min(V_3(\tau),K) \qquad (1.4b)$$
$$L(T,r(T),V(T))\equiv L_2(T,r(T),V_2(T))=L_3(T,r(T),V_2(T)+Q_T\cdot(1-\mathrm{I_A})+R_T\cdot\mathrm{I_A}) \qquad (1.4c)$$
$$L(t_0,r(t_0),V(t_0))\equiv L_1(t_0,r(t_0),V_1(t_0))=L_2(t_0,r(t_0),V_1(t_0)-P_{t_0}). \qquad (1.4d)$$

Those boundary conditions will be used in later sections 3, 4, and 5.

(8) Stock Option Price C

Similarly, the stock option price C is the function of the firm value, then, we introduce C_1 and C_2 in the following;

$$C(t,r(t),V(t))=\begin{cases} C_1(t,r(t),V_1(t)) & \text{for } t\in[0,t_0) \\ C_2(t,r(t),V_2(t)) & \text{for } t\in[t_0,T] \end{cases} \qquad (1.5a)$$

where,
$C_1: [0,t_0]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$
$C_2: [t_0,T]\times[0,+\infty)\times[0,+\infty)\rightarrow[0,+\infty)$
C_i s are twice differentiable on its domain for $i=1, 2$.

The boundary conditions at time T and t_0 are as follows;

$$C(T,r(T),V(T))\equiv C_2(T,r(T),V_2(T))=\max(S_3(T,r(T),V_2(T)+R_T)-M,0) \qquad (1.5b)$$
$$C(t_0,r(t_0),V(t_0))\equiv C_1(t_0,r(t_0),V_1(t_0))=C_2(t_0,r(t_0),V_1(t_0)-P_{t_0}). \qquad (1.5c)$$

Those boundary conditions will be used in later sections 4 and 5.

2. *Pricing Formulae for Derivative*

In this section we show the basic pricing formulae which will be used in later sections 3, 4 and 5.

Let F be the price of any derivative defined on the instantaneous spot rate (r) and the firm value (V). Furthermore , we assume that its maturity is $T^* \in [0,\tau]$ and its value at the maturity is

$$F(T^*), V(T^*)) = f(r(T^*), V(T^*)), \qquad (2.1)$$

where $f(r(T^*), V(T^*))$ is a function of $r(T^*)$ and $V(T^*)$, and it specifies the payoff of the derivative.

Here, t_0, T and τ are fixed time points, and T^* belongs to one of time intervals, $[0, t_0)$, $[t_0, T)$ or $[T,\tau]$. If $T < T^* < \tau$, as we have seen, F is defined for all three different time interval. Let $s_1 = 0$, $s_2 = t_0$, $s_3 = T$ and $s_4 = T^*$ and without loss of generality we assume $T^* \in [T,\tau]$. Similar to Miura and Kishino (1995), we obtain the following partial differential equation (PDE) which F must satisfy;

$$\begin{aligned} &\frac{\partial F}{\partial t}(t,x,y) + \{a(t,x) - \lambda(t,x,y) \cdot b(t,x)\} \frac{\partial F}{\partial x}(t,x,y) + x \cdot y \cdot \frac{\partial F}{\partial y}(t,x,y) \\ &+ \frac{1}{2} b(t,x)^2 \frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2} \beta_i(t,y)^2 \frac{\partial^2 F}{\partial y^2}(t,x,y) \\ &+ \rho \cdot b(t,x) \cdot \beta_i(t,y) \frac{\partial^2 F}{\partial x \partial y}(t,x,y) - x \cdot F(t,x,y) = 0 \\ &\quad \text{in} \quad (t,x,y) \in [s_i, s_{i+1}) \times [0, +\infty) \times [0, +\infty) \quad \text{for} \quad i = 1, 2, 3 \end{aligned} \qquad (2.2)$$

where $\lambda(t,x,y)$ denotes market price of risk, and β_i is identified according to the location of t.

By Feynman-Kac theorem (see Friedman (1975), chapter 6, Theorem 5.3), we can solve (2.2) under the boundary condition (2.1). The solution is

$$F(t, r(t), V(t)) = E\left[f(X(T^*), Y(T^*)) \cdot e^{-\int_t^{T^*} X(u)du} \,\middle|\, \begin{matrix} X(t) = r(t) \\ Y(t) = V(t) \end{matrix} \right] \qquad (2.3a)$$

where $X(u)$, $Y(u)$ for $u \in [t, T^*]$ are the solutions of the stochastic differential equation ;

$$\begin{bmatrix} dX(u) \\ dY(u) \end{bmatrix} = \begin{bmatrix} a(u,X(u)) - \lambda(u,X(u),Y(u)) \cdot b(u,X(u)) \\ X(u) \cdot Y(u) \end{bmatrix} du + \mathbf{B} \begin{bmatrix} dZ_1(u) \\ dZ_2(u) \end{bmatrix}, \qquad (2.3b)$$

Z_1 and Z_2 are two independent standard Brownian motions which are different from W_1 and W_2, and

$$\mathbf{B}'\mathbf{B} = \begin{bmatrix} (b(u,X(u)))^2 & \rho \cdot b(u,X(u)) \cdot \beta_i(u,Y(u)) \\ \rho \cdot b(u,X(u)) \cdot \beta_i(u,Y(u)) & (\beta_i(u,Y(u)))^2 \end{bmatrix}. \qquad (2.3c)$$

3. S_3 and L_3 in the third time interval

In this section, we express the pricing formulae, S_3 and L_3 for stock and zero coupon bond in the general form. When stochastic processes r and V are specified, the pricing formulae will be specific accordingly. An example will be shown in section 7.

Let $t \in [T,\tau]$ and we recall that $X(u)$ and $Y(u)$ for $u \in [t,\tau]$ are the solutions of stochastic differential equation (2.3b).

(1) Stock price for $t \in [T,\tau]$

Since S_3 is the solution of PDE (2.2) under the boundary condition (1.3b), we obtain , from (2.3a), that

$$S_3(t,r(t),V_3(t)) = E\left[\frac{1}{N_1}\max(Y(\tau)-K,0) \cdot e^{-\int_t^\tau X(u)du} \,\middle|\, \begin{array}{l} X(t)=r(t) \\ Y(t)=V_3(t) \end{array}\right]. \tag{3.1}$$

(2) Zero coupon bond price for $t \in [T,\tau]$

At time τ , the payoff of the zero coupon bond is

$$\begin{aligned} L_3(\tau,r(\tau),V_3(\tau)) &= \min(V_3(\tau),K) = \min(0,K-V_3(\tau)) + V_3(\tau) \\ &= -\max(V_3(\tau)-K,0) + V_3(\tau). \end{aligned}$$

Since L_3 is the solution of PDE (3.2) under the preceding boundary condition, we obtain, from (2.3a), that

$$\begin{aligned} L_3(t,r(t),V_3(t)) &= E\left[\{-\max(Y(\tau)-K,0)+Y(\tau)\} \cdot e^{-\int_t^\tau X(u)du} \,\middle|\, \begin{array}{l} X(t)=r(t) \\ Y(t)=V_3(t) \end{array}\right] \\ &= -N_1 \cdot S_3(t,r(t),V_3(t)) + V_3(t) \end{aligned} \tag{3.2}$$

where final equality follows by (3.1) and

$$E\left[Y(\tau) \cdot e^{-\int_t^\tau X(u)du} \,\middle|\, \begin{array}{l} X(t)=r(t) \\ Y(t)=V_3(t) \end{array}\right] = V_3(t).$$

(3) Firm value

From (3.2), we have that for any $t \in [T,\tau]$,

$$V_3(t) = L_3(t,r(t),V_3(t)) + N_1 \cdot S_3(t,r(t),V_3(t)). \tag{3.3}$$

This equation represents the firm's balance sheet during the time interval $[T,\tau]$.

4. S_2, C_2, and L_2 in the second time interval

In this section, we derive the pricing formulae for S_2, C_2,and L_2 in the general form. When stochastic processes r and V are specified, the corresponding pricing formulae will be shown in section 7. Before writing them down, we show that the random variables Q_T and R_T are expressed in terms of $r(T)$ and $V_2(T)$, and we specify the range of $V_2(T)$ such that the stock options are exercised. These are preliminaries for obtaining S_2, C_2, and L_2.

4.1 The value of the firm's special position

For any $t \in [0,\tau]$, $G(t)$ denotes the value of the firm's special position which consists of issuing the stock options at time 0, keeping the short position of the stock option issued at time 0, buying N_2 units of the treasury stocks at time t_0, holding N_2 units of the treasury stocks during the time period $[t_0,T]$, selling the N_2 units of the treasury stocks in the market at time T in case that the stock options are not exercised, and allocating a sum of cash, which will flow into the firm at time T, to a part of V (the firm's investment portfolio).

To show the firm's action towards the stock option clearly, We deal with the firm's special position as if it is the firm's subsidiary (conceptual subsidiary) which specializes in the business concerning the firm's stock option plan. Then, from a view point of the subsidiary, we describe the special position again. During the time period $[0,t_0]$, it will be keeping the short position of the stock option issued at time 0. At time t_0, to buy N_2 units of the treasury stocks in the market, it will raise funds by short selling N_2 units of the treasury stocks to the firm. At time T, if the stock options would be exercised, the subsidiary must exchange N_2 units of the treasury stocks for $N_2 \cdot M$ units of cash with the stock option holders, and if otherwise, it will sell them in the market. In any case, the subsidiary will make the firm to allocate the amount of money in a part of V (the firm's investment portfolio) during $[T,\tau]$. At time τ, the subsidiary must repay $(N_2/N_1) \cdot \max(V_3(\tau)-K,0)$ to the firm.

There are cash flows at three time points. The cash out-flow at time t_0 is

$$-N_2 \cdot S_2(t_0,r(t_0),V_1(t_0)-P_{t_0})$$

which flows from the special position in the firm (the subsidiary) to the market. The cash in-flow at time T is

$$\begin{aligned}
&N_2 \cdot S_3(T,r(T),V_2(T)+Q_T) \cdot (1-\mathrm{I_A})+N_2 \cdot M \cdot \mathrm{I_A} \\
&=N_2\begin{Bmatrix} S_3(T,r(T),V_2(T)+Q_T) \cdot (1-\mathrm{I_A})+S_3(T,r(T),V_2(T)+R_T)\mathrm{I_A} \\ -S_3(T,r(T),V_2(T)+R_T)\mathrm{I_A}+M \cdot \mathrm{I_A} \end{Bmatrix} \\
&=N_2 \cdot S_3(T,r(T),V_2(T)+Q_T \cdot (1-\mathrm{I_A})+R_T \cdot \mathrm{I_A})-N_2 \cdot \max\big(S_3(T,r(T),V_2(T)+R_T)-M,0\big)
\end{aligned}$$

which flows from the market or the stock option holder to the special position in the firm (the subsidiary). The cash out-flow at time τ is

$$-N_2 \frac{1}{N_1} \max(V_3(\tau)-K,0)$$

which flows from the special position in the firm (the subsidiary) to the market.
Here, we define $G_{3,1}$. For any $t \in [T,\tau]$, $G_{3,1}(t)$ denotes a time t value of sum of the cash which was flowed into the firm's special position (the subsidiary) at time T and has been allocated in

a part of V (the firm's investment portfolio) during time period $[T,\mathrm{t}]$, where

$$G_{3,1}(T)=N_2 \cdot S_3(T,r(T),V_2(T)+Q_T) \cdot (1-\mathrm{I}_A)+N_2 \cdot M \cdot \mathrm{I}_A$$

The value of the special position at time τ is

$$G(\tau)=G_{3,1}(\tau)-N_2 \frac{1}{N_1} \max(V_3(\tau)-K,0).$$

Then, from the assumption of no arbitrage for any $t\in[T,\tau]$, the value of the special position at time t is

$$G(t)=G_{3,1}(t)-N_2 \cdot S_3(t,r(t),V_3(t)).$$

Hence, the value of the special position at time T is

$$\begin{aligned}G(T)&=G_{3,1}(T)-N_2 \cdot S_3(T,r(T),V_2(T)+Q_T \cdot (1-\mathrm{I}_A)+R_T \cdot \mathrm{I}_A)\\&=N_2 \cdot S_3(T,r(T),V_2(T)+Q_T \cdot (1-\mathrm{I}_A)+R_T \cdot \mathrm{I}_A)\\&\quad -N_2 \cdot \max(S_3(T,r(T),V_2(T)+R_T)-M,0)\\&\quad -N_2 \cdot S_3(T,r(T),V_2(T)+Q_T \cdot (1-\mathrm{I}_A)+R_T \cdot \mathrm{I}_A),\end{aligned}$$

then, for any $t\in[t_0,T]$, the value of the special position at time t is

$$\begin{aligned}G(t)&=N_2 \cdot S_2(t,r(t),V_2(t))-N_2 \cdot C_2(t,r(t),V_2(t))-N_2 \cdot S_2(t,r(t),V_2(t))\\&=-N_2 \cdot C_2(t,r(t),V_2(t)).\end{aligned}$$

Similarly, the value of the firm's special position at time t_0 is

$$\begin{aligned}G(t_0)&=N_2 \cdot S_2(t_0,r(t_0),V_1(t_0)-P_{t_0})-N_2 \cdot S_2(t_0,r(t_0),V_1(t_0)-P_{t_0})\\&\quad -N_2 \cdot C_2(t_0,r(t_0),V_1(t_0)-P_{t_0})\\&=-N_2 \cdot C_2(t_0,r(t_0),V_1(t_0)-P_{t_0}),\end{aligned}$$

then for any $t\in[0,t_0]$, the value of the special position at time t is

$$G(t)=-N_2 \cdot C_1(t,r(t),V_1(t)).$$

Therefore for any $t\in[0,T]$

$$G(t)=-N_2 \cdot C(t,r(t),V(t)).$$

This equation represents that for any $t\in[0,T]$ the value of the firm's special position at time t is equal to the value of the short position in N_2 units of the stock option at time t.

N_2 units of treasury stocks which the firm is holding during time period $[t_0,T]$ is used in exchange for an amount of money at time T in either case that the stock options are exercised or not exercised. Here, we assume that the firm exchange new issued N_2 units of stocks instead of N_2 units of the treasury stocks for an amount of money with investors in the market or stock option holders at time T. We describe the firm's position which consists of the special position and issuing N_2 units of the stocks at time T as the firm's special position 2. There are no differences in the cash flows which will appear at time T and τ between the firm's special position and the firm's special position 2. Then, for any $t \in [t_0, T]$,the value of the firm's special position at time t is equal to the value of the firm's special position2 at time t. Accordingly in our framework there are no meaning in firm's holding N_2 units of treasury stocks during the period (t_0, T). Therefore, the firm's repurchasing N_2 treasury stocks at time t_0 is not buying something for the investment, but only the repayment of the securities which were issued to raise funds at time 0.

4.2 The pricing formulae for S_2, C_2, and L_2

(1) Increment of the firm value when stock options are not exercised at time T

Fix $x \in [0, +\infty)$ and $y \in [0, +\infty)$. Here, x denotes a value of $r(T)$ and y denotes a value of $V_2(T)$.

If the stock options are not exercised at time T,the firm sells N_2 stocks in the market. In other words the firm issues N_2 stocks at the market price. Therefore,

(cash-flow into the firm per one stock) = (market price of stock at time T).

Let q be the total value of N_2 stocks which the firm sells in the market and then,

$$\frac{1}{N_2} q = S_3(T, x, y+q), \qquad (4.1)$$

since the firm value increases by q at time T. Since the preceding equation for q has unique solution (See APPENDIX for the proof), the trade that the firm sells N_2 stocks in the market is accomplished. Let $q^*(x, y)$ be the unique solution of (4.1), then, we have

$$Q_T = q^*(r(T), V_2(T)). \qquad (4.2)$$

(2) Increment of the firm value when stock options are exercised at time T

If the stock options are exercised at time T,the firm exchange N_2 stocks for price M per one stock with the stock option holders. In other words the firm issues N_2 stocks to the stock option holders at price M per one stock. By the raising equity, the firm value increases by $N_2 \cdot M$. Then,

$$R_T = N_2 \cdot M. \qquad (4.3)$$

(3) The range of the firm value such that stock options are exercised

Let's fix $x \in [0, +\infty)$ and $y \in [0, +\infty)$ arbitrarily. x denotes a value of $r(T)$ and y denotes a value of $V_2(T)$.

Under the assumption of no arbitrage and the assumption of frictionless market , it is proved that $S_3(t, x, y : K)$ is a decreasing function of K (the face value of the zero coupon bond) in Miura (1989) chapter 3 pp78. From (3.1), we can see that $S_3(t, x, y : K)$ is linear homogeneous in y and K. Using these properties of S_3, we can prove that $S_3(t, x, y : K)$ is an increasing function of y (See Miura (1989) chapter 3 pp79-81 for the proof). Then the range of the firm

value such that stock options are exercised is as follows;
for any $x \in [0,+\infty)$, there exists $\bar{V}(x) \in [0,+\infty)$,

$$\{y \in [0,+\infty) \mid S_3(T,x,y+N_2 \cdot M) > M\} = (\bar{V}(x),+\infty). \qquad (4.4)$$

Here, $\bar{V}(r(T))$ is the boundary such that the stock options are exercised at time T if $V_2(T)$ is greater than it. Then

$$\mathrm{A} = \{V_2(T) > \bar{V}(r(T))\}. \qquad (4.5)$$

For any $D \subseteq \mathbf{R}$, we denote an indicator function defined on $\mathbf{R}$ as

$$I_D(z) = \begin{cases} 1 & z \in D \\ 0 & z \in D \end{cases}.$$

From (4.5),

$$\mathrm{I_A} = I_{A(r(T))}(V_2(T)) \qquad (4.6)$$

where $A(x) := \{u \in [0,+\infty) \mid S_3(T,x,u+N_2 \cdot M{:}K) > M\}$, for $x \in [0,+\infty)$.

In the following sub-sections (4), (5) and (6), we let $t \in [t_0,T)$. We recall that $X(u)$ and $Y(u)$ for $u \in [t,T]$ are the solutions of stochastic differential equation (2.3b).

(4) Stock price for $t \in [t_0,T)$

From (1.3c), (4.2), (4.3) and (4.6),

$$\begin{aligned} S_2(T,r(T),V_2(T)) = & S_3(T,r(T),V_2(T)+q^*(r(T),V_2(T))) \cdot (1-I_{A(r(T))}(V_2(T))) \\ & + S_3(T,r(T),V_2(T)+N_2 \cdot M) \cdot I_{A(r(T))}(V_2(T)). \end{aligned}$$

This is the boundary condition in order to obtain the pricing formulae of S_2 as the solution of PDE (2.2). From (2.3a), we have that

$S_2(t,r(t),V_2(t))$

$$= E\left[\left\{\begin{array}{l} S_3(T,X(T),Y(T)+q^*(X(T),Y(T))) \cdot (1-I_{A(X(T))}(Y(T))) \\ +S_3(T,X(T),Y(T)+N_2 \cdot M) \cdot I_{A(X(T))}(Y(T)) \end{array}\right\} \cdot e^{-\int_t^T X(u)du} \,\middle|\, \begin{array}{l} X(t)=r(t) \\ Y(t)=V_2(t) \end{array}\right] \qquad (4.7)$$

(5) Stock option price for $t \in [t_0,T)$

From (1.5b) and (4.3),

$$C_2(T,r(T),V_2(T)) = \max(S_3(T,r(T),V_2(T)+N_2 \cdot M) - M, 0)$$

This is the boundary condition in order to obtain the pricing formulae of C_2 as the solution of PDE (2.2). From (2.3a), we have that

$$C_2(t,r(t),V_2(t))$$
$$=E\left[\max(S_3(T,X(T),Y(T)+N_2\cdot M)-M,0)\cdot e^{-\int_t^T X(u)du}\,\middle|\,\begin{matrix}X(t)=r(t)\\ Y(t)=V_2(t)\end{matrix}\right]. \tag{4.8}$$

(6) Zero coupon bond price for $t\in[t_0,T)$

Using (1.4c), (4.2), (4.3) and (4.6), we have

$$\begin{aligned}L_2(T,r(T),V_2(T))=&L_3(T,r(T),V_2(T)+q^*(r(T),V_2(T)))\cdot(1-I_{A(r(T))}(V_2(T)))\\&+L_3(T,r(T),V_2(T)+N_2\cdot M)\cdot I_{A(r(T))}(V_2(T))\end{aligned}$$

here, we substitute the right hand side for (3.2),

$$\begin{aligned}=&-N_1\cdot S_3(T,r(T),V_2(T)+q^*(r(T),V_2(T)))\cdot(1-I_{A(r(T))}(V_2(T)))\\&+(V_2(T)+q^*(r(T),V_2(T)))\cdot(1-I_{A(r(T))}(V_2(T)))\\&-N_1\cdot S_3(T,r(T),V_2(T)+N_2\cdot M)\cdot I_{A(r(T))}(V_2(T))\\&+(V_2(T)+N_2\cdot M)\cdot I_{A(r(T))}(V_2(T))\\=&-N_1\left\{\begin{matrix}S_3(T,r(T),V_2(T)+q^*(r(T),V_2(T)))\cdot(1-I_{A(r(T))}(V_2(T)))\\ S_3(T,r(T),V_2(T)+N_2\cdot M)\cdot I_{A(r(T))}(V_2(T))\end{matrix}\right\}\\&+q^*(r(T),V_2(T))\cdot(1-I_{A(r(T))}(V_2(T))+N_2\cdot M\cdot I_{A(r(T))}(V_2(T))+V_2(T)\end{aligned}$$

and then from (4.1),

$$\begin{aligned}=&-(N_1-N_2)\left\{\begin{matrix}S_3(T,r(T),V_2(T)+q^*(r(T),V_2(T)))\cdot(1-I_{A(r(T))}(V_2(T))\\ S_3(T,r(T),V_2(T)+N_2\cdot M)\cdot I_{A(r(T))}(V_2(T))\end{matrix}\right\}\\&-N_2\{S_3(T,r(T),V_2(T)+N_2\cdot M)-M\}\cdot I_{A(r(T))}(V_2(T))+V_2(T).\end{aligned}$$

This is the boundary condition in order to obtain the pricing formulae of L_2 as the solution of PDE (2.2). From (2.3a), we have that

$$L_2(t,r(t),V_2(t))$$
$$=E\left[\left\{\begin{matrix}-(N_1-N_2)\left\{\begin{matrix}S_3(T,X(T),Y(T)+q^*(X(T),Y(T)))\\ \times(1-I_{A(X(T))}(Y(T))\\ +S_3(T,X(T),Y(T)+N_2\cdot M)\\ \times I_{A(X(T))}(Y(T))\end{matrix}\right\}\\ -N_2\{S_3(T,X(T),Y(T)+N_2\cdot M)-M\}I_{A(X(T))}(Y(T))+Y(T)\end{matrix}\right\}\cdot e^{-\int_t^T X(u)du}\,\middle|\,\begin{matrix}X(t)=r(t)\\ Y(t)=V_2(t)\end{matrix}\right]$$

$$=-(N_1-N_2)\cdot S_2(t,r(t),V_2(t))-N_2\cdot C_2(t,r(t),V_2(t))+V_2(t), \tag{4.9}$$

where last equality uses (4.7), (4.8) and

$$E\left[Y(T)\cdot e^{-\int_t^T X(u)du}\,\middle|\,\begin{array}{l}X(t)=r(t)\\ Y(t)=V_2(t)\end{array}\right]=V_2(t).$$

(7) Firm value

From (4.9) we have that for any $t\in[t_0,T)$,

$$V_2(t)=L_2(t,r(t),V_2(t))+(N_1-N_2)\cdot S_2(t,r(t),V_2(t))+N_2\cdot C_2(t,r(t),V_2(t)). \quad (4.10)$$

This equation represents that the firm's balance sheet during the time interval $[t_0,T)$.

5. S_1, C_1, and L_1 in the first time interval

In this section, we derive the pricing formulae of S_1, C_1, and L_1 in the general form. When stochastic processes r and V are specified, the pricing formulae will be shown in section 7. Before writing them down, we show that the random variable P_{t_0} is expressed in terms of $r(t_0)$ and $V_1(t_0)$. This will be utilized for obtaining the pricing formulae for S_1, L_1, and C_1.

(1) Increment of the firm value caused by repurchasing the treasury stocks at time t_0

Let's fix $x\in[0,+\infty)$ and $y\in[0,+\infty)$ arbitrarily. Here, x denotes a value of $r(t_0)$ and y denotes a value of $V_1(t_0)$.

As mentioned in 4.1, the firm's repurchasing N_2 units of treasury stocks at time t_0 is the firm's repayment of N_2 stocks. Therefore,

(cash-flow out of the firm per one stock) = (market price of stock at time t_0).

Let p be total value of N_2 units of treasury stocks which the firm repurchases in the market and then,

$$\frac{1}{N_2}p=S_2(t_0,x,y-p) \qquad (5.1)$$

holds since the value of the firm decreases by p at time t_0. The preceding equation for p has unique solution and it belongs to $[0,y]$ (See APPENDIX for the proof), So the trade that the firm repurchases N_2 stocks in the market is accomplished. Let $p^*(x,y)$ be the unique solution of (5.1), then, we have

$$P_{t_0}=p^*(r(t_0),V_1(t_0)). \qquad (5.2)$$

We let $t\in[0,t_0)$, in the following sub-sections (2), (3) and (4). We recall that $X(u)$ and $Y(u)$ for $u\in[t,t_0]$ are the solutions of stochastic differential equation (2.3b).

(2) Stock price for $t\in[0,t_0)$

From (1.3d) and (5.2),

$$S_1(t_0,r(t_0),V_1(t_0))=S_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big).$$

This is the boundary condition in order to obtain the pricing formulae of S_1 as the solution of PDE (2.2). From (2.3a), we have that

$$\begin{aligned} &S_1(t,r(t),V_1(t))\\ &\quad=E\left[S_2\big(t_0,X(t_0),Y(t_0)-p^*(X(t_0),Y(t_0))\big)\cdot e^{-\int_t^{t_0}X(u)du}\;\middle|\;\begin{matrix}X(t)=r(t)\\ Y(t)=V_1(t)\end{matrix}\right] \end{aligned} \tag{5.3}$$

(3) Stock option price for $t\in[0,t_0]$
From (1.5c) and (5.2),

$$C_1(t_0,r(t_0),V_1(t_0))=C_2(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))).$$

This is the boundary condition in order to obtain the pricing formulae of C_1 as the solution of PDE (2.2). From (2.3a), we have that

$$\begin{aligned} &C_1(t,r(t),V_1(t))\\ &\quad=E\left[C_2\big(T,X(T_0),Y(T_0)-p^*(X(t_0),Y(t_0))\big)\cdot e^{-\int_t^{t_0}X(u)du}\;\middle|\;\begin{matrix}X(t)=r(t)\\ Y(t)=V_1(t)\end{matrix}\right]. \end{aligned} \tag{5.4}$$

(4) Zero coupon bond price for $t\in[0,t_0]$
From (1.4d) and (5.2)

$$\begin{aligned} &L_1(t_0,r(t_0),V_1(t_0))\\ &\quad=L_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big) \end{aligned}$$

here, we substitute the right hand side for (4.9),

$$\begin{aligned} &=-(N_1-N_2)\cdot S_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big)\\ &\quad-N_2\cdot C_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big)+V_1(t_0)-p^*(r(t_0),V_1(t_0)) \end{aligned}$$

and then, by (5.1)

$$\begin{aligned} &=-N_1\cdot S_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big)\\ &\quad-N_2\cdot C_2\big(t_0,r(t_0),V_1(t_0)-p^*(r(t_0),V_1(t_0))\big)+V_1(t_0). \end{aligned}$$

This is the boundary condition in order to obtain the pricing formulae of L_1 as the solution of PDE (2.2). From (2.3a), we have that

$$L_1(t,r(t),V_1(t))$$

$$=E\left[\left\{\begin{array}{l}-N_1 \cdot S_2\big(t_0, X(t_0), Y(t_0)-p^*(X(t_0), Y(t_0))\big) \\ -N_2 \cdot C_2\big(t_0, X(t_0), Y(t_0)-p^*(X(t_0), Y(t_0))\big) \\ +Y(t_0)\end{array}\right\} \cdot e^{-\int_t^{t_0} X(u)du} \,\middle|\, \begin{array}{l}X(t)=r(t) \\ Y(t)=V_1(t)\end{array}\right]$$

$$=-N_1 \cdot S_1\big(t, r(t), V_1(t)\big)-N_2 \cdot C_1\big(t, r(t), V_1(t)\big)+V_1(t). \qquad (5.5)$$

where last equality uses (5.3), (5.4) and

$$E\left[Y(t_0) \cdot e^{-\int_t^{t_0} X(u)du} \,\middle|\, \begin{array}{l}X(t)=r(t) \\ Y(t)=V_1(t)\end{array}\right]=V_1(t). \qquad (5.3)$$

(5) Firm value

From (5.5), we have that for any $t \in [0, t_0)$

$$V_1(t)=L_1\big(t, r(t), V_1(t)\big)+N_1 \cdot S_1\big(t, r(t), V_1(t)\big)+N_2 \cdot C_1\big(t, r(t), V_1(t)\big). \qquad (5.6)$$

This equation represents the firm's balance sheet during the time interval $[0, t_0)$.

6. *Consideration*

(1) Balance sheet

We write down (4.10) and (5.6) again. For any $t \in [t_0, T)$,

$$V_2(t)=L_2\big(t, r(t), V_2(t)\big)+(N_1-N_2) \cdot S_2\big(t, r(t), V_2(t)\big)+N_2 \cdot C_2\big(t, r(t), V_2(t)\big) \qquad (6.1)$$

and, for any $t \in [0, t_0]$,

$$V_1(t)=L_1\big(t, r(t), V_1(t)\big)+N_1 \cdot S_1\big(t, r(t), V_1(t)\big)+N_2 \cdot C_1\big(t, r(t), V_1(t)\big). \qquad (6.2)$$

The last two equations show the balance sheet of the firm based on market value accounting. The credit side of the balance sheet consists of three kinds of debts, namely, stock, zero coupon bond and stock option. It means that how much each one of stockholders, bond holders or stock option holders get in case of liquidation of the firm at time $t \in [0, T)$. Then, at the maturity of the stock option, the wealth of the stock holders do not transfer to the stock option holders. The profit in the stock option belong to the executives (stock option holders) from the beginning because the stock options are in the part of executive compensation.

(2) Effect of introducing the stock option on the management

The functions which represent stock price or stock option price,

$$\begin{cases} S_1(t,x,y) & \text{for}(t,x,y) \in [0, t_0] \times [0, +\infty) \times [0, +\infty) \\ S_2(t,x,y) & \text{for}(t,x,y) \in [t_0, T] \times [0, +\infty) \times [0, +\infty) \end{cases}$$

$$\begin{cases} C_1(t,x,y) & \text{for}(t,x,y) \in [0,t_0] \times [0,+\infty) \times [0,+\infty) \\ C_2(t,x,y) & \text{for}(t,x,y) \in [t_0,T] \times [0,+\infty) \times [0,+\infty) \end{cases}$$

are increasing functions of y (See APPENDIX). In other words, an increase of firm's value increases both stockholders' wealth and stock option holders' wealth. Then it follows that the stock option (treasury stock type) gives the stock option holders the incentive for better firm's performance which brings the stock holders the more profit.

(3) Risk from N_2 units of treasury stocks the firm is holding during $[t_0,T]$

It might be accepted by some people that the firm suffers losses caused by the treasury stocks in case that the stock price falls down under the acquisition value after repurchasing and the stock options are not exercised. We show this can be seen an force idea. In the above view, N_2 units of the treasury stocks are assets for the firm and the firm's balance sheet at time $t \in [t_0,T]$ is

$$\begin{aligned} &V_2(t)+N_2 \cdot S_2(t,r(t),V_2(t)) \\ &\quad =L_2(t,r(t),V_2(t))+N_1 \cdot S_2(t,r(t),V_2(t))+N_2 \cdot C_2(t,r(t),V_2(t)). \end{aligned} \tag{6.3}$$

Still more, in that view, the event

$$\{S(t_0,r(t_0),V(t_0)) > S(T,r(T),V(T))\}$$

is interpreted as the firm's losses caused by the treasury stocks, or the risk from the treasury stocks which the firm is holding during the time interval $[t_0,T]$. But as we have mentioned in 4.1, the firm's holding N_2 units of the treasury stocks during the time interval $[t_0,T]$ is no meaning. Therefore, (6.3) is merely anonymous expression of the balance sheet, and for the firm the above event means only that the capital financed at time T is less than the repayment at time t_0.

With respect to this discussion, from an accounting viewpoint, Ito (1997a,1997b) indicated that increasing paid-in capital by stock issue is not consistent with increasing assets by repurchasing treasury stocks, which is the reversing trade of the stock issue. Ito's view agrees with our explanation.

7. *Example*

In this section we specify stochastic processes r and V and calculate (3.1), (4.7), (4.8), (5.3) and (5.4).

Assuming that

$$r(t) \equiv r(\text{positive constant}), \text{ and}$$

$$dV_i(t) = \mu \cdot V_i(t) \cdot dt + \sigma \cdot V_i(t) \cdot dW_2(t) \qquad i=1,\ 2,\ 3$$

where μ and σ are positive constant,

we calculate S_1, S_2, S_3, C_1 and C_2.

For $t \in [T,\tau]$ and $v \in [0,+\infty)$,

$$S_3(t,r,v)=\frac{1}{N_1}v\cdot\Phi(h(t,\tau,r,v,K))-\frac{1}{N_1}e^{-r(\tau-t)}K\cdot\Phi\left(h(t,\tau,r,v,K)-\sigma\sqrt{\tau-t}\right)$$

$$\text{where } \Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,du, \text{ and}$$

$$h(t,\tau,r,v,K)=\frac{\log\frac{v}{K}+(r+\frac{1}{2}\sigma^2)(\tau-t)}{\sigma\sqrt{\tau-t}}.$$

For $(t,v)\in[t_0,T]\times[0,+\infty)$,

$$S_2(t,r,v)=e^{-r(T-t)}\int_{A(r)^c}S_3(T,r,y+q^*(r,y))\frac{1}{\sigma\sqrt{2\pi(T-t)}\cdot y}e^{-\frac{1}{2}\left(h(t,T,r,v,y)-\sigma\sqrt{T-t}\right)^2}dy$$

$$+e^{-r(T-t)}\int_{A(r)}S_3(T,r,y+N_2\cdot M)\frac{1}{\sigma\sqrt{2\pi(T-t)}\cdot y}e^{-\frac{1}{2}h(t,T,r,v,y)^2}dy$$

$$\text{where } A(r)=\left\{y\in[0,+\infty)\mid S_3(T,r,y+N_2\cdot M)>M\right\}, \text{ and}$$

$$C_2(t,r,v)=e^{-r(T-t)}\int_{A(r)}S_3(T,r,y+N_2\cdot M)\frac{1}{\sigma\sqrt{2\pi(T-t)}\cdot y}e^{-\frac{1}{2}\left(h(t,T,r,v,y)-\sigma\sqrt{T-t}\right)^2}dy$$

$$-e^{-r(T-t)}\cdot M\cdot\Phi\left(h(t,T,r,v,T,\bar{V}(r))-\sigma\sqrt{\tau-t}\right).$$

For $(t,v)\in[0,t_0]\times[0,+\infty)$,

$$S_1(t,r,v)=e^{-r(t_0-t)}\int_0^{+\infty}S_2(t_0,r,y-p^*(r,y))\frac{1}{\sigma\sqrt{2\pi(t_0-t)}\cdot y}e^{-\frac{1}{2}\left(h(t,t_0,r,v,y)-\sigma\sqrt{t_0-t}\right)^2}dy$$

and

$$C_1(t,r,v)=e^{-r(t_0-t)}\int_0^{+\infty}C_2(t_0,r,y-p^*(r,y))\frac{1}{\sigma\sqrt{2\pi(t_0-t)}\cdot y}e^{-\frac{1}{2}\left(h(t,t_0,r,v,y)-\sigma\sqrt{t_0-t}\right)^2}dy.$$

Acknowledgment

The first author was supported by a research grant from the Ministry of Education.

Appendix

Assuming 'No Arbitrage', 'Frictionless Market' and 'If the rate of returns of two assets are identically distributed and the other conditions of two derivatives whose underlying asset is each of assets, are same, then the prices of the two derivatives are equal.' (Hereafter, (aa) denotes this assumption.), then we can prove following properties. We need not assume that r, V_1, V_2, and V_3 are Ito processes. Namely we can prove them in general setting about stochastic processes of r and V.

A1. Properties of S_3

From the definition, S_3 is considered as a function of face value K. If necessary , we will use the notation $S_3(t,x,y:K)$ instead of $S_3(t,x,y)$.

Let $t \in [T,\tau]$, $x \in [0,+\infty)$ and $y \in [0,+\infty)$.

(1) $\max(y-K,0) \leq N_1 \cdot S_3(t,x,y) \leq y$ (a1.1)

(2) $S_3(t,x,y:K)$ is a decreasing function of K.

(3) $S_3(t,x,y:K)$ is linear homogeneous in y and K.

(4) $S_3(t,x,y:K)$ is an increasing function of y.

(5) $S_3(t,x,y)$ is a convex function of y.

See Miura(1989) chapter 3 pp67-81 for the proofs.

(6) For any y_0, $y_1 \in [0,+\infty)$, $y_0 < y_1$, $\dfrac{N_1 \cdot S_3(t,x,y_1) - N_1 \cdot S_3(t,x,y_0)}{y_1 - y_0} \leq 1$ (al.2)

[proof of (6)]

We use (1) and (5) to prove (a1.2) by the reduction to absurdity.

Suppose that there exists y_0, $y_1 \in [0,+\infty)$, $y_0 < y_1$ such that

$$\frac{N_1 \cdot S_3(t,x,y_1) - N_1 \cdot S_3(t,x,y_0)}{y_1 - y_0} > 1.$$

Let $(y_1 <) y_2 < y_3 < \cdots$ be increasing sequence. Then, using (5), we have

$$1 < \frac{N_1 \cdot S_3(t,x,y_1) - N_1 \cdot S_3(t,x,y_0)}{y_1 - y_0} \leq \frac{N_1 \cdot S_3(t,x,y_2) - N_1 \cdot S_3(t,x,y_1)}{y_2 - y_1} \leq \cdots.$$

For n$\in$**N**, set

$$a_n := \frac{N_1 \cdot S_3(t,x,y_n) - N_1 \cdot S_3(t,x,y_{n-1})}{y_n - y_{n-1}} \ (>1), \qquad \text{(a1.3)}$$

then for n$\in$**N**,

$$\begin{aligned} N_1 \cdot S_3(t,x,y_n) &= a_n(y_n - y_{n-1}) + N_1 \cdot S_3(t,x,y_{n-1}) \\ &= a_n(y_n - y_{n-1}) + a_{n-1}(y_{n-1} - y_{n-2}) + N_1 \cdot S_3(t,x,y_{n-2}) \\ &\ \vdots \\ &= \sum_{j=1}^{n} a_j(y_j - y_{j-1}) + N_1 \cdot S_3(t,x,y_0). \end{aligned}$$

Since $(a_n)_{n \in \mathrm{N}}$ is the increasing sequence,

$$N_1 \cdot S_3(t,x,y_n) \geq a_1(y_n - y_0) + N_1 \cdot S_3(t,x,y_0),$$

then we obtain that for any $y \in (y_1, +\infty)$

$$N_1 \cdot S_3(t,x,y) \geq a_1(y - y_0) + N_1 \cdot S_3(t,x,y_0). \qquad \text{(a1.4)}$$

We use following two straight lines;

$$\begin{cases} z_1(y) := y \\ z_2(y) := a_1(y - y_0) + N_1 \cdot S_3(t,x,y_0). \end{cases}$$

As

$$y_0 \geq 0, \ -a_1 \cdot y_0 + N_1 \cdot S_3(t,x,y_0) \leq 0 \text{ and } a_1 > 1,$$

there exists the intersection point of the above straight lines. Let (y^*, y^*) denote the coordinates of the intersection point. From (a1.3) and (a1.1)

$$a_1(y_1 - y_0) + N_1 \cdot S_3(t,x,y_0) = N_1 \cdot S_3(t,x,y_1) \leq y_1, \text{ and } a_1 > 1.$$

Then we have that $y^* \in [y_1, +\infty)$. By (a1.4), it follows that for any $y \in (y^*, +\infty)$

$$y < a_1(y - y_0) + N_1 \cdot S_3(t,x,y_0) \leq N_1 \cdot S_3(t,x,y).$$

This contradicts (1). Therefore (a1.2) holds.

A2. (4.1) has an unique solution which belongs to $[0, +\infty)$.

Fix $x \in [0, +\infty)$ and $y \in [0, +\infty)$ arbitrarily.

Multiply both sides of (4.1) by N_1,

$$\frac{N_1}{N_2} q = N_1 \cdot S_3(T,x,y+q). \qquad \text{(a2.1)}$$

Set $u = y + q$ and change the variable in equation (a2.1),

$$\frac{N_1}{N_2}(u - y) = N_1 \cdot S_3(T,x,u). \qquad \text{(a2.2)}$$

Set

$$\begin{cases} z_1(u) := \dfrac{N_1}{N_2}(u - y) \\ z_2(u) := N_1 \cdot S_3(T,x,u), \end{cases}$$

and it is sufficient to show that there exists an unique intersection point of z_1 and z_2,and that the u coordinates of the point belongs to $[y, +\infty]$.

We first show that there exists an intersection point of z_1 and z_2. To show this, we use two straight lines;

$$\begin{cases} z_3(u) := u \\ z_1(u) = \dfrac{N_1}{N_2}(u-y). \end{cases}$$

From the assumption $\dfrac{N_1}{N_2} > 1$, then ,the two straight lines cross each other and the coordinates of the intersection point is

$$\left(\frac{N_1}{N_1-N_2}y, \frac{N_1}{N_1-N_2}y\right).$$

From A1.(1),

$$0 = \frac{N_1}{N_2}(y-y) \leq N_1 \cdot S_3(T,x,y) \quad \text{(a2.3) , and}$$

$$N_1 \cdot S_3(T,x,u_0) \leq u_0 = \frac{N_1}{N_2}(u_0-y) \quad \text{(a2.4)}$$

$$\text{where } u_0 := \frac{N_1}{N_1-N_2}y.$$

Therefore, from (a2.3) and (a2.4), the u coordinates of the point in which z_1 and z_2 intersect belongs to $[y, +\infty)$. It remains to show the uniqueness of the point. Suppose that z_1 and z_2 intersect in two distinct points. Suppose u_1 and $u_2 (u_1 < u_2)$ are the u coordinates of the two distinct intersection points. Then

$$\frac{N_1 \cdot S_3(t,x,u_2) - N_1 \cdot S_3(t,x,u_1)}{u_2-u_1} = \frac{\frac{N_1}{N_2}(u_2-y) - \frac{N_1}{N_2}(u_1-y)}{u_2-u_1} = \frac{N_1}{N_2} > 1.$$

This contradicts A1(6). Therefore the intersection point of z_1 and z_2 is unique.

A3. Properties of q^*

Fix $x \in [0, +\infty)$ and $y \in [0, +\infty)$ arbitrarily.

Since $q^*(x,y)$ is a solution of following equation for q;

$$\frac{1}{N_2}q = S_3(T,x,y+q:K),$$

q^* is considered as a function of face value K. If necessary, we will use the notation $q^*(x,y:K)$ instead of $q^*(x,y)$.

(1) $q^*(x,y:K)$ is an increasing function of y.

[proof]

To show that for $0<y_1<y_2$, $q^*(x,y_1)<q^*(x,y_2)$, we use the reduction to absurdity and apply A1(4) and A1(6). Suppose that there exists $y_1, y_2 \in [0,+\infty)$, $y_1<y_2$ such that

$$q^*(x,y_1)>q^*(x,y_2).$$

From the definition of q^* and A1(2),

$$\frac{N_1}{N_2}q^*(x,y_1)=N_1\cdot S_3(T,x,y_1+q^*(x,y_1))\leq N_1\cdot S_3(T,x,y_2+q^*(x,y_1)) \text{ and}$$
$$\frac{N_1}{N_2}q^*(x,y_2)=N_1\cdot S_3(T,x,y_2+q^*(x,y_2))\geq N_1\cdot S_3(T,x,y_1+q^*(x,y_2))$$

then,

$$\frac{N_1\cdot S_3(T,x,y_1+q^*(x,y_1))-N_1\cdot S_3(T,x,y_1+q^*(x,y_2))}{q^*(x,y_1)-q^*(x,y_2)}$$
$$=\frac{N_1\cdot S_3(T,x,y_1+q^*(x,y_1))-N_1\cdot S_3(T,x,y_1+q^*(x,y_2))}{N_2\cdot S_3(T,x,y_1+q^*(x,y_1))-N_2\cdot S_3(T,x,y_2+q^*(x,y_2))}$$
$$\geq\frac{N_1\cdot S_3(T,x,y_1+q^*(x,y_1))-N_1\cdot S_3(T,x,y_2+q^*(x,y_2))}{N_2\cdot S_3(T,x,y_1+q^*(x,y_1))-N_2\cdot S_3(T,x,y_2+q^*(x,y_2))}=\frac{N_1}{N_2}>1,$$

and this contradicts A1(6). Apparently it does not hold that for $0<y_1<y_2$, $q^*(x,y_1)=q^*(x,y_2)$.

(2) Fix $x\in[0,+\infty)$ arbitrarily, then
for any $y\in\{u\in[0,+\infty) \mid S_3(T,x,u+N_2\cdot M)>M\}$, $q^*(x,y)>N_2\cdot M$, and
for any $y\in\{u\in[0,+\infty) \mid S_3(T,x,u+N_2\cdot M)\leq M\}$, $q^*(x,y)\leq N_2\cdot M$.

[proof]

From A1(1) and A1(4), Apparently, following equation for y;

$$M=S_3(T,x,y+N_2\cdot M) \qquad \text{(a3.2)}$$

has an unique solution and the solution is $\overline{V}(x)$. From A1(4),

$$\begin{aligned}&\{y\in[0,+\infty) \mid S_3(T,x,y+N_2\cdot M)>M\}=(\overline{V}(x),\infty),\\ &\{y\in[0,+\infty) \mid S_3(T,x,y+N_2\cdot M)\leq M\}=[0,\overline{V}(x)],\end{aligned} \qquad \text{(a3.3)}$$

and

$$\frac{1}{N_2}N_2\cdot M=S_3(T,x,\overline{V}(x)+N_2\cdot M).$$

Then

$$q^*(x,\overline{V}(x))=N_2 \cdot M$$

Therefore, from (1) and (a3.3), we have that

for any $y\in(\overline{V}(x),\infty)$, $q^*(x,y)>q^*(x,\overline{V}(x))=N_2 \cdot M$ and

for any $y\in[0,\overline{V}(x)]$, $q^*(x,y)\leq q^*(x,\overline{V}(x))=N_2 \cdot M$.

(3) $q^*(x,y:K)$ is a decreasing function of K.

[proof]

Fix $x\in[0,+\infty)$ and $y\in[0,+\infty)$ arbitrarily. For $0<K_1<K_2$, set

$$q^*_j:=q^*(x,y:K_j)\quad j=1,\ 2.$$

Suppose $q^*_1<q^*_2$. From the definition of q^* and A1(2),

$$\frac{N_1}{N_2}q^*_1=N_1 \cdot S_3(T,x,y+q^*_1:K_1),\text{ and}$$

$$N_1 \cdot S_3(T,x,y+q^*_2:K_1)\geq N_1 \cdot S_3(T,x,y+q^*_2:K_2)=\frac{N_1}{N_2}q^*_2.$$

Then

$$\frac{N_1 \cdot S_3(T,x,y+q^*_2:K_1)-N_1 \cdot S_3(T,x,y+q^*_1:K_1)}{q^*_2-q^*_1}\geq\frac{\frac{N_1}{N_2}q^*_2-\frac{N_1}{N_2}q^*_1}{q^*_2-q^*_1}=\frac{N_1}{N_2}>1,$$

and this contradicts A1(6). Therefore $q^*_1\geq q^*_2$ holds.

(4) $q^*(x,y:K)$ is linear homogeneous in y and K.

[proof]

Fix $c\in[0,+\infty)$, $x\in[0,+\infty)$ and $y\in[0,+\infty)$ arbitrarily. $q^*(x,y:K)$ is the solution of the following equation for q;

$$\frac{1}{N_2}q=S_3(T,x,y+q:K),\qquad \text{(a3.4)}$$

then

$$\frac{1}{N_2}q^*(x,y:K)=S_3(T,x,y+q^*(x,y:K):K).$$

From A1(3),

$$\frac{1}{N_2}c \cdot q^*(x,y:K)=S_3(T,x,c \cdot y+c \cdot q^*(x,y:K):c \cdot K).$$

Here, $c \cdot q^*(x,y{:}K)$ is the solution of the following equation for u:

$$\frac{1}{N_2} u = S_3(T,x,c \cdot y+u{:}c \cdot K). \qquad \text{(a3.5)}$$

From the definition of q^*, the solution of (a3.5) is

$$q^*(x,c \cdot y{:}c \cdot K).$$

Therefore

$$q^*(x,c \cdot y{:}c \cdot K) = c \cdot q^*(x,y{:}K).$$

A4. Properties of S_2

From the definition , S_2 is considered as a function of face value K and exercise price M. If necessary, we will use the notation $S_2(t,x,y{:}K,M)$ instead of $S_2(t,x,y)$.

Let $t \in [t_0,T]$, $x \in [0,+\infty)$ and $y \in [0,+\infty)$.

(1) $0 \leq S_2(t,x,y)$ (a4.1)

It follows immediately from the assumption of no arbitrage.

(2) $\dfrac{1}{N_1} \max(y-K,0) \leq S_2(t,x,y)$ (a4.2)

[proof]

From (1.3c) and (4.6)

$$S_2(T,x,y) = S_2(T,x,y+q^*(x+y)) \cdot (1-I_{A(x)}(y)) + S_3(T,x,y+N_2 \cdot M) \cdot I_{A(x)}(y).$$

Using A1(1),

$$S_2(T,x,y) \geq \frac{1}{N_1} \max(y+q^*(x,y)-K,0) \cdot (1-I_{A(x)}(y)) + \frac{1}{N_1} \max(y+N_2 \cdot M-K,0) \cdot I_{A(x)}(y)$$

and since $q^*(x,y) \geq 0$, $N_2 \cdot M > 0$,

$$S_2(T,x,y) \geq \frac{1}{N_1} \max(y-K,0).$$

(3) $S_2(t,x,y) \leq \dfrac{1}{N_1 - N_2} y$ (a4.3)

[proof]

From the assumption of no arbitrage, it is sufficient to show that

$$S_2(T,x,y) \leq \frac{1}{N_1 - N_2} y.$$

From A1(1),

$$S_3(T,x,y+q^*(x,y)) \leq \frac{1}{N_1} \{y+q^*(x,y)\}, \quad \text{(a4.4)}$$

and from the definition of q^*,

$$q^*(x,y) = N_2 \cdot S_3(T,x,y+q^*(x,y)). \qquad \text{(a4.5)}$$

Here we recall that

$$\begin{aligned} S_2(T,x,y) = & S_3(T,x,y+q^*(x,y)) \cdot (1 - \mathrm{I}_{A(x)}(y)) \\ & + S_3(T,x,y+N_2 \cdot M) \cdot \mathrm{I}_{A(x)}(y), \end{aligned} \qquad \text{(a4.6)}$$
$$\text{where } A(x) := \{u \in [0,\infty) \mid S_3(T,x,u+N_2 \cdot M{:}K) > M\}.$$

Applying A3(2) to (a4.6), we obtain

$$S_2(T,x,y) \leq S_3(T,x,y+q^*(x,y)). \qquad \text{(a4.7)}$$

Substituting (a4.4) and (a4.5) for the right hand side of (a4.7) repeatedly ,

$$\begin{aligned} & S_2(T,x,y) \\ & \leq \frac{1}{N_1} y + \frac{1}{N_1} q^*(x,y) \\ & = \frac{1}{N_1} y + \frac{1}{N_1} N_2 \cdot S_3(T,x,y+q^*(x+y)) \\ & \leq \frac{1}{N_1} y + \frac{N_2}{N_1^2} y + \frac{N_2}{N_1^2} q^*(x,y) \\ & \quad \vdots \\ & \leq \frac{1}{N_1} y + \frac{N_2}{N_1^2} y + \frac{N_2^2}{N_1^3} y + \cdots = \frac{\frac{1}{N_1}}{1 - \frac{N_2}{N_1}} y = \frac{1}{N_1 - N_2} y. \end{aligned}$$

(4) $S_2(t,x,y{:}K,M)$ is a decreasing function of K.
[proof]
From the assumption of no arbitrage, it is sufficient to show that for $0 < K_1 < K_2$,

$$S_2(T,x,y{:}K_1,M) \geq S_2(T,x,y{:}K_2,M). \qquad \text{(a4.8)}$$

Let $0<K_1<K_2$. For $j=1, 2$,

$$S_2(T,x,y:K_j,M)=S_3(T,x,y+q^*(x,y:K_j):K_j)\cdot(1-I_{A_j(x)}(y))+S_3(T,x,y+N_2\cdot M\cdot K_j)\cdot I_{A_j(x)}(y), \tag{a4.9}$$
$$\text{where } A_j(x):=\{u\in[0,\infty]\mid S_3(T,x,u+N_2\cdot M:K_j)>M\}.$$

For $x\in[0,+\infty)$ and $y\in A_2(x)$, from A1(2),

$$S_3(T,x,y+N_2\cdot M:K_1)\geq S_3(T,x,y+N_2\cdot M:K_2)>M,$$

and hence $A_2(x)\subset A_1(x)$.
Fix $x\in[0,+\infty)$ arbitrarily.
① If $y\in A_2(x)$, from (a4.9),
$S_2(T,x,y:K_1,M)=S_3(T,x,y+N_2\cdot M:K_1)$, and
$S_2(T,x,y:K_2,M)=S_3(T,x,y+N_2\cdot M:K_2)$.

Therefore,

$$S_2(T,x,y:K_1,M)\geq S_2(T,x,y:K_2,M).$$

② If $y\in A_1(x)\cap A_2(x)^c$, from (a4.9),
$S_2(T,x,y:K_1,M)=S_3(T,x,y+N_2\cdot M:K_1)$, and
$S_2(T,x,y:K_2,M)=S_3(T,x,y+q^*(x,y:K_2):K_2)$,

and from A3(2),

$$q^*(x,y:K_2)\leq N_2\cdot M$$

then

$$S_2(T,x,y:K_1,M)=S_3(T,x,y+N_2\cdot M:K_1)\geq S_3(T,x,y+q^*(x,y:K_2):K_2)=S_2(T,x,y:K_2,M)$$

③ If $y\in A_1(x)^c$, from (a4.9),

$S_2(T,x,y:K_1,M)=S_3(T,x,y+q^*(x,y:K_1):K_1)$, and
$S_2(T,x,y:K_2,M)=S_3(T,x,y+q^*(x,y:K_2):K_2)$,

and from A3(3),

$$q^*(x,y:K_1)\geq q^*(x,y:K_2),$$

then

$$S_2(T,x,y:K_1,M)=S_3(T,x,y+q^*(x,y:K_1):K_1)$$

$$\begin{aligned}&\geq S_3(T,x,y+q^*(x,y:K_2):K_1)\\&\geq S_3(T,x,y+q^*(x,y:K_2):K_2)\\&=S_2(T,x,y:K_2,M).\end{aligned}$$

Therefore, (a4.8) holds.

(5) $S_2(t,x,y:K,M)$ is an increasing function of M.
[proof]

From the assumption of no arbitrage, it is sufficient to show that for $0<M_1<M_2$,

$$S_2(T,x,y:K,M_1)\leq S_2(T,x,y:K,M_2). \quad \text{(a4.10)}$$

Let $0<M_1<M_2$. For $j=1,2$,

$$\begin{aligned}S_2(T,x,y:K,M_j)=&S_3(T,x,y+q^*(x,y:K):K)\cdot(1-I_{A_j(x)}(y))\\&+S_3(T,x,y+N_2\cdot M_j:K)\cdot I_{A_j(x)}(y)\end{aligned}, \quad \text{(a4.11)}$$

where $A_j(x):=\{u\in[0,\infty)\mid S_3(T,x,u+N_2\cdot M_j:K)>M_j\}$.

First, we show that $A_2(x)\subset A_1(x)$ for $x\in[0,+\infty)$.
Fix $x\in[0,+\infty)$ arbitrarily. For $y\in A_2(x)$, from A1(6),

$$1\geq\frac{N_1\cdot S_3(T,x,y+N_2\cdot M_2:K)-N_1\cdot S_3(T,x,y+N_2\cdot M_1:K)}{N_2\cdot M_2-N_2\cdot M_1}.$$

Then

$$\begin{aligned}&M_2-M_1\geq S_3(T,x,y+N_2\cdot M_2:K)-S_3(T,x,y+N_2\cdot M_1:K)\\&S_3(T,x,y+N_2\cdot M_1:K)\geq M_1+S_3(T,x,y+N_2\cdot M_2:K)-M_2.\end{aligned}$$

Since $S_3(T,x,y+N_2\cdot M_2:K)-M_2>0$, then

$$S_3(T,x,y+N_2\cdot M_1:K)>M_1.$$

Therefore $y\in A_1(x)$, and $A_2(x)\subset A_1(x)$.

Fix $x\in[0,+\infty)$ arbitrarily.

① If $y\in A_2(x)$, from (a4.11) and the preceding result,

$$\begin{aligned}&S_2(T,x,y:K,M_1)=S_3(T,x,y+N_2\cdot M_1:K), \text{ and}\\&S_2(T,x,y:K,M_2)=S_3(T,x,y+N_2\cdot M_2:K)\end{aligned}$$

and hence

$$S_2(T,x,y:K,M_1)\leq S_2(T,x,y:K,M_2).$$

② If $y\in A_1(x)\cap A_2(x)^c$, by (a4.11),

$$S_2(T,x,y:K,M_1)=S_3(T,x,y+N_2\cdot M_1:K), \text{ and}$$
$$S_2(T,x,y:K,M_2)=S_3(T,x,y+q^*(x,y:K):K),$$

and using A3(2),

$$N_2\cdot M_1<q^*(x,y:K).$$

Then from A1(4),

$$S_2(T,x,y:K,M_1)\leq S_2(T,x,y:K,M_2).$$

③ If $y\in A_1(x)^c$, from (a4.11),

$$S_2(T,x,y:K,M_1)=S_3(T,x,y+q^*(x,y:K):K)=S_2(T,x,y:K,M_2).$$

Therefore (a4.10) holds.

(6) $S_2(t,x,y:K,M)$ is linear homogenous in y, K and M.
[proof]
From the assumption of no arbitrage and (aa), it is sufficient to show that for any $c\geq 0$,

$$c\cdot S_2(T,x,y:K,M)=S_2(T,x,c\cdot y:c\cdot K,c\cdot M). \quad \text{(a4.12)}$$

Fix $c>0$ arbitrarily. From A1(3) and A3(4),

$$\begin{aligned}
&c\cdot S_2(T,x,y:K,M)\\
&=c\cdot S_3(T,x,y+q^*(x,y:K):K)\cdot(1-I_{A(x)}(y))+c\cdot S_3(T,x,y+N_2\cdot M:K)\cdot I_{A(x)}(y)\\
&=S_3(T,x,c\cdot y+q^*(x,c\cdot y:c\cdot K):c\cdot K)\cdot(1-I_{A(x)}(y))\\
&\quad+S_3(T,x,c\cdot y+N_2\cdot c\cdot M:c\cdot K)\cdot I_{A(x)}(y)
\end{aligned}$$

Set

$$A_c(x):=\{u\in[0,\infty]\mid S_3(T,x,u+N_2\cdot c\cdot M:c\cdot K)>c\cdot M\}.$$

Since

$$\begin{aligned}
A(x)&=\{u\in[0,\infty)\mid S_3(T,x,u+N_2\cdot M:K)>M\}\\
&=\{u\in[0,\infty)\mid c\cdot S_3(T,x,u+N_2\cdot M:K)>c\cdot M\}\\
&=\{u\in[0,\infty)\mid S_3(T,x,c\cdot u+N_2\cdot c\cdot M:c\cdot K)>c\cdot M\}, \qquad \text{(a4.13)}
\end{aligned}$$

then for any $y\in[0,+\infty)$

$$I_{A(x)}(y)=I_{A_c(x)}(c\cdot y),$$

and hence

$$\begin{aligned}
&c \cdot S_2(T,x,y{:}K,M) \\
&=S_3(T,x,c \cdot y+q^*(x,c \cdot y{:}c \cdot K){:}c \cdot K) \cdot (1-I_{A_c(x)}(c \cdot y)) \\
&\quad +S_3(T,x,c \cdot y+N_2 \cdot c \cdot M{:}c \cdot K) \cdot I_{A_c(x)}(c \cdot y) \\
&=S_2(T,x,c \cdot y{:}c \cdot K,c \cdot M).
\end{aligned}$$

If $c=0$, it is easy to check that (a4.12) holds.

If we assume that stochastic processes r and V are Ito processes as mentioned in 1(3) and 1(5), we can derive (3.1) and (4.7). Then we can prove (6) by using (3.1) and (4.7) instead of the assumption (aa).

(7) For any $c\in(0,1)$, $c \cdot S_2(t,x,y{:}K,M)<S_2(t,x,y{:}K,c \cdot M)$.
[proof]
From the assumption of no arbitrage, it is sufficient to show that for $\forall c\in(0,1)$

$$c \cdot S_2(T,x,y{:}K,M)<S_2(T,x,y{:}K,c \cdot M). \quad \text{(a4.14)}$$

Fix $c\in(0,1)$ arbitrarily. From (1.3c) and (4.6),

$$\begin{aligned}
S_2(T,x,y{:}K,c \cdot M)=&S_3(T,x,y+q^*(x,y{:}K){:}K) \cdot (1-I_{A_1(x)}(y)) \\
&+S_3(T,x,y+N_2 \cdot c \cdot M{:}K) \cdot I_{A_1(x)}(y)
\end{aligned}, \quad \text{(a4.15)}$$

where $A_1(x):=\{u\in[0,\infty] \mid S_3(T,x,u+N_2 \cdot c \cdot M{:}K)>c \cdot M\}$, and

$$\begin{aligned}
S_2(T,x,y{:}K,M)=&S_3(T,x,y+q^*(x,y{:}K){:}K) \cdot (1-I_{A_2(x)}(y)) \\
&+S_3(T,x,y+N_2 \cdot M{:}K) \cdot I_{A_2(x)}(y))
\end{aligned}, \quad \text{(a4.16)}$$

where $A_2(x):=\{u\in[0,\infty] \mid S_3(T,x,y+N_2 \cdot M{:}K)>M\}$.

In the same manner as (5), since $c \cdot M<M$, we have $A_2(x)\subset A_1(x)$.
Fix $x\in[0,+\infty)$ arbitrarily.
① If $y\in A_2(x)$, from (a4.15) and (a4.16),

$$\begin{aligned}
&S_2(T,x,y{:}K,c \cdot M)=S_3(T,x,y+N_2 \cdot c \cdot M{:}K) \text{ and} \\
&S_2(T,x,y{:}K,M)=S_3(T,x,y+N_2 \cdot M{:}K).
\end{aligned}$$

From A1(6),

$$\begin{aligned}
&1\geq \frac{N_1 \cdot S_3(T,x,y+N_2 \cdot M{:}K)-N_1 \cdot S_3(T,x,y+N_2 \cdot c \cdot M{:}K)}{N_2 \cdot M-N_2 \cdot c \cdot M} \\
&1\geq \frac{N_1 \cdot S_3(T,x,y+N_2 \cdot M{:}K)-N_1 \cdot S_3(T,x,y+N_2 \cdot c \cdot M{:}K)}{N_1 \cdot M(1-c)} \\
&c \cdot M(1-c)\geq c \cdot S_3(T,x,y+N_2 \cdot M{:}K)-c \cdot S_3(T,x,y+N_2 \cdot c \cdot M{:}K) \\
&c \cdot S_3(T,x,y+N_2 \cdot c \cdot M{:}K)+c \cdot M(1-c)\geq c \cdot S_3(T,x,y+N_2 \cdot M{:}K).
\end{aligned}$$

Here, from $A_2(x) \subset A_1(x)$, $S_2(T,x,y:K,c \cdot M) > c \cdot M$. Applying this to the left hand side of the above inequality, we have

$$c \cdot S_3(T,x,y+N_2 \cdot c \cdot M:K) + S_3(T,x,y+N_2 \cdot c \cdot M:K)(1-c) > c \cdot S_3(T,x,y+N_2 \cdot M:K)$$
$$S_3(T,x,y+N_2 \cdot c \cdot M:K) > c \cdot S_3(T,x,y+N_2 \cdot M:K)$$

and hence $S_2(T,x,y:K,c \cdot M) > c \cdot S_2(T,x,y:K,M)$.

② If $y \in A_1(x) \cap A_2(x)^c$, From (a4.15) and (a4.16),

$$S_2(T,x,y:K,c \cdot M) = S_3(T,x,y+N_2 \cdot c \cdot M:K) \text{ and}$$
$$S_2(T,x,y:K,M) = S_3(T,x,y+q^*(x,y:K):K).$$

and using A3(2),

$$S_3(T,x,y+q^*(x,y:K):K) \leq S_3(T,x,y+N_2 \cdot M:K).$$

Then, similar to ①, $S_2(T,x,y:K,c \cdot M) > c \cdot S_2(T,x,y:K,M)$.

③ If $y \in A_1(x)^c$, from (a4.15) and (a4.16),

$$S_2(T,x,y:K,c \cdot M) = S_3(T,x,y+q^*(x,y:K):K) = S_2(T,x,y:K,M).$$

Apparently $S_2(T,x,y:K,c \cdot M) > c \cdot S_2(T,x,y:K,M)$.

Therefore we obtain that for $x \in [0,+\infty)$ and $y \in [0,+\infty)$,

$$S_2(T,x,y:K,c \cdot M) > c \cdot S_2(T,x,y:K,M).$$

(8) $S_2(t,x,y:K,M)$ is an increasing function of y.
[proof]

Fix $t \in [t_0,T]$, $x \in [0,+\infty)$ and $y \in [0,+\infty)$ arbitrarily. For any $c \in (0,1)$, from (4), (6), and (7),

$$\begin{aligned} S_2(t,x,c \cdot y:c \cdot K,c \cdot M) &= c \cdot S_2(t,x,y:K,M) \\ &< S_2(t,x,y:K,c \cdot M) \\ &\leq S_2(t,x,y:c \cdot K,c \cdot M) \end{aligned}$$

then $S_2(t,x,y:K,M)$ is an increasing function of y.

A5. (5.1) has an unique solution.

Fix $x \in [0,+\infty)$ and $y \in [0,+\infty)$ arbitrarily.
We show that the following equation for p;

$$\frac{1}{N_2} p = S_2(t_0,x,y-p) \qquad \text{(a5.1)}$$

has an unique solution and the solution belongs to $[0,y]$.

Set $u=y-p$ and change the variable in equation (a5.1)

$$\frac{1}{N_2}(-u+y)=S_2(t_0,x,u). \text{ (a5.2)}$$

If the solution of (a5.2) belongs to $[0,y]$, the solution of (a5.1) belongs to $[0,y]$.
Set

$$\begin{cases} z_1(u):=\dfrac{1}{N_2}(-u+y) \\ z_2(u):=S_2(t_0,x,u) \end{cases}$$

and it is sufficient to show that there exists an unique intersection point of z_1 and z_2, and that the u coordinates of the intersection point belongs to $[0,y]$.
From A4(3),

$$z_1(0)=\frac{1}{N_2}(0+y)\geq\frac{1}{N_1-N_2}0\geq S_2(t_0,x,0)=z_2(0),$$

and from A4(1),

$$z_1(y)=\frac{1}{N_2}(-y+y)=0\leq S_2(t_0,x,y)=z_2(y),$$

then the u coordinates of the intersection point belongs to $[0,y]$. Since $S_2(t,x,y)$ is an increasing function of y, it is easy to check that the intersection point of z_1 and z_2 is unique.

A6. Properties of p^*

Let $x\in[0,+\infty)$ and $y\in[0,+\infty)$.
Since $p^*(x,y)$ is a solution of following equation for p;

$$\frac{1}{N_2}p=S_2(t_0,x,y-p:K,M),$$

p^* is considered as a function of face value K and exercise price M. If necessary, we will use the notation $p^*(x,y:K,M)$ instead of $p^*(x,y)$.

(1) $p^*(x,y)$ is an increasing function of y.
[proof]

Suppose there exists $y_1,y_2\in[0,+\infty),y_1<y_2$ such that

$$p^*(x,y_1)\geq p^*(x,y_2).$$

From the definition of p^* and A4(8),

$$\begin{aligned}\frac{1}{N_2}p^*(x,y_2)&=S_2(t_0,x,y_2-p^*(x,y_2))\\&>S_2(t_0,x,y_1-p^*(x,y_2))\\&\geq S_2(t_0,x,y_1-p^*(x,y_1))=\frac{1}{N_2}p^*(x,y_1)\end{aligned}$$

but this contradicts the assumption.

(2) $p^*(x,y{:}K,M)$ is a decreasing function of K.
[proof]
Fix $x\in[0,+\infty)$ and $y\in[0,+\infty)$ arbitrarily. Let $0<K_1<K_2$, and set for $j=1,2$,

$$p^*_j=p^*(x,y{:}K_j,M).$$

Suppose $p^*_1<p^*_2$, then

$$\begin{aligned}\frac{1}{N_2}p^*_1&=S_2(t_0,x,y-p^*_1{:}K_1,M)\\&\geq S_2(t_0,x,y-p^*_2{:}K_1,M)\\&\geq S_2(t_0,x,y-p^*_2{:}K_2,M)=\frac{1}{N_2}p^*_2.\end{aligned}$$

But this contradicts the assumption. Therefore $p^*_1\geq p^*_2$.

(3) $p^*(x,y{:}K,M)$ is an increasing function of M.
[proof]
Fix $x\in[0,+\infty)$ and $y\in[0,+\infty)$ arbitrarily. Let $0<M_1<M_2$, and set for $j=1,2$,

$$p^*_j{:}=p^*(x,y{:}K,M_j).$$

Suppose $p^*_1>p^*_2$, then

$$\begin{aligned}\frac{1}{N_2}p^*_1&=S_2(t_0,x,y-p^*_1{:}K,M_1)\\&\leq S_2(t_0,x,y-p^*_2{:}K,M_1)\\&\leq S_2(t_0,x,y-p^*_2{:}K,M_2)=\frac{1}{N_2}p^*_2.\end{aligned}$$

But this contradicts the assumption. Therefore $p^*_1\leq p^*_2$.

(4) $p^*(x,y{:}K,M)$ is liner homogenous in y,K and M.
It is obvious from the definition of p^* and A4(6).

A7. Properties of S_1

From the definition, S_1 is considered as a function of face value K and exercise price M. If necessary, we will use the notation $S_1(t,x,y:K,M)$ instead of $S_1(t,x,y)$.

Let $t \in [0,t_0], x \in [0,+\infty)$, and $y \in [0,+\infty)$.

(1) $S_1(t,x,y:K,M)$ is a decreasing function of K.

[proof]

From the assumption of no arbitrage, it is sufficient to show that for $0<K_1<K_2$,

$$S_1(t_0,x,y:K_1,M) \geq S_1(t_0,x,y:K_2,M). \quad \text{(a7.1)}$$

Let $0<K_1<K_2$. From (1.3d), the definition of p^* and A6(2),

$$\begin{aligned} S_1(t_0,x,y:K_1,M) &= S_2(t_0,x,y-p^*(x,y:K_1,M):K_1,M) \\ &= \frac{1}{N_2} p^*(x,y:K_1,M) \\ &\geq \frac{1}{N_2} p^*(x,y:K_2,M) \\ &= S_2(t_0,x,y-p^*(x,y:K_2,M):K_2,M) \\ &= S_1(t_0,x,y:K_2,M). \end{aligned}$$

(2) $S_1(t,x,y:K,M)$ is an increasing function of M.

[proof]

From the assumption of no arbitrage, it is sufficient to show that for $0<M_1<M_2$,

$$S_1(t_0,x,y:K,M_1) \leq S_1(t_0,x,y:K,M_2). \quad \text{(a7.2)}$$

Let $0<M_1<M_2$. From (1.3d), the definition of p^* and A6(4),

$$\begin{aligned} S_1(t_0,x,y:K,M_1) &= S_2(t_0,x,y-p^*(x,y:K,M_1):K,M_1) \\ &= \frac{1}{N_2} p^*(x,y:K,M_1) \\ &\leq \frac{1}{N_2} p^*(x,y:K,M_2) \\ &= S_2(t_0,x,y-p^*(x,y:K,M_2):K,M_2) \\ &= S_1(t_0,x,y:K,M_2). \end{aligned}$$

(3) $S_1(t,x,y:K,M)$ is liner homogenous in y,K and M.

It is obvious from the assumption of no arbitrage, the assumption (aa), A4(6) and A6(4).

If we assume that stochastic processes r and V are Ito processes as mentioned in 1(3) and 1(5), we can derive (3.1) ,(4.7) and (5.3). Then we can prove (3) by using (3.1) ,(4.7) and (5.3) instead of the assumption (aa).

(4) For any $c \in (0,1)$, $c \cdot S_1(t,x,y{:}K,M) < S_1(t,x,y{:}K,c \cdot M)$
[proof]
From the assumption of no arbitrage, it is sufficient to show that for $c \in (0,1)$,

$$c \cdot S_1(t_0,x,y{:}K,M) < S_1(t_0,x,y{:}K,c \cdot M). \quad (a7.3)$$

Fix $c \in (0,1)$ arbitrarily. From A4(6),

$$\begin{aligned} c \cdot S_1(t_0,x,y{:}K,M) &= c \cdot S_2(t_0,x,y-p^*(x,y{:}K,M){:}K,M) \\ &= S_2(t_0,x,c \cdot y - c \cdot p^*(x,y{:}K,M){:}c \cdot K,c \cdot M) \end{aligned}$$

Here we use A4(8), A6(3), and A4(6),

$$\begin{aligned} &\leq S_2(t_0,x,c \cdot y - c \cdot p^*(x,y{:}K,c \cdot M){:}c \cdot K,c \cdot M) \\ &= c \cdot S_2(t_0,x,y-p^*(x,y{:}K,c \cdot M){:}K,M) \end{aligned}$$

and we use A4(7),

$$\begin{aligned} &< S_2(t_0,x,y-p^*(x,y{:}K,c \cdot M){:}K,c \cdot M) \\ &= S_1(t_0,x,y{:}K,c \cdot M) \end{aligned}$$

Therefor (a7.3) holds.

(5) $S_1(t,x,y{:}K,M)$ is an increasing function of y.
It follows by (1), (3) and (4).

A8. Properties of C_2 and C_1
As similar to S_2 and S_1, we can prove following two properties.

(1) Fix $t \in [t_0,T]$ and $x \in [0,+\infty)$ arbitrarily. For $y \in [0,+\infty)$, $C_2(t,x,y)$ is an increasing function of y.

(2) Fix $t \in [0,t_0]$ and $x \in [0,+\infty)$ arbitrarily. For $y \in [0,+\infty)$, $C_1(t,x,y)$ is an increasing function of y.

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