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A NOTE ON LOOK-BACK OPTIONS BASED ON ORDER STATISTICS

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Abstract

Average options and minimum (or maximum) options are well-known look-back options. In this paper we define new look-back options which use the order statistics of the stock prices for the exercise prices and/or for the underlying variables of the options.

The probability distributions of the order statistics which are required for pricing are obtained here. Though we have not yet reached the pricing formula of these options, we are close to it. What is left to be done for pricing will be pointed out.

I. *Introduction*

Average options and minimum (or maximum) options are the well-known look-back options. They are also called path-dependent options (or contingent claims) while the ordinary options are called path-independent since they depends only on the price of the underlying asset at the time of maturity. In defining a look-back option, any statistic which is a function of the prices or a functional of the price path of the underlying asset can be used. Arithmetic average, geometric average, maximum and minimum are the simple statistics, and they are already used.

For the European average options, there are several studies in the literature including Bergman [1], Kemna and Vorst [6], Kunitomo and Takahashi [7] and Tumbull and Wakeman [9]. They determined the probability distribution of the geometric average of the prices when the underlying asset price follows the log-normal distributilon, and the closed form for the option prices were obtained. However, the closed form pricing formula for the arithmetic average options do not seem to be derived yet except for a special case in Bergman [1]. The approximated pricing formula and the algorithms for them are quite well studied. The diffiiculty seems to be in deriving the exact distribution function of the average price.

For the European minimum (or maximum) options, Goldman et al. [3] defined and derived the closed form pricing formula. The exact distribution of the maximum and the minimum of the prices-path had been available among the established results in the field of mathematics (Probability Theory).

In this paper the order statistics of the prices of the underlying asset are used for the state variables: the i-th order statistic is the i-th smallest of the n observed prices for the discrete time representation as seen in Section 2, and α-percentile point for a continuous price path for $0<\alpha<1$ is the level of the price to which the price path stays below, for the $100\times\alpha$ percent of the time during the option's contract period. The exact distribution of the α-percentile point or equivalently the asymptotic ($n\rightarrow\infty$) distribution of the i-th order statistics are obtained here. However, the proofs are informal. Using these distributions, the pricing of options in a risk-neutral economic world is possible. It is indicated in Section 4 but the calculation is not performed there. Before using the Preference-free pricing method, we wanted to check the hedgability of these options. We have checked the necessary requirement for the hedgability except one point. We have not seen yet the independence of the conditional distribution of the whole-period α-percentile on the present (conditioned) value of the α-percentile. This may be obtained by some more effort.

Now we outline the present paper. In Section II, the notations and the definitions are given. In Section III, the unconditional and the conditional distribution of the i-th order statistic of the prices of the underlying asset are obtained in the asymptotic manner. Their joint distributions with the price of the underlying asset at the maturity time are also sketched. This is the main body of the paper. In Section IV, the hedgability of the type of options treated in this paper is examined.

We may call our options the α-percentile options. They can be regarded as an extension or soothed version of maximum or minimum options since α-percentile are in the middle rather than at the extremes. α-percentiles may find a certain needs in the investment society since they may be more stable than the arithmetic average.

II. *Model and Definitions*

The underlying asset could be anything which satisfy the following price behavior. For simplicity, let it be a stock. We begin with discrete times and stock prices at these times rather than with continuous time setting. We will later on take the time intervals approach to zero in order to approximate the probability distribution of the order statistics.

Let the time period for options be $[0, T]$ and the stock be traded at times $t_0, t_1, \ldots,$ t_n where.

$$o=t_0<t_1<t_2<\ldots<t_n=T$$

$$t_i-t_{i-1}=T/n.$$

Let S_t denote the stock price at time t. For simplicity, we write S_i instead of S_{t_i}, $i=0, 1, 2, \ldots, n$.

The Model.

Assume that the behavior of the stock price follow the multiplicative stochastic process: for $i=1, 2, \ldots, n-1$.

$$\begin{aligned} S_{i+1}&=S_i\cdot\exp\{X_{i+1}\}\\ &=S_0\cdot\exp\{X_1+X_2+\ldots+X_{i+1}\}\end{aligned}$$

where $X_1, \ldots, X_n$ are independent random variables and are identically distributed with a continuous distribution function F. F is not necessarily normal. We assume that for $i=1, 2, \ldots, n-1$.

$$E[X_i]=(T/n) \cdot \mu \stackrel{\rightarrow}{=} \mu_n$$

$$\mathrm{Var}[X_i]=(T/n) \cdot \sigma^2 \stackrel{\rightarrow}{=} \sigma_n{}^2$$

where μ and σ^2 are constants.

Order Statistics.

We denote. by $S_{(i)}$, the i-th smallest value in the set $\{S_1, S_2, \ldots, S_n\}$;

$$S_{(1)} < S_{(2)} < \ldots < S_{(n)}.$$

We exclude S_0 from the set for simplicity. $S_{(i)}$ is called the i-th order statistic of $\{S_1, \ldots, S_n\}$

For $i=1, 2, \ldots, n$, let

$$Y_i = X_1 + X_2 + \ldots + X_i$$

and $Y_{(i)}$ be the i-th order statistic of $\{Y_1, Y_2, \ldots, Y_n\}$. Note that for i=i, 2, . . . , n, we have

$$S_{(i)} \equiv \exp\{Y_{(i)}\}$$

since $S_i = S_0 \cdot \exp\{Y_i\}$ and the transformation $y \rightarrow \exp\{y\}$ does not change the order of magnitude: that is $S_i < S_j$ if and only if $Y_i < Y_j$.

Option Contract.

(a) For any i, $S_{(i)}$ can be used as an underlying state variable. A new deposit may be an European option whose pay-off at the maturity date is $S_{(i)}$. The price of this option at time t_0 is the amount deposited at the beginning of the period and $S_{(i)}$ is the amount of return at the end of the period. A call (put) option can be defined in an ordinary manner by setting its pay-off as $\max\{S_{(i)}-K,0\}$ $(\max\{K-S_{(i)},0\})$.

(b) For any i, $S_{(i)}$ can be used as an uncertain exercise price of ordinary call or put options. The pay-off of the call and put options will be $\max\{S_T-S_{(i)},0\}$, and $\max\{S_{(i)}-S_T,0\}$ respectively.

III. *Asymptotic Distributions of Order Statistics*

Unconditional Distribution of Order Statistics.

We define a step function $G_n(\cdot)$ for $\{Y_1, \ldots, Y_n\}$ as follows. For an arbitrary real number y, define

$$G_n(y)=(1/n)\cdot\sum_{i=1}^{n} I\{Y_i<y\},$$

where $I\{\cdot\}$ is the indicator function such that

$$I\{Y\leqq y\}=\begin{cases}1 & \text{if } Y\leqq y\\ 0 & \text{otherwise.}\end{cases}$$

Note that $G_n(\cdot)$ is different from the ordinary empirical distribution function since the Y_i's are not independent at all.

We first give an asymptotic distribution of $G_n(\cdot)$.

Lemma 1.

For an arbitrary, fixed real number y, the distribution of $G_n(y)$ converges to the distribution of

$$\int_0^1 I\{W(t)\leqq(y-t\cdot\mu\cdot T)/(\sigma\cdot T^{1/2})\}\,dt$$

as $n\to\infty$, where $W(t)$ is a standard Wiener process,

Proof.

When i/n converges to $t\in(0,1)$ as $n\to\infty$, it is known that the sum of the standardized X_j's converges in distribution to a standard Wiener process $W(t)$. Therefore we have, as $n\to\infty$

$$\begin{aligned}G_n(y)&=\frac{1}{n}\cdot\sum_{i=1}^{n} I\{Y_i\leqq y\}\\&=\frac{1}{n}\cdot\sum_{i=1}^{n} I\left\{\frac{Y_i-i\cdot\mu_n}{\sigma_n}\leqq\frac{y-i\cdot\mu_n}{\sigma_n}\right\}\\&=\frac{1}{n}\cdot\sum_{i=1}^{n} I\left\{\frac{1}{n^{1/2}}\cdot\sum_{j=1}^{i}\left(\frac{X_j-\mu_n}{\sigma_n}\right)\leqq\frac{y-(i/n)\cdot\mu\cdot T}{\sigma\cdot T^{1/2}}\right\}\\&\xrightarrow[\text{(in distribution)}]{}\int_0^1 I\left\{W(t)\leqq\frac{y-t\cdot\mu\cdot T}{\sigma\cdot T^{1/2}}\right\}dt\end{aligned}$$

(We denote this limit by $G(y:\mu,\sigma)$). ■

We owe, for the validity of this convergence, to the argument shown in Shorack-Wellner [8] pp. 59–62.

Now we show the asymptotic distribution of order statistics $Y_{(i)}$ and $S_{(i)}$.

Theorem 1.

Let i/n converge to $t\in(0,1)$ as $n\to\infty$. For an arbitrary, fixed real number y, we have, as $n\to\infty$,

$$P\{Y_{(i)}\leqq y\}\longrightarrow P\{t\leqq G(y:\mu,\sigma)\}$$

and for an arbitrary, fixed positive real number x, we also have, as $n \to \infty$

$$P\{S_{(t)} \leqq x\} \longrightarrow P\{t \leqq G(\log(x/S_0): \mu, \sigma).$$

Proof.

Let $G_n^{-1}(\cdot)$ be defined by

$$G_n^{-1}(t) = \inf\{y: G_n(y) \geqq t\}.$$

Then, we have, as $n \to \infty$, by lemma 1

$$\begin{aligned} P\{Y_{(t)} \leqq y\} &= P\left\{G_n^{-1}\left(\frac{i}{n+1}\right) \leqq y\right\} \\ &= P\{i/n \leqq G_n(y)\} \longrightarrow P\{t \leqq G(y: \mu, \sigma)\}. \end{aligned}$$

Similarly, we have as $n \to \infty$

$$\begin{aligned} P\{S_{(t)} \leqq x\} &= P\{S_0 \cdot \exp\{Y_{(t)}\} \leqq x\} \\ &= P\{Y_{(t)} \leqq \log(x/S_0)\} \\ &= P\{i/n \leqq G_n(\log(x/S_0))\} \longrightarrow P\{t \leqq G(\log(x/S_0): \mu, \sigma)\}. \quad \blacksquare \end{aligned}$$

The explicit form of the distribution function of $G(y: \mu, \sigma)$ is given in the following theorems. As for the special case where $\mu=0$ and $\sigma=1$, we introduce a new notation.

Define for any real number y,

$$\begin{aligned} G^*(y) &\equiv G(y \cdot T^{1/2}: 0, 1) \\ &= \int_0^1 I\{W(t) \leqq y\}\, dt. \end{aligned}$$

Theorem 2.

For $-\infty < y < \infty$ and $0 < x < 1$, we have

$$\begin{aligned} P\{G^*(y) \leqq x\} &= \int_0^x P\left\{\int_0^{1-s} I\{W(t) \leq 0\}\, dt \leqq x - s\right\} h_y^*(s) ds \\ &= \int_0^x \frac{2}{\pi} \cdot \sin^{-1}\left(\left(\frac{x-t}{1-t}\right)^{1/2}\right) \cdot h_y^*(t) dt \end{aligned}$$

where

$$h_y^*(t) = \frac{y}{(2\pi t^3)^{1/2}} \cdot e^{-\frac{y^2}{2t}}, \qquad \text{for} \quad 0 < t < \infty.$$

Proof.

By definition, we see that $0 \leqq G^*(y) \leqq 1$.

We prove the statement for $y \geqq 0$. The proof for $y \leqq 0$ can be obtained by using the following relation.

$$P\{G^*(y)\leqq x\}=P\left\{\int_0^1 I\{-W(t)\geqq -y\}\,dt\leqq x\right\}$$

$$=P\left\{\int_0^1 I\{W(t)\geqq -y\}\,dt\leqq x\right\}$$

$$=P\left\{\int_0^1 [1-I\{W(t)\leqq -y\}]dt\leqq x\right\}$$

$$=P\{G^*(y)\geqq 1-x\}$$

$$=1-P\{G^*(-y)\leqq 1-x\}.$$

Let $\tau_y{}^*$, for $y<0$, be the first time for the process $W(t)$ to hit y, i.e.

$$\tau_y{}^*=\inf\{t:\ W(t)\geqq y\}.$$

The density function $h_y{}^*(t)$ of the distribution of $\tau_y{}^*$ is given by

$$h_y{}^*(t)=\frac{y}{(2\pi t^3)^{1/2}}e\cdot^{-\frac{y^2}{2t}},\qquad \text{for}\quad 0<t<\infty\,.$$

(See for example, Shorack and Weller [8], p. 33)
Then

$$P\{G^*(y)\leqq x\}$$

$$=P\left\{\int_0^1 I\{W(t)\leqq y\}\,dt\leqq x\right\}$$

$$=P\left\{\int_0^1 I\{W(t)\leqq y\}\,dt\leqq x\,,\quad\text{and}\quad \tau_y{}^*\leqq x\right\}$$

(because the inclusive relation of the events holds:

$$\left\{\int_0^1 I\{W(t)\leqq y\}\,dt\leqq x\right\}\subset\{\tau_y{}^*\leqq x\}.\Big)$$

$$=\int_0^x P\left\{\int_0^1 I\{W(t)\leqq y\}\,dt\leqq x|\tau_y{}^*=s\right\}\cdot h_y{}^*(s)ds$$

$$=\int_0^x P\left\{\int_0^{1-s} I\{W(t)\leqq 0\}\,dt+s\leqq x\right\}\cdot h_y{}^*(s)ds$$

(because the process $W(t)$ reaches y for the first time at time s, and then the behavior of $W(t)$ afterward is the same as that of $W(\cdot)$ restarting from that point, and we check if it is below zero.)

$$=\int_0^x \frac{2}{\pi}\sin^{-1}\left(\left(\frac{x-s}{1-s}\right)^{1/2}\right)\cdot h_y{}^*(s)ds\,.\quad \blacksquare$$

(This is the Arc Sine Law. See Billingsley or Shorack & Wellner [8] p. 34, for example.)
Let $\tilde{W}(t)=\mu\cdot T\cdot t+\sigma\sqrt{T}\cdot W(t)$, for $0\leq t<\infty$, and define, for $y>0$.

$$\tau_y = \inf\{t: \tilde{W}(t) \geqq y\}.$$

The density $h_y(t)$ of the distribution of τ_y is given by

$$h_y(t) = \frac{y}{\sigma(2\pi t^3)^{1/2}} \exp\left[-\frac{(y-\mu t)^2}{2\sigma^2 t}\right], \quad \text{for} \quad t>0.$$

(See, for example. Karlin and Taylor [5] p. 363.)

Theorem 2′.

For $-\infty<y<\infty$ and $0<x<1$, we have

$$P\{G(y: \mu, \sigma) \leqq x\}$$
$$= \int_0^x P\left\{\int_0^{1-s} I\{\tilde{W}(t) \leqq 0\}\, dt \leqq x-s\right\} h_y(s) ds$$

Proof.

The proof is exactly the same as that for Theorem 2 when $\tilde{W}(\cdot)$ is replaced by $W(\cdot)$.

$$P\{G(y: \mu, \sigma) \leqq x\}$$
$$= P\left\{\int_0^1 I\{\tilde{W}(t) \leqq y\}\, dt\,, \quad \text{and} \quad \tau_y \leqq x\right\}$$
$$= \int_0^x P\left\{\int_0^1 I\{\tilde{W}(t) \leqq y\}\, dt \leqq x \mid \tau_y = s\right\} h_y(s) ds$$
$$= \int_0^x P\left\{\int_0^{1-s} I\{\tilde{W}(t) \leqq 0\}\, dt \leqq x-s\right\} h_y(s) ds. \quad \blacksquare$$

Various forms of pay-off's can be defined, based on the orderst atistics. In evaluating options with these pay-offs, we need to know the joint distribution of the random variables used in the pay-off functions.

Here we deal with the joint distribution of S_n and $S_{(i)}$ which is a simplest case of this kind and was described in the Section 2.

Let a and b be arbitrary, fixed positive real numbers. Then

$$P\{S_{(i)} \leqq a, \quad S_n \leqq b\}$$
$$= P\{Y_{(i)} \leqq \log (a/S_0), \quad Y_n \leqq \log (b/S_0)\}$$
$$= \int_{-\infty}^{\log(b/s_0)} P\{Y_{(i)} \leqq \log (a/S_0) \mid Y_n = y\}\, f_{Y_n}(y) dy$$

where $f_{Y_n}(\cdot)$ is the density function of the distribution of the random variable Y_n which is asymptotically normal.

The probability in the integrand can be obtained in the following way. Define a conditional empirical distribution function;

$$G_n(x \mid Y_n = y) = \frac{1}{n} \sum_{i=1}^{n} I[Y_i \leqq x \mid Y_n = y]$$

Then this converges, as $n\to\infty$, in distribution to

$$\int_0^1 I\left\{W(t)-tW(1)\leqq \frac{x-t\cdot y}{\sigma T^{1/2}}\right\}dt\ .$$

(Denote this by $G(x|y)$.)
To see this, we just rewrite the definition;

$$\begin{aligned}
&G_n(x|Y_n=y)\\
&=\frac{1}{n}\sum_{i=1}^{n} I\left\{Y_i-\frac{i}{n}(Y_n-y)\leqq x|y\right\}\\
&=\frac{1}{n}\sum_{i=1}^{n} I\left\{\frac{Y_i-i\cdot\mu_n}{\sigma_n}-\frac{1}{\sigma_n}\left(\frac{i}{n}Y_n-\frac{i}{n}\cdot n\cdot\mu_n\right)\leqq\frac{x-i\cdot\mu_n}{\sigma_n}\,\middle|\, Y_n=y\right\}\\
&=\frac{1}{n}\sum_{i=1}^{n} I\left\{\frac{1}{\sqrt{n}}\cdot\sum_{j=1}^{i}\left(\frac{X_j-\mu_n}{\sigma_n}\right)-\frac{i}{n}\cdot\frac{1}{\sqrt{n}}\cdot\sum_{j=1}^{n}\left(\frac{X_j-\mu_n}{\sigma_n}\right)\right.\\
&\left.\leqq\frac{1}{\sqrt{n}}\cdot\frac{x-\frac{i}{n}y}{\sigma_n}\,\middle|\, Y_n=y\right\}\\
&\longrightarrow\int_0^1 I\left\{W(t)-t\cdot W(1)\leqq\frac{x-t\cdot y}{\sigma_n T^{1/2}}\right\}dt, \qquad \text{in distribution}\ .
\end{aligned}$$

Then

$$\begin{aligned}
&P\{Y_{(i)}\leqq\log(a/S_0)\,|\,Y_n=y\}\\
&=P\left\{\frac{i}{n+1}\leqq G_n(\log(a/S_0)\,|\,Y_n=y\right\}\\
&\longrightarrow P\{\alpha\leqq G(\log(a/S_0)\,|\,y)\}
\end{aligned}$$

where $\frac{i}{n+1}\to\alpha\in(0,1)$, as $n\to\infty$.

Thus, we have

$$P(S_{(i)}\leqq a,\quad S_n\leqq b)\longrightarrow\int_{-\infty}^{\log(b/S_0)} P\{\alpha\leqq G(\log(a/S_0)\,|\,y)\}\cdot f_{Y_\infty}(y)dy\ .$$

Conditional Distribution of Order Statistics.

After the look-back option is issued, the stock prices will incur one after another as time passes by. On the middle way to the maturity of the option's period, the conditional distribution of the order statistic $S_{(i)}$ at the time of the maturity, given the stock prices $S_1=s_1,\dots,S_k=s_k$, is required for evaluation of the value of the option at the time $t=t_k$. We will briefly derive this conditional distribution here.

Let $S_1=s_1,\ .\ .\ .\ ,\ S_k=s_k$ be the realized stock prices by the time $t=t_k$. Then we denote the realized Y's by $Y_1=y_1, \ldots, Y_k=y_k$. Define the conditional empirical distribution in a similar way as follows.

$$G_n(y \mid y_1, \ldots, y_k)=\frac{1}{n+1}\left[\sum_{i=1}^{k} I\{y_i \leqq y\}+\sum_{i=k+1}^{n} I\{Y_i \leqq y\}\right]$$

Note that given $Y_1=y_1, \ldots Y_k=y_k$, we have $Y_i=y_k+X_{k+1}+\ldots+X_i$, for $i>k$.
Let's write

$$G_{n,k}(y)=\frac{1}{k+1}\sum_{i=1}^{k} I\{y_i \leqq y\}$$

and

$$G_{n,k}(y)=\frac{1}{n-k+1}\sum_{i=k+1}^{n} I\{Y_i \leqq y\}=\frac{1}{n-k+1}\sum_{j=1}^{n-k} I\{\tilde{Y}_j \leqq y-y_k\}$$

where $\tilde{Y}_j=Y_i-y_k$, and $j=i-k$, for $i=k+1, \ldots, n$.
Then

$$G_n(y: y_1, \ldots, y_k)=\beta_n \cdot g_{n,k}(y)+(1-\beta_n)\cdot G_{n,k}(y)$$

where

$$\beta_n=\frac{k}{n+1}.$$

Assume that

$$\frac{i}{n+1} \longrightarrow \alpha \in (0, 1)$$

$$\frac{k}{n+1} \longrightarrow \beta \in (0, 1)$$

$$g_{n,k}(y) \longrightarrow g_\beta(y) \qquad \text{for each fixed } y,$$

as $n \rightarrow \infty$.

Then, $G_{n,k}(y)$ converges in distribution to $G_\beta(y-y_\beta)$ where y_β is the limit of y_k, as $n \rightarrow \infty$, which has the same distribution as $G(y-y_\beta: \mu, \sigma)$. Now we will briefly sketch the conditional distribution of $Y_{(i)}$ and $S_{(i)}$.

Theorem 3.

Under the assumptions mentioned in the above, we have, as $n \rightarrow \infty$

$$P\{Y_{(i)} \leqq y \mid Y_1=y_1, \ldots, Y_k=y_k\} \longrightarrow P\{\alpha-\beta \cdot g_\beta(y) \leqq (1-\beta)\cdot G_\beta(y-y_\beta)\}$$

and

$$P\{S_{(i)} \leqq x \mid S_1 = s_1, \ldots, S_k = s_k\}$$
$$\longrightarrow P\left\{\alpha - \beta \cdot g_\beta\left(\log\left(\frac{x}{S_0}\right)\right) \leqq (1-\beta) \cdot G_\beta\left(\log\left(\frac{x}{S_0}\right) - y_\beta\right)\right\}$$

Proof.

Define as before, for any $t \in (0, 1)$

$$G_n^{-1}(t \mid y_1, \ldots, y_k) = \inf\{y : G_n(y \mid y_1, \ldots, y_k) \geqq t\}$$

Then for any fixed real number y, as $n \to \infty$

$$\begin{aligned} &P\{Y_{(i)} \leqq y \mid y_1 = y_1, \ldots, Y_k = y_k\} \\ &= P\left\{G_n^{-1}\left(\frac{i}{n+1} \,\middle|\, y_1, \ldots, y_k\right) \leqq y\right\} \\ &= P\left\{\frac{i}{n+1} \leqq G_n(y \mid y_2, \ldots, y_k)\right\} \\ &= P\left\{\frac{i}{n+1} - \beta_n \cdot g_{n,k}(y) \leqq (1-\beta_n) \cdot G_{n,k}(y)\right\} \\ &\quad \longrightarrow P\{\alpha - \beta \cdot g_\beta(y) \leqq (1-\beta) \cdot G_\beta(y - y_\beta)\}. \end{aligned}$$

The second statement is obvious since

$$P\{S_{(i)} \leqq x \mid S_1 = s_1, \ldots, S_k = s_k\} = P\left\{Y_{(i)} \leqq \log\left(\frac{x}{S_0}\right) \,\middle|\, Y_1 = y_1, \ldots, Y_k = y_k\right\}. \quad \blacksquare$$

The conditional (given $Y_1 = y_1, \ldots, Y_k = y_k$) joint distribution of $S_{(i)}$ and S_n can also be obtained in a similar way.

Define the conditional empirical distribution,

$$\begin{aligned} &G_n(x \mid Y_1 = y_1, \ldots, Y_k = y_k, Y_n = y) \\ &= \frac{1}{n}\left[\sum_{j=1}^{k} I\{y_i \leqq x \mid Y_n = y\} + \sum_{j=k+1}^{n} I\{Y_i \leqq x \mid Y_n = y\}\right]. \end{aligned}$$

Then, this converges as $n \to \infty$, in distribution, to

$$\beta \cdot g_\beta(x) + (1-\beta) \cdot G_\beta(x - y_\beta \mid y)$$

since it can be rewritten as follows:

$$\begin{aligned} &G_n(x \mid Y_1 = y_1, \ldots, Y_k = y_k, Y_n = y) \\ &= \beta_n \cdot g_{n,k}(x) + (1-\beta_n) \cdot \frac{1}{n-k+1} \sum_{j=k+1}^{n} I\left\{Y_j - \frac{j}{n}(Y_n - y) \leqq x \mid Y_n = y\right\}. \end{aligned}$$

Note that $G_\beta(x - y_\beta \mid y)$ has the same distribution as $G(x - y_\beta \mid y)$ does.

Now the conditional distribution of $S_{(i)}$, and S_n can be given as follows:

$$
\begin{aligned}
&P\{S_{(i)}\leqq a,\ S_n\leqq b\,|\,Y_1=y_1,\ \dots,\ Y_k=y_k\}\\
&\quad=P\{Y_{(i)}\leqq\log(a/S_0),\ Y_n\leqq\log(b/S_0)\,|\,Y_1=y_1,\ \dots,\ Y_k=y_k\}\\
&\quad=\int_{-\infty}^{\log(b/S_0)}P\{Y_{(i)}\leqq\log(a/S_0)\,|\,Y_n=y,\ Y_1=y_1,\ \dots,\ Y_k=y_k\}f_{Y_n}(y)dy\\
&\quad=\int_{-\infty}^{\log(b/S^0)}P\left\{\frac{i}{n+1}\leqq G_n(\log(a/S_0)\,|\,Y_n=y,\ Y_1=y_1,\ \dots,\ Y_k=y_k)\right\}f_{Y_n}(y)dy\\
&\quad\longrightarrow\int_{-\infty}^{\log(b/S_0)}P\left\{\alpha\leqq\beta\cdot g_\beta(\log(a/S_0))+(1-\beta)\cdot G_\beta(\log(a/S_0)-y_\beta\,|\,y)\right\}f_{Y_\infty}(y)dy
\end{aligned}
$$

IV. *Hedgability and Option Pricing*

In this section we examine the hedgability of the options based on the order statistics. Once the hedgability is shown for these options, the pricing can be done in the risk-neutral economic world to give the correct price for the ordinary economic world. Though we have not completely shown their hedgability in this section, we will point out what is left to study.

Let, for $0<\alpha<1$, $F_\alpha(x,\beta)$ be the approximated conditional distribution function of $S_{(i)}$ derived in Theorem 3, i.e.

$$F_\alpha(x,\beta)=P\left\{\alpha-\beta\cdot g_\beta\left(\log\left(\frac{x}{S_0}\right)\right)\leqq(1-\beta)\cdot G_\beta\left(\log\left(\frac{x}{S_0}\right)-y_\beta\right)\right\}$$

Let $f_\alpha(x,\beta)$ be its density function.

In the risk-neutral economic world, the option price is given by taking the expectation of its pay-off at the maturity and multiplying the time discount factor. For example, for the option with the pay-off $S_{(i)}$, the approximated price at time t_k is

$$
\begin{aligned}
&E[S_{(i)}\,|\,S_1,\ \dots,\ S_k,\ \text{given}]\cdot e^{-r(T-t_k)}\\
&\quad=\int_0^\infty x\cdot f_\alpha(x,\beta)dx\cdot e^{-r(T-t_k)}
\end{aligned}
$$

where $\alpha=\dfrac{i}{n+1}$, $\beta=\dfrac{k}{n+1}$, and n is the total number of observed stock prices which are used for determining the order statistics.

To examine the hedgability, we introduce a continuous time version of the order statistics. Let $m_\alpha(t)$, for $t>0$, and $0<\alpha<1$, be the number such that

$$\alpha=\frac{1}{t}\int_0^t I\{u\colon S_u\leqq m_\alpha(t)\}\,du\,,$$

Call this the α-percentile for the continuous path S_u, $0<u<t$.
Similarly, let $m_\alpha(T_0,T)$, for $0\leqq T_0<T$ and $0<\alpha<1$, be the number such that,

$$\alpha=\frac{1}{T}\left[\int_0^{T_0} I\{u: S_u \leqq m_\alpha(T_0, T)\}\,du+\int_{T_0}^{T} I\{u: S_u \leqq m_\alpha(T_0, T)\}\,du\right]$$

provided that the path S_u up to the time T_0 is given. Call $m_\alpha(T_0,T)$ the conditional α-percentile given S_u, up to the time T_0. Note that $m_\alpha(t)\equiv m_\alpha(0,t)$.

What we obtained in the section III were the probability distribution of $m_\alpha(t)$ and $m_\alpha(T_0,T)$, i.e.

$$P\{m_\alpha(T)\leqq x\} \equiv F_\alpha(x,0)$$

$$P\{m_\alpha(T_0,T)\leqq x\} \equiv F_\alpha(x,\beta)$$

where $\alpha\approx\dfrac{i}{n+1}$, $\beta\approx\dfrac{k}{n+1}$ and $t_k\approx T_0$.

We follow the argument in Ingersall [4], pp. 376–379, for examination of the hedgability.

The price of options based on the α-percentile of the continuous path is a function of the state variables; s_t, $m_\alpha^{-}(t,T)$ and t.

To express the evolution of m_α, we introduce some notations.

Let $l(m,t)=\int_0^t I\{u:\ S_u\leqq m\}\,du$. Then $l(m,t)$ is an increasing function of m. Let $\left.\dfrac{dl(m,t)}{d_m}\right|_{m=m_\alpha}$ denote the ratio of the amount of instantaneous increment of $l(m,t)$ to dm at the level $m=m_\alpha$.

Then, the evaluations of S_t and m_α are

$$dS_t=\left(\mu+\frac{1}{2}\sigma^2\right)S_t\cdot dt+\sigma\cdot S_t\cdot dW_t\ ,$$

$$dm_\alpha=\frac{\alpha-1}{\left.\dfrac{dl(m,t)}{dm}\right|_{m=m_\alpha}}\cdot \mathrm{dt},\qquad \text{for } S_t<m_\alpha(t)\,,$$

$$\frac{\alpha}{\left.\dfrac{dl(m,t)}{dm}\right|_{m=m_\alpha}}\cdot dt,\qquad \text{for } S_t>m_\alpha(t)\,,$$

$$\frac{\alpha}{\dfrac{ds}{dt}\cdot\left.\dfrac{dl(m,t)}{dm}\right|_{m=m_\alpha^+}+1}\cdot ds,\qquad \text{for } S_t=m_\alpha(t) \text{ and } dS_t>0$$

$$\frac{\alpha-1}{\dfrac{ds}{dt}\cdot\left.\dfrac{dl(m,t)}{dm}\right|_{m=m_\alpha^-}-1}\cdot ds,\qquad \text{for } S_t=m_\alpha(t) \text{ and } dS_t<0\,.$$

Therefore the hedge is possible only when $S_t \neq m_\alpha(t)$. If we have $\frac{\partial c}{\partial m_\alpha}=0$ when $S_t=m_\alpha(t)$, hedging is possible since the option's price dynamics are,

$$dc=\frac{\partial c}{\partial t}\cdot dt+\frac{\partial c}{\partial S_t}\cdot dS_t+\frac{1}{2}\frac{\partial^2 c}{\partial S_t^2}\cdot(dS_t)^2+\frac{\partial c}{\partial m_\alpha}\cdot dm_\alpha+\frac{1}{2}\frac{\partial^2 c}{\partial m_\alpha^2}\cdot(dm_\alpha)^2$$
$$+\frac{\partial^2 c}{\partial m_\alpha\cdot\partial S_t}\cdot(dm_\alpha)(dS_t),$$

letting c denote the option price,

As the original proof by Goldman et al. [3] goes, the independence of the probability distribution of $m_\alpha(t,T)$ on the present value of $m_\alpha(t)$, if proved, will imply $\frac{\partial c}{\partial m_\alpha}=0$. However I have not been able to prove this yet. The examination of the conditional distribution of $m_\alpha(t, T)$ may or may not imply this property when the exact functional form of the distribution is obtained. Note that $m_\alpha(t)=S_t\equiv S_0\cdot\exp\{y_\beta\}$.

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