GLOBAL OPTIMIZATION
IN A
DECENTRALIZED DECISION MAKING PROCESS

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I. *Introduction*

Issues surrounding centralization v.s. decentralization of the decision-making process in an organization have been examined by many over the years. One of the basic characteristics of decentralization is that while a proper decentralization scheme could bring about various tangible and intangible benefits to organizations, it could also introduce a risk of suboptimization.

This paper examines the issue of global optimization in a simple decentralized environment. More specifically, following Itami and Kaplan [3] we consider a reasonably realistic linear programming (*LP*) model of a decentralized process in which sales and production decisions are made virtually separately. We shall show that a global optimum can be achieved, at least in principle, without unduly compromising autonomy of each department's decision-making process, if suffcient information is exchanged and if proper coordination mechanism implemented between the two departments.

It turns out that the key mechanism ensuring the successful global optimization is a certain "conservatism" in choosing a right set of marginal costs of goods used by the sales department. This result seems to have an implication to the following open question in a more general context: When a "marginal cost" or "shadow price" interpretation of a dual optimal solution is used in a LP model, and when there are more than one dual optimal solution, which solution should be used to base one's decision on? Our analysis indicates that choosing the most "conservative" one ensures a desirable result in a global system, where the global system consists of sections with conflicting interests.

To explain our basic formulation consider a hypothetical company with sales and production departments. A model for the global optimization (which we denote by G) of this company is that of finding a vector x of sales amounts and a vector z of production levels which maximize the sales revenue minus the production cost under the following three sets of requirements: (a) the sales amounts satisfy sales restrictions, (b) enough goods be produced for the sale, and (c) the production satisfy capacity/resource limitations.

A sales/production model, which may be more true to the reality than the model G above is the following "two-stage" process considered in [3]. In this model, the global problem G is decomposed into two problems, one corresponding to sales and the other

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to production. First, the sales department receives a set of goods (from the accounting department) and solves an optimization problem in which the net profit (based on the given cost figures) is maximized under a set of sales restriction ((a) above). We let S or $S(v)$ denote the sales optimization problem, where v is the set of costs used; also let x solve $S(v)$. The sales department then informs the production department that such-and-such amounts of goods (i.e., x) are needed. The problem of the production department (which is denoted by P or $P(x)$) is then that of minimizing the production cost under the requirements that the production quota x be fulfilled ((b) above), and that a set of own capacity/resource limitations ((c) above) be satisfied. This decentralization model nicely reflects a common notion of a sales department being a profit center while a production department a cost center, as noted in [3].

This two-stage procedure may or may not yield a global optimal solution. The obvious reason for which the two-stage decentralization process described above may not work is that the information flow is one-way. A coordination effort in the opposite direction could be achieved indirectly through cost adjustments in the literative process described below. This cycle of an information flow is implicit in the discussion given in Itami and Kaplan [p. 837,3], but we shall make it more explicit:

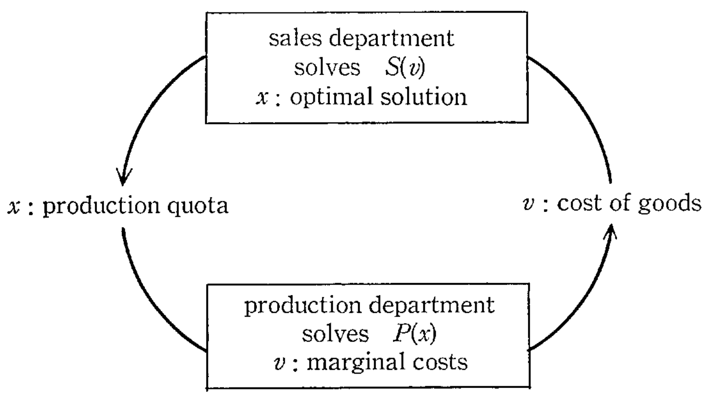
(0) Initially a set of costs, v, is given by the accounting department to the sales department.

(i) The sales department solves $S(v)$, and gives the computed optimal solution, x, to the production department as a production quota.

(ii) The production department solves $P(x)$, and gives the marginal costs, v, associated with the production quota constraints to the accounting department. The accounting department gives this v to the sales department as an updated set of costs. The process (i) and (ii) repeats itself.

Figure 1 depicts the exchange of information between the departments.

FIGURE 1. INFORMATION FLOW



The iterative procedure suggested above appears to be a relatively realistic model of what goes on, at least in an idealized sense, in real-world situations.

It is well-known (see e.g., Ha [4] and Van Roy [11]) that a decomposition procedure of the type such as the above two-stage process does not always succeed in finding a global optimal solution of G (c.f. Section II). It will be shown, however, that if one modifies the procedure in such a way that more information is exchanged and a stricter coordination implemented between the two departments, then a convergence to a true global optimum will indeed take place. This guarantee of success, however, is achieved at a cost, i.e., the modified version of the procedure which is guaranteed to find a global optimum is considerably more complex than the original iterative two-stage procedure. In a sense, this added complexity is consistent with common sense. That is, if one wants to achieve a global optimum in a decentralized environment, one must be prepared to compensate for the segmentation of decision-making. In Section IV, we will find out exactly how much the compensation of decentralization is in our simplified model.

The best-known "decentralization" scheme for an LP activity analysis is that based on the Dantzig-Wolfe decomposition algorithm. It is well-known, however, that a scheme based on the decomposition algorithm does not really represent a common notion of decentralization with divisional autonomy (c.f. Baumol and Fabian [1]).

In each iteration of the Dantzig-Wolfe algorithm, each "division" submits a "solution proposal" to "the company" or a central decision unit. The part of the optimal solution corresponding to each division is a certain weighted average of past proposals made by the division, but the division has no idea as to why this particular set of weights are selected by the central decision unit. For an iterative scheme to "qualify" as a decentralized procedure preserving essential divisional autonomy, we consider that at least the following two properties must be satisfied:

(A) The only variables involved in the optimization problem solved by each division (at each iteration) are those relevant to its own division; and

(B) A global optimum, when it is achieved, is a simple aggregate of the "current" optimal solutions of individual divisions. For instance, if there are two divisions and a global optimum (x_1, x_2) is generated at some iteration, where x_1 and x_2 correspond to the first and second divisions, respectively, then x_1 and x_2 themselves must be optimal solutions of the respective two divisions at the current iteration.

Clearly, the scheme based on Dantzig-Wolfe decomposition satisfies (A) but not (B).

In the next section we specify the basic models, problems G, S and P mentioned above, and present a numerical example. After the problem G is reformulated as a maximin problem in Section III, we describe in Section IV a modified iterative two-stage procedure which is guaranteed to find a global optimum and which satisfies both (A) and (B) above. We will also discuss relationships of the proposed procedure with some existing results in Section IV. The solution procedure we shall propose allows some interesting economic and managerial interpretations; these interpretations are given in the fifth and final section. An appendix deals with some mathematical aspects.

II. *Preliminaries*

The three optimization problem models, the global problem G, the sales problem S and the production problem P mentioned in Section I are given respectively as follows (for the purpose of reference, the dual of P is also shown). These models are borrowed from Itami and Kaplan [3].

The global optimization model G:

$$
\begin{array}{lll}
(G.1) & \text{maximize} & qx - cAz \\
(G.2) & \text{subject to} & Hx \leqq h \\
(G.3) & & Pz \geqq x \\
(G.4) & & Bz \leqq d \\
(G.5) & & x,\ z \geqq 0.
\end{array}
$$

The sales optimization model $S(v)$:

$$
\begin{array}{lll}
 & \text{maximize} & (q-v)x \\
(S(v)) & \text{subject to} & Hx \leqq h \\
 & & x \geqq 0.
\end{array}
$$

The production optimization model $P(x)$ and its dual $DP(x)$:

$$
\begin{array}{lll}
 & \text{minimize} & cAz \\
(P(x)) & \text{subject to} & Pz \geqq x \\
 & & Bz \leqq d \\
 & & z \geqq 0.
\end{array}
\qquad
\begin{array}{lll}
 & \text{maximize} & xv - dw \\
(DP(x)) & \text{subject to} & P^T v - B^T w \leqq A^T c \\
 & & v,\ w \geqq 0
\end{array}
$$

Here, x is the vector variable representing amounts of goods to be sold, and z is the vector variable of activity levels in the production department; $(G.2)$ represents the set of sales restrictions, $(G.3)$ denotes the requirement that the output quota, x, be satisfied by the production department, and $(G.4)$ is the set of production restrictions (capacity and resource limitations); q is the vector of selling prices, c is the vector of unit prices of variable input factors, and A is a matrix such that Az is the vector of amounts of the variable input factors consumed by production at level z. A vector is a column or a row, depending on the context, and superscript T is used to denote matrix transposition.

We denote by X, $Z(x)$ and Y the feasible regions of $S(v)$, $P(x)$ and $DP(x)$ respectively. Note that X is independent of v and that Y is constant for any x. We adopt the following informal abbreviations. The "x-part" of a solution of G refers to the component x in a solution (x, z) of G; the "v-part" of a solution of $DP(x)$ is v in a solution (v, w) of $DP(x)$. For the sake of simplicity we shall adopt the following assumption throughout the rest of the paper.

(i) X is nonempty and bounded; and

(ii) $Z(x)$ is nonempty and bounded for any x in X.

These assumptions are purely technical ones which can be relaxed. We shall make these assumptions to keep our presentation simple and to better focus on essentials. Note that from these two assumptions it follows that $S(v)$ has an optimal solution for any given v, and that $P(x)$ has an optimum (and hence so does $DP(x)$) for any given x in X.

We shall consider the following example in this and subsequent sections. This example is an extension of that considered in [3]. (See [3] for an explanation for the meaning of the data, except q, H and h.)

$$c = (1, 2, 2.5, 1, 1.2, 1) \qquad P = \begin{bmatrix} 1 & 1 & -0.5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2.5 & 4 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & .2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1.5 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ .1 & .15 & .05 & -1 \end{bmatrix}$$

$$d = (350, 300, 300, 20)$$

$h=(110, 140, 160, 250)$

$$H=\begin{bmatrix} 1 & 0 \\ -7/8 & 1 \\ 2 & -1 \\ 1/2 & 1 \end{bmatrix}$$

The following two values of q will be considered:

$q^1=(27.8, 18)$, and $q^2=(12.53, 10.27)$.

Figure 2 shows the feasible region X of the LP problem S. The first value of q is chosen so that the point $a=(100, 200)$ is the x-part of an optimal solution of G (along with all points on the edge between b and c). Point e in Figure 2 represents the x-part of an optimum of G for the second q. In the latter case, the problem G has a unique optimal solution. The feasible region X in Figure 2 is partitioned into four "cells" C_1 to C_4. These cells represent the sets of x in X with the same marginal cost vector (c.f. Section III). The marginal cost vector corresponding to each of the four cells is given as follows:

C_1: $v^1=(12.5, 10.25)$, C_2: $v^2=(27.5, 17.75)$

C_3: $v^3=(12.64, 10.36)$ and C_4: $v^4=(27.78, 17.96)$.

As mentioned in the previous section, the iterative two stage procedure specified in Section I may fail to find a global optimal solution of G: in particular, it may enter a cycle and "get stuck." The iterative procedure in fact fails when it is applied to the numerical example given above with q^2. The procedure enters a cycle alternating between an extremely low sales quantity, $x=(0, 0)$, and an extremely high sales quantity, $x=(110, 195)$ (point b in Figure 2).[1]

When $x=(0, 0)$, the cost vector for the sales pronlem $S(v)$ is given by $v^1=(12.5, 10.25)$. This v causes a strictly positive $q-v$ and so the sales department is encouraged to sell as much as possible, generating the high $x=(110, 195)=b$ as an optimal solution. The high demand in the production then causes high values of marginal costs of $v^4=(27.28, 17.96)$ as a result of solving $P(x)$ with the high x. This means that in the ensuing period, the net profit coeffcients for the sales department will be strictly negative and thus the low extreme $x=(0, 0)$ will result from the sales optimization. Thus, the process is drawn into an instable cycle by overcorrecting the "mistake" in the preceding period.

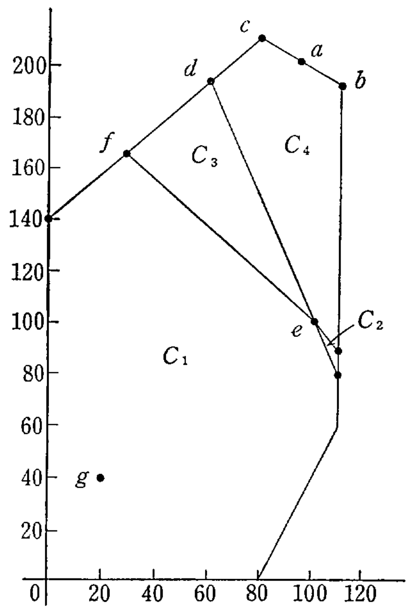
The error in this process is a failure to recognize the fact that there is a limit, in addition to the sales restrictions, on the sales volume beyond which the current costs will be invalid. In Figure 2, the cell C_1 is the limit of x such that the set of costs v^1 is valid. Similarly, when v changes to the high (27.28, 17.96) as a result of a high production quota, the sales department should be aware of the admissible region C_4 outside of which the existing costs are not meaningful.

The iterative two-stage procedure does not always fail. If the procudere is applied to G with q^1 in the above example, then the optimum of G, whose x-part is (110, 195) (point b in Figure 2) is computed in no more than two iterations.

We close this section by giving a mathematical programming interpretation of $S(v)$.

[1] Note that for q^2, G has a unique optimum (x, z) where $x=(100, 100)$ (point e in Figure 2), as mentioned above. There is no chance that the iterative two-stage procedure finds this solution since the x-part of this unique optimum is an interior point of X and by solving LPs S and P separately (presumably by the simplex method), only extreme points of X and $Z(x)$ are looked at.

FIGURE 2. THE FEASIBLE REGION FOR S



As mentioned in the first section, the sales problem, $S(v)$, reflects naturally what a sales department may do in a real situation: given sales prices and costs of goods, maximize the net profit subject to sales restrictions. It turns out that $S(v)$ has an interesting interpretation from a mathematical programming point of view. In the problem G consider the Lagrangean problem where the constraints (G.3) and (G.4) are dualized, i.e., where dual variables v and w are multiplied to (G.3) and (G.4), respectively, and each product is added to the objective. (A dual of G is obtained by optimizing this problem with respect to (v, w).) If (v, w) is feasible in $DP(x)$, then it is not difficult to show that z could be set equal to zero to maximize the objective of this Lagrangean problem. Thus z may be removed from the problem and then the Lagrangean problem reduces to $S(v)$. Thus it can be said that $S(v)$ is obtained from G by taking the Lagrangean problem of G with respect to (G.3) and (G.4), while $P(x)$ is obtained from G by fixing x.

III. *Maximin Formulation*

As a preparation for presenting a modified version of the iterative procedure, we shall, in this section, reformulate the problem G into a maximin problem using a well-known decomposition technique.

Let us recall that X, $Z(x)$, Y denote, respectively, the feasible sets of S, $P(x)$, and $DP(x)$. Assume that Y has L extreme points and let (v^i, w^i) denote the i-th one. From our assumptions (c.f. Section II) it follows that for any given x in X, there corresponds an (at least one) extreme point (v^i, w^i) which is optimal in $DP(x)$. With this, the following well-known chain of transformation of G into a maximin problem should be clear.

$$\max_{\substack{x\in X\\ z\in Z(x)}} \{qx-cAz\}$$

$$=\max_{x\in X} \{qx-\min_{z\in Z(x)} (cAz)\}$$

$$=\max_{x\in X} \{qx-\max_{(v,w)\in Y} xv-dw\}$$

$$=\max_{x\in X} \{\min_{(v,w)\in Y} \{x(q-v)+dw\}\}$$

$$=\max_{x\in X} \{\min_{1\leq i\leq L} \{x(q-v^i)+dw^i\}\}.$$

Thus, G is equivalent to the following:

(M) $\max_{x\in X} F(x)$, where

$$F(x)=\min_{1\leq i\leq L} \{x(q-v^i)+dw^i\}.$$

More precisely, x solves M and z solves $P(x)$ if and only if (x, z) solves G. In particular, for any given x, the objective value of the global problem G is given by

$$x(q-v)+dw,$$

where (v, w) solves $DP(x)$.

The definition of $F(x)$ suggests that $F(x)$ is piecewise linear: i.e., X is partitioned into "cells" in such a way that on each of these cells, $F(x)$ is linear (plus constant). More specifically, let the cell C_i be defined as the set of x in X such that i-th extreme point (v^i, w^i) is an optimal solution to the LP $DP(x)$. Then clearly $F(x)$ is linear on each such C_i. It can be shown (see appendix) that each C_i is a convex polyhedron and that to any interior point of C_i there corresponds a unique marginal cost vector v. In the numerical example given in Section II, there are 4 cells partitioning X with the corresponding unique v as shown. Note that it follows directly from its definition that C_i is the admissible region, discussed in Section II, outside of which the marginal cost vector v^i is invalid.

It is interesting to observe that a global optimal solution of G is a saddle point of the maximin problem M. For convenience, let us define the maximin function f by

$$f(x, (v, w))=x(q-v)+dw,$$

and rewrite M as

(M') $\max_{x\in X} \min_{(v,w)\in Y} f(x, (v, w)).$

It is not diffcult to prove that x^* solves G and (v^*, w^*) solves its dual (more precisely, "(x^*, z^*) solves G for some z^* and (w^*, v^*, u^*) solves its dual for some u^*") if and only if (x^*, v^*, w^*) is a saddle point of M', or

$$f(x, (v^*, w^*))\leq f(x^*, (v^*, w^*))\leq f(x^*, (v, w))$$

for any x in X and for any (v, w) in Y.

This "nash-equilibrium" property of an optimal solution of G implies the following: (x, z) solves G if and only if the production department has no incentive to change its current optimal solution x as long as the cost vector is unchanged, and the production department has no desire to alter its current optimal solution, z, as long as the output quota remains identical. This is another indication of autonomy in our decentralized model.

IV. *Modified Iterative Two-Stage Procedure*

In this section we shall present a modified version of the iterative two-stage procedure which is guaranteed to find an optimum of the global problem, G. We shall also discuss its relationships to some of the existing results.

We need to introduce some notation. For any given x in X define

$$R(x)=\{i:\ F(x)=x(q-v^i)+dw^i\}=\{i:\ (v^i,\ w^i)\text{ solves }DP(x)\}.$$

We shall sometimes use the letter k to denote the number of elements in $R(x)$. For any x in X, Δx is called a feasible direction of X at x if one can stay in X by moving along the direction Δx: i.e., if there exists a positive ε^* such that for any $\varepsilon\in[0,\varepsilon^*]$, $x+\varepsilon\Delta x$ is in X. The set of feasible directions can be characterized by inequalities corresponding to "active constraints." We let $T(x)$ denote the set of feasible directions of X at x:

$$T(x)=\left\{\Delta x:\ \begin{array}{rl} H_{i\cdot}\Delta x\leqq 0 & \text{for all } i \text{ such that } H_{i\cdot}=h_i \\ x_j\geqq 0 & \text{for all } j \text{ such that } \ x_j=0 \\ |\Delta x_j|\leqq 1 & \text{for all } j \end{array}\right\},$$

where $H_{i\cdot}$ denotes the i-th row of H.

In the modified procedure, the "accounting department" must solve a certain problem in order to choose a right set of costs, i.e., for a given x in X, $R(x)$ and all v_i for all i in $R(x)$, let E be the problem

$$(E)\qquad \max_{\Delta x\in T(x)}\ \{\min_{i\in R(x)}\ \{(q-v^i)\Delta x\}\}.$$

The sales problem to be solved must be modified to reflect the admissible region. That is, given v^i and C_i, the following replaces S.

$$(S'(v^i))\qquad \max\,(q-v^i)x \text{ subject to } x\in C_i.$$

The algorithm (modified iterative two-stage procedure):

Step 0: Assume an arbitrary x in X is given. Also assume that $R(x)$ and v^i for all i in $R(x)$ are given.

Step 1: Solve E. If the optimal value is zero, then stop: (x, z) solves G, where x and z are, respectively, the optimal solutions of the sales and production departments in the previous iteration. Otherwise, let i be an index for which the optimal value is attained, and go to Step 2.

Step 2: Solve $S'(v^i)$ and replace x by the computed optimum. Go to Step 3.

Step 3: Solve $P(x)$, and determine $R(x)$ by finding all dual extreme point optimal solutions of $P(x)$ and the corresponding cells. Rrtuen to Step 1.

This procedure terminates in a finite number of iterations generating an optimum solution of G. A proof of the validity of the algorithm is given in an appendix. It is clear that this modified procedure satisfies both properties (A) and (B) given in Srction I.

The problem E resembles M, but from a computational point of view, it is considerably simpler, provided that k, the number of elements in $R(x)$ is relatively small. In most cases, k is considerably smaller than L. For instance, in the numerical example given in Section II, the number of bases for $DP(x)$ could be (and hence L could also be) as large as 210, whereas the maximum value of k for all x in X is four. Also, the problem M can't be set up unless one knows all the extreme points of $DP(x)$ in advance. In contrast, in E, the necessary extreme points (and only necessary ones) are supplied by solving $DP(x)$ in the previous period. The best way to solve E is to use a standard LP transformation (see [9]); the resulting LP has $k+p$ constraints, where p is the number of active constraints at x. The modified sales problem, S', is a LP since C_t is polyhedral.

In Step 3, all extreme point optimal solutions of $DP(x)$ must be determined. This can be done by a well-known pivoting scheme (see e.g., Simonnard [10]) applied to an optimal simplex tableau of $DP(x)$.

The proposed procedure described above is related to several schemes appeared in the literature. The procedure is in a way a "specialization" of the algorithm given in Madsen and Schjaer-Jacobsen [6] for a nonlinear minimax problem. It also employs an overall strategy similar to that used in Kaneko [5] in a considerably different context.

There are many decomposition methods proposed in the Mathematical Programming literature which are potentially applicable to solving the global problem G in a decentralized manner. Most of these existing schemes may be grouped into the following two categories: (i) price directive decomposition, and (ii) resource directive decomposition. The major difference between these two groups of decomposition schemes is the way the central coordination of divisions is performed.

Roughly speaking, in a price directive scheme, the coordination is done by the central decision unit by issueing appropriate prices (or costs) under which each division executes its optimization. Meanwhile, the central control in a resource directive scheme consists of imposing limitations on resources (or production quotas) on individual divisons.

In most of existing decomposition schemes (with a notable exception of Cress Decomposition by Van Roy [11]; see below), the central control is either price directive with respect to all divisions, or resource directive with respect to all divisions. The proposed model differs from these existing approaches in that it employs a combination of price and resource directive coordinations; more specifically, it uses a resource directive contril with respect to the production department, and it employs essentially a price directive control with respect to the sales department.

In a decomposition model in which the central coordination is of the same type (price or resource directive) for all divisions, all divisions are assumed to be interchangeable units of the same type (e.g., all divisions are factories, or all divisions are sales offices). In the problem we are dealing with, however, the two divisions are of distinct types; the sales department is a profit center and the production department a cost center. For this reason, the model we are proposing in the present paper seems to provide a more natural setting for the problem under consideration, compared to most of the existing decomposition methods.

Cross Decomposition of Van Roy [11] decomposes G in a way which is much similar to the way our scheme decomposes G. Cross Decomposition consists of (i) a recource directed "primal" subproblem LP, (ii) a price directed "dual" subproblem LP, and a central coordination problem, which is a certain "implicit enumeration" scheme with respect to extreme points of the feasible regions of the subproblems. The decomposition procedure is executed by solving the primal and dual subproblems alternatingly, while the central problem is solved occasionally to control the sequence of subproblems to be solved, and to provide (resource and price) directives to the subproblems.

If one applies Cross Decomposition to G in a certain appropriate way, it turns out that the primal and dual subproblems in the decomposition scheme are identical to $P(x)$ and $S(v)$, respectively. Thus, Cross Decomposition is similar to the (unmodified) iterative two-stage procedure, except that solving the central coordination problem guarantees a termination of the procedure generating a global optimum of G. A difference between Cross Decomposition and the modified iterative two-stage procedure we are proposing is that Cross Cecomposition does not satisfy the autonomy requirement (B) discussed in the first section.[2]

V. *Economical and Managerial Implications*

In this final section we shall give some managerial interpretation of the modified iterative two-stage prodecure, and discuss economic implications of the interpretation.

First of all, the modified iterative two-stage procedure shows that the problem G can be solved in an autonomous, decentralized process although the process is much more complex than solving G directly (by the simplex method, say). The complexity may be regarded as the "cost" of decentralization; i.e., it represents "how much we have to pay" if we are to solve G by a decentralized process and still want to guarantee global optimality.

Compared to the original version, the modified process requires a more information flow and a more coordination effort. The added requirements may be summarized as follows:

(C) For the set of costs to be used, the corresponding admissible region be found and taken into consideration in the sales optimization.

(D) All marginal cost vectors be found in P and an appropriate one (to be used in the sales optimization) be selected among them by solving E.

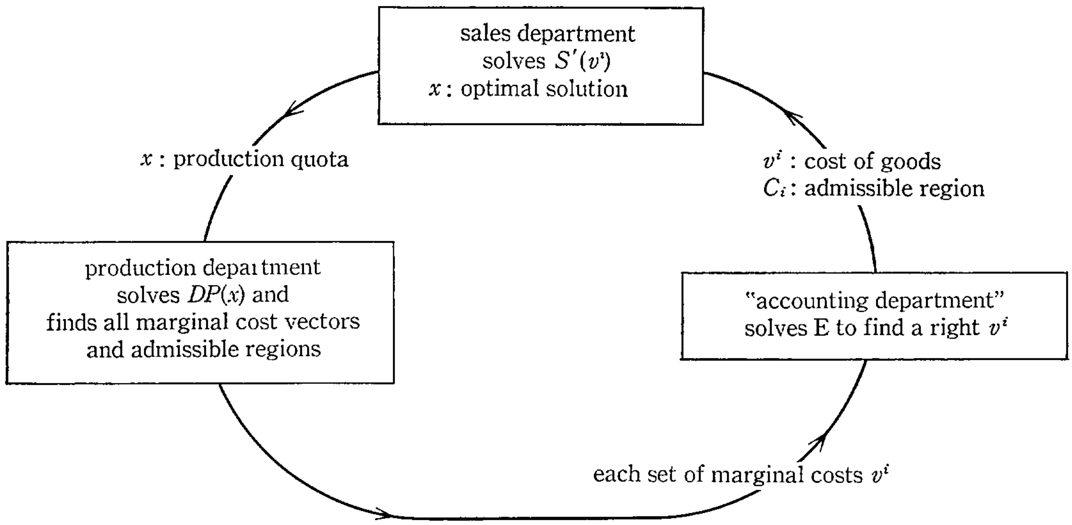
Figure 3 depicts the new information flow.

Necessity of requirement (C) was explained in Section II. Intuitively, a good planner must be aware of the fact that the current level of prices/costs will not be sustained beyond a certain limit. Since it is impossible to know ahead of time exactly how the prices/costs will change, a modest strategy is to limit one's activity within the admissible region where the current prices/costs remain valid. The modified sales optimization S' does just that.

The role of requirement (D) is less apparent, but turns out to be a crucially important one. In particular, the proper handling of multiple marginal cost vectors implied by solving E is the key mechanism ensuring the successful global optimization in the decentralized decision-making process.

[2] A reason for which Cross Decomposition does not satisfy (B) can be explained as follows. Assuming the simplex method is used to solve the sales subproblem $S(v)$, the sales vector, x, resulting from solving $S(v)$ will always be an extreme point of the feasible region X of $S(v)$. There are cases, however, in which each optimal solution (x, z) of G has the property that x is in the interior of X. The numerical example given in Section II with $q=q^2$ is one such case. In general, a decomposition procedure whose sales problem is $S(v)$ does not satisfy (B). Dantzig-Wolfe decomposition algorithm is also of this type.

FIGURE 3. NEW INFORMATION FLOW



Some readers may wonder how often there are more than one marginal cost vector; if there is only one marginal cost vector, solving E is irrelevant. Computational experience has shown that *LPs* arising in practice often possess multiple optimal solutions.[3] (Recall that a marginal cost vector is the v-part of an optimal solution of $DP(x)$.) In our case, in particular, it can be shown that the $DP(x)$ solved in our two-stage procedure *always* has multiple optimal solutions, except possibly at the first and last iterations.

The basic strategy, achieved by solving E, in the modified iterative two-stage procedure is this: In each iteration, choose a marginal cost vector v so that at the end of the iteration, i.e., after the sales department solves $S'(v)$ and the production department solves $P(x)$, where x is the computed optimum of $S'(v)$, the objective value of the global problem G increases strictly. In this way, a positive improvement is made in every iteration and due to a finite nature of the problem a true global optimum is reached after finitely many iterations.

Needless to say, it is not a trivial task to guarantee an improvement in the global objective value in a decentralized decision process. That is, a combination of two separate optimizations does not necessarily result in a better solution in the global problem. Indeed, the difficulty is compounded by the fact that interests of the two departments are basically in conflict. For example, a higher level of sales tends to generate a higher profit and so is desirable at the sales department, but it is undesirable at the production department since it tends to increase the production cost (to meet the higher demand). To see a clear manifestation of this conflict, let (for a given x) a constant xq be subtracted from the objective of $DP(x)$, and let (given v, w) dw be added to the objective of $S(v)$. Then, $(q-v)x+dw$ is maximized by the sales department, while the same is minimized by the production department. Thus, an action at the sales department tends to be countered by the production department. It is, therefore, highly unlikely that an arbitrarily chosen marginal cost vector

[3] This is so despite the fact that multiplicity of optimal solutions is a "rare" phenomenon, from a purely theoretical point of view.

will do a coordinating job resolving the conflict.

To explain why solving E yields a right cost vector, which does do the coordinating job, assume that a given production quota vector x^* corresponds to k marginal cost vectors, $v^1, \ldots, v^k$. Suppose that the sales optimization results in incrementing x^* by Δx to a new level $x^*+\Delta x$. Assuming Δx is sufficiently small (so that $x^*+\Delta x$ stays in the union of C_j, $j=1, \ldots, k$), the optimal solution of the (dual of) production optimization is (v^i, w^i), where v^i is one of the k marginal cost vectors of x^*. It follows that at the end of the iteration, the increment of the global objective value is expressed as $(q-v^i)\Delta x$. Since this v^i is (the v-part of) an optimal solution of DP (note that in DP, $(q-v)x+dw$ is minimized, as pointed out above), it has the property that $(q-v)\Delta x$ is as small as possible, i.e., that

$$(q-v^i)\Delta x=\min_{\text{all } j} (q-v^j)\Delta x.$$

To summarize, if the sales optimization results in incrementing x^* by Δx, the corresponding increment of the global objective value at the end of the iteration is the minimum of $(q-v^j)\Delta x$ for all marginal cost vectors $v^j, j=1, \ldots, k$. Thus, by choosing the v^i minimizing $(q-v^j)\Delta x$, and by using it as the cost vector in the sales optimization, an improvement of the global objective coincides with that of the sales optimization problem, and hence a global improvement is induced by the sales optimization. It is now clear what E and Step 1 of the modified procedure attempt to do. A global improvement is possible if the optimal objective value of E is strictly positive, and if we choose, as the cost vector in the sales optimization, the v^i minimizing $(q-v^j)\Delta x$.

Note that if v is the given cost vector, $(q-v)x$ is maximized in the sales optimization problem. The marginal cost vector v^i which is selected minimizes $(q-v)\Delta x$, and so is actually the worst choice among all the marginal cost vectors, as far as the sales optimization alone is concerned. That is, to ensure a global improvement, we need to adopt the following conservative policy:

> To select a cost vector among all the marginal cost vectors for the sales optimization, choose the one which corresponds to the smallest marginal increment in the objective of the sales optimization problem.

It is not difficult to see that this suggested conservatism is a direct result of the conflict between the sales and production departments mentioned above. That is, since the production department will try to minimize $(q-v)\Delta x$, the sales department must assume the "worst case scinario" and choose the v^i with the minimal $(q-v)\Delta x$.

The discussion given above concerning multiple marginal cost vectors and the suggested conservative choice among them[4] seems to possess wide implications beyond the current framework of the iterative two-stage procedure. A "marginal cost" or "shadow price" interpretation of a dual optimal solution in LP activity analysis (or pther LP) models are commonly used in many managerial situations. When there are more than one dual optimal solution, however, it appears that the conventional theory/practice is not clear as to which dual optimal solution should be used, or as to a possible ill effect of making a "wrong" choice.

As mentioned above, multiplicity of optimal solutions in LPs is not a remote possibility

[4] The notion of choosing the "best" dual optimal solution is discussed in [7] in a different context (accelerating Benders' decomposition).

in pathological cases, but is a frequent occurrence is practical cases. Clearly, a choice of dual optimal solution may have a profound effect in the underlying decision-making process. For instance, suppose in the problem of evaluating effects of adding resources, there are two sets of shadow prices suggesting contradicting decisions. Which set of shadow prices should be used? Is using one more desirable, in some sense, than using the other?

An answer is given in the case of the decentralized decision-making process we are analyzing in this paper. It is suggested that a conservative policy be adopted in choosing a right dual optimal solution in order to ensure global optimization.

Strictly speaking, the results obtained in this paper are shown to be valid only within the framework of the stated decentralized scheme. However, the results may have some implications in a wider context. Our framework provides a reasonably general and realistic model for an economy in which there are two (or possibly more) sections whose objectives are basically in conflict. Our analysis may indicate that under a certain economic situation, where conflicting forces are present, one should adopt the worst case scinario and pick a conservative choice of a marginal cost vector in order to ensure a desirable consequence in the global system.

Finally, we would like to point out that some additional insight may be obtained by considering the equivalence between optimality of G and a state of equilibrium, mentioned in Section III, in our two-stage decentralization model. Instead of defining our problem as one of optimization, we can regard our ultimate goal to be one of achieving a state of equilibrium. The validity of the modified iterative two-stage procedure in solving G is then interpreted as a proof that a state of equilibrium is eventually achieved in the two-stage process, if the required set of information is exchanged. One may look at the situation from the opposite direction. If one believes that an economic process, which follows an appropriate price guideline, will eventually attain an equilibrium, then one may argue that the modified iterative two-stage procedure achieves an equilibrium because it follows a proper price guideline, given by items (C) and (D) mentioned above.

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APPENDIX

A. The subdivision of X.

Let (v^i, w^i) be the i-th extreme point of Y. We defined the i-th cell, C_i, as the set of all x in X such that (v^i, w^i) solves the LP $DP(x)$. Clearly, C_i is given as the set of x such that there exists a dual (of $DP(x)$) feasible vector (i.e., a vector in $Z(x)$) satisfying the complementary slackness conditions with (v^i, w^i). It is not difficult to prove that such a set is a convex polyhedron. It is clear that the union of all such cells covers X since, by the assumptions in Section II, there exists an extreme point optimal solution to $DP(x)$ for any given x in X. The cell C_t permits a particularly simple representation if (v^t, w^t) corresponds to a nondegenerate basic feasible solution of $DP(x)$. In this case, (v^t, w^t) corresponds to a unique optimal basis in $DP(x)$ and so C_t is the set of x such that the relative cost coefficients with respect to the optimal basis are all nonnegative.

The uniqueness of the marginal cost vector for all interior points of a given cell can be proved by carefully examining the optimal simplex tableau for $DP(x)$. We give an outline of the proof. Let (v^*, w^*) be an extreme point of Y and let C^* be the associated cell. Let x^* be an inteiror point of C^*. Suppose x^* belongs to another cell defined by another extreme point (v', w') of Y, i.e., suppose that an extreme point (v', w') also solves the LP $DP(x^*)$. It is well-known (see Simonnard [10]) that (v', w') can be reached from the optimal simplex tableau for (v^*, w^*) by pivoting nonbasic variables with zero relative cost coefficients into basis. The proof consists of showing that in the pivoting process which transforms (v^*, w^*) into (v', w'), no pivot takes place which changes the values of the v variables. More specifically, you can prove that (i) no nonbasic v-variable is replaced by a nondegenerate basic variable, and (ii) if a nonbasic w-variable or a nonbasic slack variable enters a basis by a nondegenerate pivot, the part of the incoming column corresponding to the basic v-variables is zero. You can prove these by showing that an occurrence of (i) or (ii) would imply that a small perturbation of x^* would bring the interior point out of C^*, which is a contradiction.

B. Validity of the modified iterative two-stage procedure.

Firstly, it is clear that the objective value $F(x)$ is strictly and monotonically increasing during the execution of the algorithm. Since the number of cells is finite and since no cell is repeated, the algorithm terminates in a finite number of iterations.

We need to show that when the algorithm terminates, we have an optimal solution to G. Some notation is necessary. We rewrite the constraints in X as

$$Nx \geqq a, \text{ where } N = \begin{bmatrix} -H \\ I \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} -h \\ 0 \end{bmatrix}.$$

For any x in X, we let $R(x)$ denote the same as before, and let

$$I(x) = \{i : N_i \cdot x = a_i\},$$

where $N_i\cdot$ is the i-th row of N. By $N_{I(x)}\cdot$ we denote the submatrix of N with the rows in the index set $I(x)$. It follows then that Δx is a feasible direction of X at any given x in X if and only if

$$N_{I(x)}\cdot\Delta x \geqq 0.$$

It is convenient to introduce the concept of a stationary point for nondifferentiable problems with subgradients (see e.g., [6]). A point x in X is called a stationary point of the problem M if and only if

$$(*) \quad \begin{array}{l} \text{there exist } \delta_i \geqq 0,\ i \in R(x),\ \Sigma_{i\in R(x)}\delta_i = 1 \text{ and} \\ \text{there exist } \lambda_i \geqq 0,\ i \in I(x) \text{ such that} \\ \qquad \Sigma_{i\in R(x)}\delta_i(-q+v^i) = \Sigma_{i\in I(x)}\lambda_i(N_i\cdot)^T. \end{array}$$

Note that if $|R(x)| = 1$, then (*) is the set of complementary slackness optimality conditions. We shall show that (i) the algorithm terminates if and only if a stationary point is generated, and that (ii) if x is a stationary point and if z solves $P(x)$, then (x, z) solves G.

From (*) and from Farkas Lemma (see e.g., Mangasarian [8]) it follows that x is a stationary point of M if and only if there does not exist Δx such that $N_{I(x)}\cdot\Delta x\geqq 0$ and $(q-v^i)\Delta x>0$ for all i in $R(x)$; i.e., if and only if for any feasible direction Δx of X at x, there exists at least one i in $R(x)$ for which $(q-v^i)\Delta x$ is nonpositive. This last set of conditions is equivalent to that the Problem E in Step 1 has the zero optimal objective value.

Now let x be a stationary point and let z solve $P(x)$. Define a vector u by

$$u_i=\begin{cases}\lambda_i & \text{if } i\in I(x)\\ 0 & \text{if } i\notin I(x).\end{cases}$$

Further, let

$$v=\Sigma_{i\in R(x)}\delta_i v^i \text{ and } w=\Sigma_{i\in R(x)}\delta_i w^i.$$

Then (5) for x (plus that $x\in X$) is rewritten as

$$\begin{aligned} &q-v+H^T u\geqq 0,\ x\geqq 0,\ x(q-v+H^T u)=0\\ &h-Hx \quad\ \geqq 0,\ u\geqq 0,\ u(h-Hx) \quad\ =0; \end{aligned}$$

thus, x solves $S(v)$. Since each (v^i, w^i), $i\in R(x)$, solves $DP(x)$, so does (v, w). Thus, we have that x solves $S(v)$, (v, w) solves $DP(x)$ and z solves $P(x)$. By comparing optimality conditions of the *LPs* involved, it is not difficult to conclude that (x, z) solves G.

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