Disequilibrium Dynamics in a Single Market with Unintended Inventory

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The Disequilibrium Theory has been intensively studied for the past ten years. Some authors have already contributed to the theory; some are instructive while some are misleading. Negishi and Hahn concentrated on a pure exchange economy when they studied the non-fattonnement process; Clower, Barro and Grossman have analysed such a disequilibrium situation that dual decided firms and households in perishable-good markets affect each other, but transactions are supposed to take place at the end of the multiplier process, or the dynamic disequilibrium paths of prices and quantities are not considered. We could say they explained the ‘spill-over’ effect but neglected the ‘carry-over’ effect of disequilibrium which is the essential idea of the inventory cycle model.

The aim of this paper is to analyze the ‘carry-over’ effect of disequilibrium in a single market for a storable good, describing the dynamic paths of actual transaction price and quantity. We introduce an inventory as a buffer stock and will show the dynamic stability of the market under certain assumptions. Our model will give an idea of disequilibrium dynamics.

To deal with the storable good, we have to introduce inventory which will cause another difficulty of flow vs. stock disequilibrium. We regard inventory as a buffer stock which will decrease when
current demand exceeds current production and will increase when
current production exceeds current demand. We also can say that the
quantity is adjusted first to meet the demand, and the price adjusted
later.\(^7\) This interpretation is especially attractive when we consider
a market of a manufactured good.

We may justify our concentration on a single market. Since
the demand is almost always satisfied and the unsatisfied supply is
carried-over as an unintended inventory, the spillover effect is not
intrinsic in our model.

The good is demanded by a representative household. The usual
argument shows that in a single market the demand function depend-
ing upon its price is derived from the utility maximizing behavior
of the household. Here the amount of demand for good is assumed to
depend upon the price and an exogeneous factor;

\[ x^d = D(p,k). \tag{1} \]

A further assumption is that the demand function is a decreas-
ing function with respect to the price;

\[ D_i \equiv \partial D/\partial p < 0 \tag{1.a} \]

The good is assumed to be produced and stored by a represen-
tative firm. The firm calculates an optimal level of inventory. He
knows that demand is sometimes disturbed by an exogeneous factor.
To meet a sudden increase of demand, the firm thinks it beneficial to
have a certain amount of inventory. The optimal level will be deter-
mined so as to minimize the expected opportunity costs (demand exceeds
supply plus inventory) and the inventory cost. The firm does not
know the equilibrium price or the demand schedule. He knows that
the price was wrong when he faces the unintended increase or decrease
of his inventory. For the sake of simplicity,\(^8\) the price is assumed to
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be gradually adjusted by a market authority, i.e., it raises the price when the optimal level of inventory exceeds the current level and conversely when the current level exceeds the optimal level;

\[
dp/dt = \begin{cases} 0, & \text{if } p=0 \text{ and } x^*<x^v \\ \phi(x^* - x^v), & \text{otherwise} \end{cases}
\]

\[\phi(0)=0, \quad \phi'>0\]  

where \(x^v\) denotes the optimal level of inventory which is assumed to be constant and \(x^v\) the current level of inventory.

The firm, facing the current price and the current level of inventory, decides the current amount of production:

\[x^s = S(p, x^* - x^v)\]  

The plausible assumptions on the derivatives are shown as,

\[S_1 = \frac{\partial S}{\partial p} > 0 \]
\[S_2 = \frac{\partial S}{\partial (x^* - x^v)} \geq 0.\]  

(3, a)

The assumption (3, a) indicates that the firm's production decision may be affected by the excess inventory over the optimal level or shortage of it. If it isn't, i.e., \(S_2=0\), then the situation is supposed to be under perfect competition. The firm thinks that at the given price he can sell all he wants to sell, so that the level of inventory is nothing to do with him. On the other hand, if he gives priority to adjust the level of inventory to the optimal one rather than to pursue the marginal condition of production, then he will set;

\[x^s = \begin{cases} \infty, & \text{if } x^* > x^v \\ 0, & \text{if } x^* < x^v. \end{cases}\]

But it is more plausible to assume the physical constraints in absolute amount and changing rate of production such as (3, a).

"Non-recontract" transaction is assumed to be carried out at the
short side of the market. If the total supply (=production plus inventory) exceeds the demand, then the amount of transaction is equal to that of demand, and the difference between the total supply and demand will be stored. If the total supply is less than the demand, the transaction will be executed at the level of total supply and unsatisfied demand is assumed to go to another market and not to affect the demand schedule in the next period. Since we are going to construct the system of differential equations, we can formulate the above argument as follows;\(^{(9)}\)

\[
\dot{x} = \begin{cases} 
  x^s, & \text{if } x^v = 0 \text{ and } x^o \geq x^s \\
  x^o, & \text{otherwise}
\end{cases}
\]  

(4)

where \(\dot{x}\) denotes the realized amount of transaction.

The inventory will change to the extent of the difference between production and transaction. But the part of the inventory may deteriorate or be subtracted as the inventory cost.

\[
dx^v/dt = x^s - x - \theta x^v, \quad \theta > 0
\]  

(5)

Now we can examine the stability of the market, provided that the equilibrium and the initial points exist in the positive region on the price-inventory plane; \(x^{v*} > 0\)

\[
\exists p^* = [p > 0 | D(p, k) + \theta x^{v*} = S(p, 0)].
\]

\[
x^v(0) > 0
\]

\[
p(0) > 0.
\]

The differential system consists of (1) through (5);

\[
\begin{cases}
  dp/dt = \phi(x^{v*} - x^v) \\
  (S) dx^v/dt = S(p, x^{v*} - x^v) - D(p, k) - \theta x^v = g(p, x^v)
\end{cases}
\]

\[
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix} = \begin{pmatrix}
  0 & -\phi' \\
  S_1 - D_1 & -S_2 - \theta
\end{pmatrix}
\]
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According to the Olech's theorem\(^{(10)}\) the sufficient conditions of stability of the above system \((S)\) will be given below:

(i) \(f_1 + g_2 < 0\) \(\forall (p, x^v)\)

(ii) \(f_2 g_1 - f_2 g_1 > 0\) \(\forall (p, x^v)\)

(iii) either \(f_1 g_2 = 0\) \(\forall (p, x^v)\)

or \(f_2 g_1 = 0\) \(\forall (p, x^v)\)

We can easily verify that assumptions \((1.a)\) and \((3.a)\) are sufficient to guarantee \((i)\) \((ii)\) and \((iii)\) hold true. The system \((S)\) is asymptotically stable in the large, which means the path starting from an arbitrary point on price-inventory plane converges to the equilibrium point as time goes to infinity.

The problem to determine the level of optimal inventory was left unsolved. Someone may assume the adaptive adjustment of the optimal inventory level, i.e., the optimal level of inventory should depend on the current level of demand for the commodity. It seems plausible that the producer thinks the optimal inventory should be, say, one third of the expected sales. If the demand for the commodity has increased, then the producer would try to have more precautionary inventory. For the sake of simplicity, let us assume the optimal level of inventory \(x^v_*\) depends on the current value of demand.

\[
x^v_* = V(x^d), \quad V' > 0.
\]

Then the system \((S)\) at p. 4 should be modified as follows:

\[
(S') \begin{cases}
  \frac{dp}{dt} = \phi(V(D(p, k)) - x^v) = F(p, x^v) \\
  \frac{dx^v}{dt} = S(p, V(D(p, k)) - x^v - D(p, k) - \theta x^v = G(p, x^v)
\end{cases}
\]

\[
\begin{pmatrix}
  F_1 & F_2 \\
  G_1 & G_2
\end{pmatrix} = \begin{pmatrix}
  \phi' V'D_1 & -\phi'' \\
  S_1 + S_2 V'D_1 - D_1 & -S_2 - \theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
  F_1 & F_2 \\
  G_1 & G_2
\end{pmatrix} = \begin{pmatrix}
  - & - \\
  ? & -
\end{pmatrix}.
\]
We cannot assert the global stability from the above signs of derivatives. We, however, can apply the Olech’s theorem eventually:

\[(S'\text{-i}) \quad F_1+G_2=\phi'V'D_1-S_2-\theta<0, \quad \nabla (p, x^v)\]
\[(S'\text{-ii}) \quad F_1G_2-F_2G_1=-\phi'V'D_1S_2-\theta\phi'V'D_1+\phi'S_1-\phi'D_1 \]
\[+\phi'S_2V'D_1 \]
\[=\phi'S_1-(1+\theta V')\phi'D_1>0, \quad \nabla (p, x^v)\]
\[(S'\text{-iii}) \quad F_1G_2=0, \text{ and } F_2G_1=0, \quad \nabla (p, x^v)\]

By \((S'\text{-i}) \quad (S'\text{-ii})\) and \((S'\text{-iii})\), we can assert that the system \((S')\) is asymptotically stable in the large provided that there is an equilibrium point.

In summary, the introduction of the adaptive determination of the optimal inventory level does not affect the stability of the system \((S)\). In other words, the system \((S)\) is the special case, in the sense that \(V'=0\), of the system \((S')\), and both \((S)\) and \((S')\) are globally stable.

We should emphasize that the path on the phase diagram is the sequence of the ‘actual’ transaction. The dynamic behavior of the Walrasian market is the path of the ‘tentative’ or ‘notional’ transaction toward the equilibrium where the actual transaction is executed. Our model shows that the introduction of the buffer stock instead of the recontract apparatus makes it possible to trace the sequence of the transaction and to assert the stability of the system. According to our model, the actual fluctuation or converging after the sudden exogeneous disturbance against the system will be interpreted as the disequilibrium adjustment process toward the equilibrium. To illustrate the assertion, consider the situation that the system was in the equilibrium, i.e., \(k=k, \quad p=p^*\), \(x^v=x^v^*\), therefore \(dp/dt=0\) and \(dx^v/dt=0\). Now suppose \(k\) is changed to \(\bar{k}\), and \(D(p^*, k)<D(p^*, \bar{k})\). Then sudden
increase of demand will immediately cause the declining of the inventory because current demand exceeds current supply. Corresponding to the declining of the inventory, the price will gradually go up and the production will increase. Throughout this process the transactions are executed one after another. The above interpretation about the fluctuation of the actual trade is impossible when we adhere to the Walras’ Demon, or Tâtonnement process. Our unintended inventory model leaves many unsatisfactory assumptions. We should extend our discussion to multi-market model. We could explicitly specify the simultaneous determination of optimal inventory, price level and production. These study of more complicated and interesting cases will be a matter for further work.

Notes;
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in the paper remains only with the author.

(2) A disequilibrium analysis should be defined as an analysis which explicitly deals with the trading out of equilibrium or the ‘false’ trading. Then it includes the study of the carry-over effect or the spill-over effect.

(3) Beckman and Ryder [3] considered the disequilibrium situation only in the tâtonnement process, although they tried to analyse the system of simultaneous price and quantity adjustment. See the detail criticism by Veendorp [11]

(4) See Negishi-Hahn [9].

(5) See Ito [6], Barro and Grossman [1] [2].


(7) See Leijonhufvud [8]. He stated that the Keynes' contribution to the theory is the inversion of the adjustment speeds of quantity and price.

(8) We can think the firm, instead of the market authority, adjusts the price. We, however, must explain the firms' two-stage decision on price and production. When the firm decides the latter, he regards the former given. The case is not so attractive. When we assume the firm has the power to set the price we should consider the simultaneous monopolistic adjustment over price and quantity and various set by different firms.

(9) When we use the system of difference equations, eq. (3) will be formulated as,

\[ \dot{x}(t) = \min(x^d(t), x^s(t) + x^v(t)) \]

(10) See Olech [10] and Ito [7]

References:


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Appendix

It is a great advantage for the system (S) that we can regard it as a system of difference equations instead of differential equations and that we can show the dynamic behaviors of prices and the stability condition in discrete time. Before we construct the difference system, we must make a necessary modification on equations.

Instead of (2),

\[
\begin{cases}
  \Delta p = -p(t) & \text{if } p(t) < -\phi(x_v^*-x_v(t)) \\
  \Delta p = \phi(x_v^*-x_v(t)), & \text{otherwise}
\end{cases}
\]

(2')

We also replace (5) with:

\[
\Delta x_v = \Delta p(t), \quad x_v^*-x_v(t) - \hat{x} - \theta x_v(t),
\]

(5')

where \( \hat{x} = \min[D(p(t), k), S(p(t), x_v^*-x_v(t))+(1-\theta)x_v(t)] \),

(4')

and \( \Delta p = p(t+1) - p(t) \),

\[
\Delta x_v = x_v(t-1) - x_v(t),
\]

\[1 > \theta \geq 0.\]

For the sake of simplicity, we first assume \( p(t) \) and \( x_v(t) \) are large enough to disregard the possible cases of \( \Delta p = 0 \) or \( x_v = 0 \). We can construct the
difference system \( (T) \) from (1), \( (2') \), (3), \( (4') \) and \( (5') \).

\[
(T) \begin{cases}
DP = \phi(x^*-x^V(t)) \\
Dx^V = S(p(t), x^*-x^V(t)) - D(p(t), k) - \theta x^V(t)
\end{cases}
\]

Secondly we approximate some functions as follows,

\[
(A) \begin{cases}
\phi(x^*-x^V(t)) = \alpha'(x^*-x^V(t)), \\
S(p(t), x^*-x^V(t)) = \beta \cdot p(t) + \gamma \cdot (x^*-x^V(t)) - \varepsilon, \\
D(p(t), k) = -\delta p(t) + \sigma,
\end{cases}
\]

where \( 1 \geq \gamma > 0 \) and \( \alpha, \beta, \delta \) and \( \sigma \) are positive constants; \( \varepsilon \geq 0 \).

From substituting \( (A) \) into \( (T) \), we can derive the following difference equation;

\[
p(t+2) + [\gamma - 2]p(t+1) + [\alpha(\beta + \delta) + 1 - (\gamma + \theta)]p(t) = \alpha(\theta x^v + \sigma + \varepsilon).
\]

(7)

The equilibrium solution of (7), \( \hat{p} \), is calculated by substituting,

\[
\hat{p} = p(t) - p(t+1) = p(t+2), \text{ as } \hat{p} = (\theta x^v + \sigma + \varepsilon)(\beta + \delta).
\]

In general, there is a set of the necessary and sufficient conditions in order that the roots of the second-order characteristic equations, \( \lambda^2 + a_1 \lambda + a_2 = 0 \), be less than unity in absolute value, which means the solution path of the original equation is convergent to the equilibrium value.

The following inequalities constitute the set;

\[
\begin{align*}
1 + a_1 + a_2 &> 0 \quad \text{(i)} \\
1 - a_2 &> 0 \quad \text{(ii)} \\
1 - a_1 + a_2 &> 0 \quad \text{(iii)}
\end{align*}
\]

(See, G. Gandolfo, "Mathematical Methods and and Models in Economic Dynamics", pp.50-62, esp. p.56)

The characteristic equation of the homogeneous part of (7) is as follows;

\[
\lambda^2 + [\gamma - 2] \lambda + [\alpha(\beta + \delta) + 1 - (\gamma + \theta)] = 0.
\]

Since \( \alpha > 0 \) and \( (\beta + \delta) > 0 \), (i) is satisfied. Since \( 2 > (\gamma + \theta) > 0 \), (iii) also holds. Then the necessary and sufficient condition that (7) be convergent is

\[
(\theta + \gamma) - \alpha(\beta + \delta) > 0.
\]

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