ON THE EQUILIBRIUM OF THE GAME WITH A CONTINUUM OF PLAYERS

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I. *Background*

We can find the essential difference between social phenomena and natural phenomena when we try to describe these phenomena in mathematical methods. Let us give an example. When we analyze a natural phenomena using a system of differential equations, generally speaking its solution is decided uniquely by its initial condition which we can get through the observation to nature. Namely if we call the Creator of nature God, we can recognize all of God's will concerned with the natural phenomenon as God's will at some fixed time that is to say the initial condition at that time and the true form of expression of nature that is to say the system of differential equations with the assumption that we can find it. Accordingly there exist no decision makers except God when we describe a natural phenomenon.

On the contrary when we try to describe a social phenomenon, generally speaking we must keep in our mind that there exist many kinds of will of decision makers. Moreover their will does not have the universal validity for space and time. Their will that is to say their decision changes with the position of space and time. Therefore if we try to describe a social phenomenon with the system of differential equations, we must solve it under the ever-changing initial conditions. But to do it is nonsense, because the ever-changing initial conditions themselves are a solution of the system of differential equations. Therefore we will find the reason why we cannot use the methods which we use on analyzing natural phenomena on analysis of social phenomena.

Now let us consider only one decision maker labeled A. His decision is represented by his choice that is to say he chooses some element x from the action space X. But we use the notation $x(t)$ in place of x, because x must depend on the time t from the time interval T which is given to the decision maker A. Then the subset $\{x(t)\in X;\ t\in T\}$ or the mapping $x:T\rightarrow X$ (needless to say, these two concepts should be distinguished) is the graph or the trace of the decisions of A.

To compare our idea which we will explain in this note with the ordinary formulation in the decision theory, let us formulate the outline of the ordinary formulation concisely. Let us assume that the states of the society which we try to describe are represented by a point s of n-dimensional Euclidean space $\boldsymbol{R}^n$, and the phenomenon of the society is described by the trace of s in the time interval T; i.e. $\{s(t);\ t\in T\}$. On the other hand let us assume

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that Λ denotes the set of all decision makers and the decision of each decision maker $A_\lambda(\lambda \in \Lambda)$ is represented by the trace $\{x_\lambda(t) \in X_\lambda;\ t \in T\}$ in the time interval T. Then one of the important problems in the decision theory is how we can construct $\{s(t) \in \boldsymbol{R}^n;\ t \in T\}$ from $\{x_\lambda(t) \in X_\lambda;\ t \in T\}$. This is the ordinary formulation.

The idea which we propose in this note is different from the above. Our idea is to replace the time interval T with the set Λ of decision makers. Why is the replacement necessary? When we consider the decision problems, the most important thing is the causality in the set $\{x(t) \in X;\ t \in T\}$ i.e. the interrelationship in the set. In the game theory the interrelationship of players is skillfully described by the pay-off functions. This is the reason why we replace the time interval T with the set of decision makers.

II. *The Relation with the Market Game*

The game in which many players are infinitely participated has been discussed by lots of game theorists and mathematical economists. Namely, these studies, whose initiative was taken by Debreu, G. and Scarf, H.[1] and Aumann, R. J.[2] were inherited by Hildenbrand, W.[3] and fructified in "Journal of Economic Theory, Vol. 22, No. 2 (1980)" which was edited by Mas-Colell, A.

But this stream of studies has the fountain-head of the problems in the Edgeworth's pure exchange Economy and the discussion has been developed in the framework of economic equilibrium. Accordingly the kernel of the matter was the relationship between the core of the economy and the competitive equilibrium. And the foundamental aim of this stream is to form the situation of perfect competition in accordance with the increases of numbers of players. Namely the increase in number of players produces such situation that the decisions of other players. Thus our way of thinking which was described in the previous section essentially differs from this stream of Debreu-Scarf-Aumann on the point that the decision of each player depends on decisions of other players.

But two fundamental concepts in the stream of Debreu-Scarf-Aumann are "core" which is the set of all elements having a collective rationality and "competitive equilibrium." We try to revive these two concepts in our story. At first we take the concept of core as the set of all decisions that have a rationality at the time interval T and are decided by a decision maker. Secondarily we take Nash equilibrium which is sufficiently discussed by game theorists in the place of competitive equilibrium. Therefore we discuss the relation between the core and Nash equilibrium.

III. *The Formulation of the Model and a Theorem*

In this section we introduce some notations and formulate our model.

$\boldsymbol{R}$: the set of all real numbers,

$T=[0, 1]=\{t \in \boldsymbol{R};\ 0 \leqq t \leqq 1\}$

We interpret the element of T as the decision maker. So there are infinitely many decision makers in our model.

X: the set of all alternatives that are chosen by each decision maker of T in common,

$\mathscr{F}(T, X)=\{\varphi, \psi, \ldots\}$: the set of all mappings from T to X.

For $\varphi \in \mathscr{F}(T, X)$ and $t \in T$, $\varphi(t)$ means that a decision maker t chooses the element $\varphi(t)$ of X. Let us call the element of $\mathscr{F}(T, X)$ a profile. Now we introduce the pay-off function f_t for each decision maker $t \in T$ and each profile $\varphi \in \mathscr{F}(T, X)$, namely,

$$f_t: \mathscr{F}(T, X) \rightarrow \boldsymbol{R}.$$

We consider the game

$$\Gamma=(T, X, \{f_t\}_{t \in T})$$

which is described by the normal form. Our discussions are restricted to this game.

We call the subset of T which is Lebesque measurable a coalition and the coalition which Lebesque measure is not zero a nonnull coalition. For two profiles φ and ψ and nonnull coalition S, the definition of "ψ dominates φ via S" is

$$f_s(\psi)>f_s(\varphi) \text{ for all } s \in S$$

and we abbreviate this concept to

$$\psi \operatorname{dom}_S \varphi.$$

Using this concept of domination, we can define the core of game Γ

$$\{\varphi \in \mathscr{F}(T, X);\ \{\phi \in \mathscr{F}(T, X);\ \psi \operatorname{dom}_S \varphi\}=\phi \text{ for all nonnull coalition } S\}$$

and denote this set by the notation $C(\Gamma)$.

In order to define Nash equilibrium of the game Γ, let us introduce a notation. For $\varphi \in \mathscr{F}(T, X)$, $t \in T$ and $x \in X$, we define $\varphi_{t,x} \in \mathscr{F}(T, X)$ by

$$\varphi_{t,x}(s)=\begin{cases} x & ; s=t \\ \varphi(s); & s \neq t \end{cases}$$

It is clear that $\varphi_{t,\varphi(t)}$ equals to φ. Now for an arbitrary nonnull coalition S, Nash equilibrium φ of game Γ is defined by

$$\sup_{x \in X} f_t(\varphi_{t,x})=f_t(\varphi) \text{ for all } t \in S$$

and we denote the set of all Nash equilibrium $E(\Gamma)$.

Hereupon let us introduce the following two assumptions.

[assumption 1] If for some profile φ and some nonnull coalition S there exists a subset $\{x_s \in X;\ s \in S\}$ of X and

$$f_s(\varphi_{s,x_s})>f(\varphi) \text{ for all } s \in S$$

holds, then

$$f_s(\psi)>f_s(\varphi) \text{ for all } s \in S$$

where the profile ψ is defined by

$$\psi(s)=\begin{cases} x_s; & s \in S \\ \varphi(s); & s \notin S. \end{cases}$$

[assumption 2] If for two profiles φ and ψ and for some nonnull coalition S,

$f_s(\psi)>f_s(\varphi)$ for all $s\in S$

holds, then

$f_s(\varphi_{S,\psi(s)})>f_s(\varphi)$ for all $s\in S$.

These two assumptions shall be examined in the section 6.

Using these notations and assumptions, we can prove the following

[theorem] $C(\Gamma)=E(\Gamma)$

IV. *The Proof of the Theorem*

(1) At first let us prove $C(\Gamma)\subset E(\Gamma)$. If there exists a profile φ which is an element of $C(\Gamma)$ but not an element of $E(\Gamma)$, for some nonnull coalition S

$$\forall_s\in S,\ \exists x_s\in X;f_s(\varphi_{S,x_s})>f_s(\varphi)$$

holds. Defining a profile ψ by

$$\psi(s)=\begin{cases} x_s; & s\in S \\ \varphi(s); & s\notin S\end{cases}$$

it is clear that

$$\psi \ \mathrm{dom}_S\ \varphi$$

holds by [assumption 1] in the previous section. Then $\varphi\in C(\Gamma)$ and it is a contradiction.

(2) In the second place let us prove $E(\Gamma)\subset C(\Gamma)$. If the profile φ is an element of $E(\Gamma)$ but not an element of $C(\Gamma)$, there exist the profile ψ and a nonnull coalition S such that

$\psi \ \mathrm{dom}_S\ \varphi$, i.e.

$f_s(\psi)>f_s(\varphi)$ for all $s\in S$.

Since φ is an element of $E(\Gamma)$, for all nonnull coalitions, accordingly for T,

$[f_s(\varphi_{S,x})\leqq f_s(\varphi)$ for all $x\in X]$ for all $s\in T$

holds. Then

$f_s(\varphi_{S,x})\leqq f_s(\varphi)$ for all $s\in T$

means a contradiction.

V. *The Examination of Two Assumptions*

(1) It is possible to interpret the assumption 1 as follows; namely let us assume that there exists the set S of decision makers whose amounts are not negligible. Each member s of the set S, supposing that the decision of other members of the set S except himself remain in the profile φ, believes that it is more favorable for him to choose x_s than to choose $\varphi(s)$. When we imagine the profile in the case that all members of the set S change their

decisions from $\varphi(s)$ $(s \in S)$ to x_s $(s \in S)$ at a time, all members of the set S must believe that ϕ is more favorable than φ.

(2) It is possible to interpret the assumption 2 as follows; namely let us assume that there exists the set S of decision makers whose amounts are not negligible. If each member s of the set S believes that for two profiles φ and ϕ the profile ϕ is more favorable for him than the profile φ, he must considers that the profile $\varphi_{s,\phi(s)}$ is more favorable for him than the profile φ; where $\varphi_{s,\phi(s)}$ means that each member s of the set S, believing that other members of the set S except himself remain in the profile φ, chooses $\varphi(s)$ in the place of $\varphi(s)$.

VI. *Conclusion*

We want to end this story with the examination of the theorem which we presented in the section III and proved in the section IV.

As mentioned in the section II, if we interpret the core $C(\Gamma)$ of the game Γ as all decisions having collective rationality and the set $E(\Gamma)$ of Nash equilibria of the game Γ as all decisions having individual rationality, it is persisted in the theorem that the above two should coincide each other. Therefore if each decision maker pursues his individual rationality separately, without any consideration for the whole society to which he belongs, it comes to a conclusion that the society takes possession of collective rationality automatically: as it were it means that the world of Adam Smith where unseen hands dominate is realized.

Let us apply this interpretation to our problem described in the section I that only one decision maker A chooses the optimal solution $x(t)$ from the space X at any time t of the time interval T. Then if the decision maker A can discover the optimal solution $x(t)$ from X every time at the time interval T, it means that he gets $E(\Gamma)$. By the theorem

$$E(\Gamma) = C(\Gamma).$$

Therefore the momentarily optimal solution automatically becomes the global optimal one in the time interval T.

It means that we can get the overtime optimal solution only by automatically connecting the optimal solution which we get everytime. It denies the casuality. That is a queer conclusion.

References

[1] Debreu, G., and Scarf, H., "A Limit Theorem on the Core of an Economy," International Economic Review, 1963.

[2] Aumann, R. J., "Markets with a Continuum of Traders," Econometrica, vol. 32, 1964.

[3] Hildenbrand, W., "Core and Equilibria of a Large Economy," Princeton University Press, 1974.