ON THE SYSTEM CHARACTERIZED BY THE INTERACTION AND AN EXAMPLE OF CONTROL PROBLEM

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This paper consists of four sections, namely

1. Introduction,
2. The definitions of the influence, the interaction and the system,
3. An example of the system—a model of the disarmament problem—,
4. The qualitative nature of the control problem.

The following notations are used in this paper:

$\boldsymbol{R}$ = the set of all real numbers,

$\boldsymbol{R}_+$ = the set of all positive real numbers,

$\boldsymbol{R}^n$ = the n-dimensional Euclidean space,

$\mathfrak{M}_2(\boldsymbol{R})$ = the set of all 2×2 matrices with real elements.

The notion of a set is taken in this paper as a primitive concept. Set will be denoted by X, Y, Z or S. Correspondence will be denoted by Γ with some suffix. Mapping will be denoted by f or g with some suffix.

1. *Introduction*

In this introductory section I shall present the purposes of this paper and sketch the outlines of the following sections.

Two purposes exist in this paper. The first purpose is to define the concepts of the influence, the interaction and the system. We can find a lot of explanations of these concepts in various investigations, for example, in T. Parsons [6], L. von Bertalanffy [2], and S. Kumon [3]. The definitions of these concepts which I will give in the second section are mathematically formulated referring to these investigations. The second purpose is to construct a model of a system controlled by one agent and to show a qualitative character of the solution of the above-mentioned control problem.

Now let us sketch the outlines of the following sections.

In the second section I will attain my first purpose mentioned above. Namely considering two agents A and B, at first I will define the concept of the influence $\Gamma_{B\to A,A}$ (respectively $\Gamma_{A\to B,B}$) from A to B expected by A (respectively B) and similarly the concept of the influence $\Gamma_{B\to A,B}$ (respectively $\Gamma_{B\to A,A}$) from B to A expected by B (respectively A). Secondly I will introduce such concept of the interaction between A and B as the quartre $(\Gamma_{A\to B,A},\ \Gamma_{A\to B,B},\ \Gamma_{B\to A,B},\ \Gamma_{B\to A,A})$. Thirdly I will define the concept of the system containing two agents. Namely the system $\{A, B\}$ is not a mere set $\{A, B\}$ but the set $\{A, B\}$

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which has a structure given by the above-mentioned concept of the interaction. In this paper I will not touch the concept of the system which contains more than two agents. It is too complicated.

In the third section considering three agents A, B, and C, I will show a model of control problem. Namely at first I will construct a system $\{A, B\}$ introducing an interaction between A and B and secondly I will introduce the concept of the state of the system, which is a point in two dimensional Euclidean space. Thirdly I will construct a control problem in which the agent C tries to transfer the state point of the above-mentioned system $\{A, B\}$ from the present state point to some target. The motional equations of this model are the system of linear ordinary differential equations with order 2.

In the fourth section I will deal with the above-mentioned control problem, and show a qualitative character of the solution of this problem. Namely when S denotes the set of all state points that can be transferred to the target by some control of the agent C, the qualitative nature of S about boundedness can be classified into three groups, i.e.

(i) S is a bounded set in two dimensional Euclidean space,

(ii) S is not a bounded set but does not coincide with two dimensional Euclidean space,

and (iii) S coincides with two dimensional Euclidean space.

The result described in the fourth section explains which case occurs among three above-mentioned cases in accordance with the variation of the constant coefficients of our motional equations.

2. *The definition of the influence, the interaction and the system*

In this section I will define the concepts of the influence, the interaction and the system which contains two agents. They are denoted by A and B. I imagine them to be an individual, a firm, a political party or a state. The set X (respectively Y) denotes all actions a priori available to the agent A (respectively B). When the agent A adopts the action x in X the subset $\Gamma_{A\to B,A}(x)$ (respectively $\Gamma_{A\to B,B}(x)$) of Y is the set of all actions that the agent A (respectively B) expects that the agent B should adopt. The correspondence $\Gamma_{A\to B,A}$ (respectively $\Gamma_{A\to B,B}$) from X to Y defined by

$$X \ni x \longrightarrow \Gamma_{A\to B,A}(x) \text{ (respectively } \Gamma_{A\to B,B}(x)) \subset Y$$

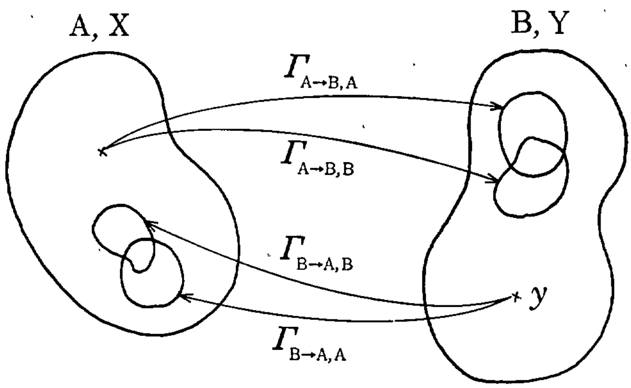
is named such *influence* that A (respectively B) *expects to be* the influence *from A to B*. It is natural that the correspondence $\Gamma_{A\to B,A}$ should not always be the same as the correspondence $\Gamma_{A\to B,B}$. If $\Gamma_{A\to B,A}$ is the same as $\Gamma_{A\to B,B}$, let us use the simpler notation $\Gamma_{A\to B}$ instead of $\Gamma_{A\to B,A}$ or $\Gamma_{A\to B,B}$ and call the correspondence $\Gamma_{A\to B}$ the *influence from A to B*.

Similarly when the agent B adopts the action y in Y the subset $\Gamma_{B\to A,B}(y)$ (respectively $\Gamma_{B\to A,A}(y)$) of X is the set of all actions that the agent B (respectively A) expects that the agent A should adopt. The correspondence $\Gamma_{B\to A,B}$ (respectively $\Gamma_{B\to A,A}$) from X to Y defined by

$Y \ni y \longrightarrow \Gamma_{B\to A,B}(y)$ (respectively $\Gamma_{B\to A,A}(y)) \subset X$ is named such *influence* that B (respectively A) *expects to be* the influence *from B to A*.

It is natural that $\Gamma_{B\to A,B}$ should not always be the same as $\Gamma_{B\to A,A}$. If $\Gamma_{B\to A,B}$ is the same as $\Gamma_{B\to A,A}$, let us use the simpler notation $\Gamma_{B\to A}$ instead of $\Gamma_{B\to A,B}$ or $\Gamma_{B\to A,A}$ and call

FIG. 1.



the correspondence $\Gamma_{B\to A}$ the *influence from B to A*.

Now when $\Gamma_{A\to B,A}$ and $\Gamma_{A\to B,B}$ (respectively $\Gamma_{B\to A,B}$ and $\Gamma_{B\to A,A}$) satisfy the relation

$$\phi \neq \Gamma_{A\to B,A}(x), \quad \Gamma_{A\to B,B}(x) \subsetneqq Y \qquad \text{for some } x \in X,$$
$$\text{(respectively } \phi \neq \Gamma_{B\to A,B}(y), \quad \Gamma_{B\to A,A}(y) \subsetneqq X \qquad \text{for some } y \in Y),$$

and $\Gamma_{B\to A,B}$ and $\Gamma_{B\to A,A}$ (respectively $\Gamma_{A\to B,A}$ and $\Gamma_{A\to B,B}$) satisfy the relation

$$\Gamma_{B\to A,B}(y) = \Gamma_{B\to A,A}(y) = X \qquad \text{for all } y \in Y,$$
$$\text{(respectively } \Gamma_{A\to B,A}(x) = \Gamma_{A\to B,B}(x) = Y \qquad \text{for all } x \in X),$$

I say that the influence from A to B (respectively from B to A) *exists* but the influence from B to A (respectively A to B) *does not exist*.

The quatre ($\Gamma_{A\to B,A}$, $\Gamma_{A\to B,B}$, $\Gamma_{B\to A,B}$, $\Gamma_{B\to A,A}$) constructed by four correspondences is named the *interaction between two agents A and B*. If $\Gamma_{A\to B,A}$ is equal to $\Gamma_{A\to B,B}$ and $\Gamma_{B\to A,B}$ is equal to $\Gamma_{B\to A,A}$, the pair ($\Gamma_{A\to B}$, $\Gamma_{B\to A}$) is named the *interaction between two agents A and B*.

Using the above-mentioned concepts, let us define the concept of the system which contains two agents. The set $\{A, B\}$ of two agents A and B which have reciprocally an interaction between them is named a system $\{A, B\}$. Therefore the system is not only a mere set $\{A, B\}$ but also the structural set $\{A, B\}$. When the number of agents is more than two, the extension of our definitions is considerably complicated. In the next section I will show one example of the system which contains three agents.

3. *An example of the system—a model of the disarmament problem—*

In this section I will show one example of the system defined in section 2. This example is a model to analyse the disarmament problem between U.S.A. and U.S.S.R. In this model three agents A, B and C appear on the scene. The present time is denoted by t_0 and the set X (respectively Y) of all a priori available actions of A (respectively B) is defined by

$$X=Y=\{\zeta:[t_0,t_1]\longrightarrow \boldsymbol{R};\ \zeta \text{ is a piecewise continuously differentiable mapping}\},$$

where t_1 depends on ζ. The set Z of all a priori available actions of C is defined by

$$Z=\{(\xi,\eta);\ \xi,\ \eta:[t_0,\ t_1]\longrightarrow \boldsymbol{R},\ \xi,\ \eta \text{ are piecewise continuously differentiable mappings}\}$$

where t_1 depends on ξ and η.

Now let us introduce the interaction between C and A and between C and B using the matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathfrak{M}_2(\boldsymbol{R})$, where $\beta,\ \gamma \in \boldsymbol{R}_+$. The influence $\Gamma_{C\to A,C}$ from C to A expected by C is the same as the influence $\Gamma_{C\to A,A}$ from C to A expected by A, and they are denoted by the common notation $\Gamma_{C\to A}$ as defined in the previous section, and the influence $\Gamma_{C\to A}$ from C to A is defined as follows:

$$Z \ni (\xi,\ \eta)\longrightarrow \Gamma_{C\to A}(\xi,\ \eta)=\text{the set of all solutions of}$$

$$\frac{d^2x}{dt^2}-(\alpha+\delta)\frac{dx}{dt}+(\alpha\delta-\beta\gamma)x=\frac{d\xi}{dt}-\delta\xi+\beta\eta.$$

Similarly the influence $\Gamma_{C\to B,C}$ from C to B expected by C is the same as the influence $\Gamma_{C\to B,B}$ from C to B expected by B, and they are denoted by the common notation $\Gamma_{C\to B}$ as defined in the previous section, and the influence $\Gamma_{C\to B}$ from C to B is defined as follows:

$$Z \ni (\xi,\ \eta)\longrightarrow \Gamma_{C\to B}(\xi,\ \eta)=\text{the set of all solutions of}$$

$$\frac{d^2y}{dt^2}-(\alpha+\delta)\frac{dy}{dt}+(\alpha\delta-\beta\gamma)y=\frac{d\eta}{dt}+\gamma\xi-\alpha\eta.$$

Conversely let us define that the influences from A to C and from B to C do not exist, namely

$$\Gamma_{A\to C,A}(x)=\Gamma_{A\to C,C}(x)=\Gamma_{A\to C}(x)=Z \qquad \text{for all } x\in X$$

and

$$\Gamma_{B\to C,B}(y)=\Gamma_{B\to C,C}(y)=\Gamma_{B\to C}(y)=Z \qquad \text{for all } y\in Y.$$

The interactions between C and A and between C and B have been defined as mentioned above. In the next place let us define the interaction between A and B. It depends on the influences $\Gamma_{C\to A}$ and $\Gamma_{C\to B}$. Namely the influence $\Gamma_{A\to B,A}$ from A to B expected by A is the same as the influence $\Gamma_{A\to B,B}$ from A to B expected by B, and they are denoted by the common notation $\Gamma_{A\to B}$ as defined in the previous section, and for any element $(\xi,\ \eta)$ of Z the influence $\Gamma_{A\to B}$ from A to B is defined as follows:

$$X\ni\varphi\longrightarrow\Gamma_{A\to B}(\varphi)=\left\{\begin{array}{ll}\left\{\dfrac{1}{\beta}\dfrac{d\varphi}{dt}-\dfrac{\alpha}{\beta}\varphi-\dfrac{1}{\beta}\xi\right\}\subset Y; & \text{if } \varphi\in\Gamma_{C\to A}(\xi,\ \eta)\\ Y & ;\ \text{if } \varphi\notin\Gamma_{C\to A}(\xi,\ \eta)\end{array}\right\}$$

Let us denote the element of $\Gamma_{A\to B}(\varphi)$ by $f_{(\xi,\eta)}(\varphi)$ when φ belongs to $\Gamma_{C\to A}(\xi,\ \eta)$. Namely

$$f_{(\xi,\eta)}(\varphi)=\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi.$$

Now it is clear that $f_{(\xi,\eta)}(\varphi)$ is a solution of $\dfrac{d^2y}{dt}-(\alpha+\delta)\dfrac{dy}{dt}+(\alpha\delta-\beta\gamma)y=\dfrac{d\eta}{dt}+\gamma\xi-\alpha\eta$ when φ belongs to $\Gamma_{C\to A}(\xi,\ \eta)$, since

$$\frac{d^2}{dt^2}(f_{(\xi,\eta)}(\varphi))-(\alpha+\delta)\frac{d}{dt}(f_{(\xi,\eta)}(\varphi))+(\alpha\delta-\beta\gamma)f_{(\xi,\eta)}(\varphi)$$
$$=\frac{1}{\beta}\frac{d^3\varphi}{dt^3}-\frac{\alpha}{\beta}\frac{d^2\varphi}{dt^2}-\frac{1}{\beta}\frac{d^2\xi}{dt^2}-(\alpha+\delta)\left(\frac{1}{\beta}\frac{d^2\varphi}{dt^2}-\frac{\alpha}{\beta}\frac{d\varphi}{dt}-\frac{1}{\beta}\frac{d\xi}{dt}\right)$$
$$+(\alpha\delta-\beta\gamma)\left(\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi\right)$$
$$=\frac{1}{\beta}\left\{(\alpha+\delta)\frac{d^2\varphi}{dt^2}-(\alpha\delta-\beta\gamma)\frac{d\varphi}{dt}+\frac{d^2\xi}{dt^2}-\delta\frac{d\xi}{dt}+\beta\frac{d\eta}{dt}\right\}-\frac{\alpha}{\beta}\frac{d^2\varphi}{dt^2}-\frac{1}{\beta}\frac{d^2\xi}{dt^2}$$
$$-(\alpha+\delta)\left(\frac{1}{\beta}\frac{d^2\varphi}{dt^2}-\frac{\alpha}{\beta}\frac{d\varphi}{dt}-\frac{1}{\beta}\frac{d\xi}{dt}\right)+(\alpha\delta-\beta\gamma)\left(\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi\right)$$
$$=-\frac{\alpha}{\beta}\left\{\frac{d^2\varphi}{dt^2}-(\alpha+\delta)\frac{d\varphi}{dt}+(\alpha\delta-\beta\gamma)\varphi\right\}+\frac{\alpha}{\beta}\frac{d\xi}{dt}+\frac{d\eta}{dt}-\frac{\alpha\delta-\beta\gamma}{\beta}\xi$$
$$=-\frac{\alpha}{\beta}\left(\frac{d\xi}{dt}-\delta\xi+\beta\eta\right)+\frac{\beta}{\alpha}\frac{d\xi}{dt}+\frac{d\eta}{dt}-\frac{\alpha\delta-\beta\gamma}{\beta}\xi$$
$$=\frac{d\eta}{dt}+\gamma\xi-\alpha\eta.$$

Accordingly $f_{(\xi,\eta)}$ is a mapping from $\Gamma_{C\to A}(\xi,\eta)$ to $\Gamma_{C\to B}(\xi,\eta)$.

Similarly the influence $\Gamma_{B\to A,B}$ from B to A expected by B is the same as the influence $\Gamma_{B\to A,A}$ from B to A expected by A, and they are denoted by the common notation $\Gamma_{B\to A}$ as defined in the previous section, and for any element (ξ,η) of Z the influence $\Gamma_{B\to A}$ from B to A is defined as follows:

$$Y\ni\psi\longrightarrow\Gamma_{B\to A}(\psi)=\begin{Bmatrix}\left\{\frac{1}{\gamma}\frac{d\psi}{dt}-\frac{\delta}{\gamma}\psi-\frac{1}{\gamma}\eta\right\}\subset X; & \text{if } \psi\in\Gamma_{C\to B}(\xi,\eta)\\ X & ;\text{ if } \psi\notin\Gamma_{C\to B}(\xi,\eta)\end{Bmatrix}$$

Let us denote the element of $\Gamma_{B\to A}(\psi)$ by $g_{(\xi,\eta)}(\psi)$ when ψ belongs to $\Gamma_{C\to B}(\xi,\eta)$. Namely

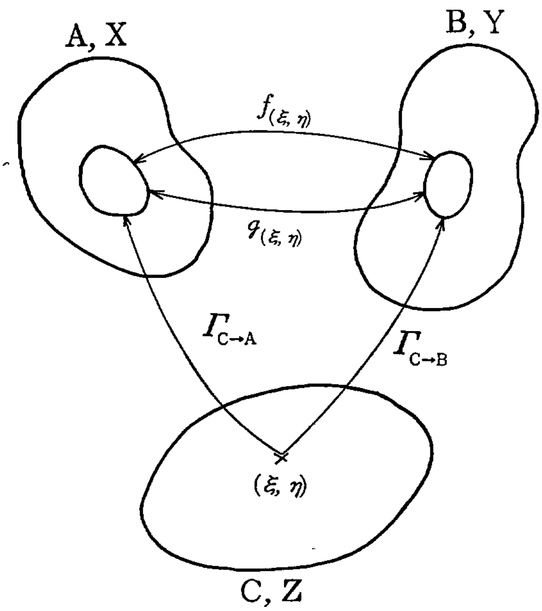
$$g_{(\xi,\eta)}(\psi)=\frac{1}{\gamma}\frac{d\psi}{dt}-\frac{\delta}{\gamma}\psi-\frac{1}{\gamma}\eta.$$

Now it is clear that $g_{(\xi,\eta)}(\psi)$ is a solution of $\frac{d^2x}{dt^2}-(\alpha+\delta)\frac{dx}{dt}+(\alpha\delta-\beta\gamma)x=\frac{d\xi}{dt}-\delta\xi+\beta\eta$ when ψ belongs to $\Gamma_{C\to B}(\xi,\eta)$. Accordingly $g_{(\xi,\eta)}$ is a mapping from $\Gamma_{C\to B}(\xi,\eta)$ to $\Gamma_{C\to A}(\xi,\eta)$. Thus the interaction between A and B has been decided according to each element (ξ,η) of Z.

Here it should be remarked that $f_{(\xi,\eta)}$ is a bijection from $\Gamma_{C\to A}(\xi,\eta)$ to $\Gamma_{C\to B}(\xi,\eta)$ and also $g_{(\xi,\eta)}$ is a bijection from $\Gamma_{C\to B}(\xi,\eta)$ to $\Gamma_{C\to A}(\xi,\eta)$ and furthermore $g_{(\xi,\eta)}{}^{-1}$ is equal to $f_{(\xi,\eta)}$ and $f_{(\xi,\eta)}{}^{-1}$ is equal to $g_{(\xi,\eta)}$, because for any element φ of $\Gamma_{C\to A}(\xi,\eta)$

$$(g_{(\xi,\eta)}\circ f_{(\xi,\eta)})(\varphi)=g_{(\xi,\eta)}(f_{(\xi,\eta)}(\varphi))$$
$$=g_{(\xi,\eta)}\left(\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi\right)$$
$$=\frac{1}{\gamma}\left(\frac{1}{\beta}\frac{d^2\varphi}{dt^2}-\frac{\alpha}{\beta}\frac{d\varphi}{dt}-\frac{1}{\beta}\frac{d\xi}{dt}\right)-\frac{\delta}{\gamma}\left(\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi\right)-\frac{1}{\gamma}\eta$$
$$=\frac{1}{\beta\gamma}\left\{\frac{d^2\varphi}{dt^2}-(\alpha+\delta)\frac{d\varphi}{dt}+(\alpha\delta-\beta\gamma)\varphi\right\}+\varphi-\frac{1}{\beta\gamma}\frac{d\xi}{dt}+\frac{\delta}{\beta\gamma}\xi-\frac{1}{\gamma}\eta$$
$$=\frac{1}{\beta\gamma}\left(\frac{d\xi}{dt}-\delta\xi+\beta\eta\right)+\varphi-\frac{1}{\beta\gamma}\frac{d\xi}{dt}+\frac{\delta}{\beta\gamma}\xi-\frac{1}{\gamma}\eta=\varphi$$

FIG. 2.



and for any element ψ of $\Gamma_{C\to B}(\xi, \eta)$

$$\begin{aligned}
(f_{(\xi,\eta)}\circ g_{(\xi,\eta)})(\psi) &= f_{(\xi,\eta)}(g_{(\xi,\eta)}(\psi))\\
&= f_{(\xi,\eta)}\left(\frac{1}{\gamma}\frac{d\psi}{dt}-\frac{\delta}{\gamma}\psi-\frac{1}{\gamma}\eta\right)\\
&= \frac{1}{\beta}\left(\frac{1}{\gamma}\frac{d^2\psi}{dt^2}-\frac{\delta}{\gamma}\frac{d\psi}{dt}-\frac{1}{\gamma}\frac{d\eta}{dt}\right)-\frac{\alpha}{\beta}\left(\frac{1}{\gamma}\frac{d\psi}{dt}-\frac{\delta}{\gamma}\psi-\frac{1}{\gamma}\eta\right)-\frac{1}{\beta}\xi\\
&= \frac{1}{\beta\gamma}\left\{\frac{d^2\psi}{dt^2}-(\alpha+\delta)\frac{d\psi}{dt}+(\alpha\delta-\beta\gamma)\psi\right\}+\psi-\frac{1}{\beta\gamma}\frac{d\eta}{dt}+\frac{\alpha}{\beta\gamma}\eta-\frac{1}{\beta}\xi\\
&= \frac{1}{\beta\gamma}\left(\frac{d\eta}{dt}+\gamma\xi-\alpha\eta\right)+\psi-\frac{1}{\beta\gamma}\frac{d\eta}{dt}+\frac{\alpha}{\beta\gamma}\eta-\frac{1}{\beta}\xi=\psi.
\end{aligned}$$

For any $(\xi, \eta)\in Z$ let us denote by $\Sigma_{(\xi,\eta)}$ the system $\{A, B\}$ which has the interaction $(\Gamma_{A\to B}, \Gamma_{B\to A})$ mentioned above.

Next we must pay attention to the following fact that when (ξ, η) belongs to Z, for an arbitrary element φ (respectively ψ) of $\Gamma_{C\to A}(\xi, \eta)$ (respectively $\Gamma_{C\to B}(\xi, \eta)$) $(\varphi, f_{(\xi,\eta)}(\varphi))$ (respectively $(g_{(\xi,\eta)}(\psi), \psi)$) is a solution of the system of linear differential equations,

(3-1) $$\begin{cases} \dfrac{dx}{dt}=\alpha x+\beta y+\xi,\\ \dfrac{dy}{dt}=\gamma x+\delta y+\eta, \end{cases}$$

because

$$\alpha\varphi+\beta f_{(\xi,\eta)}(\varphi)+\xi=\alpha\varphi+\left(\frac{d\varphi}{dt}-\alpha\varphi-\xi\right)+\xi=\frac{d\varphi}{dr},$$

$$\begin{aligned}\frac{d}{dt}(f_{(\xi,\eta)}(\varphi))&=\frac{1}{\beta}\frac{d^2\varphi}{dt^2}-\frac{\alpha}{\beta}\frac{d\varphi}{dt}-\frac{1}{\beta}\frac{d\xi}{dt}\\&=\frac{1}{\beta}\left\{(\alpha+\delta)\frac{d\varphi}{dt}-(\alpha\delta-\beta\gamma)\varphi+\frac{d\xi}{dt}-\delta\xi+\beta\eta\right\}-\frac{\alpha}{\beta}\frac{d\varphi}{dt}-\frac{1}{\beta}\frac{d\xi}{dt}\\&=\gamma\varphi+\delta\left(\frac{1}{\beta}\frac{d\varphi}{dt}-\frac{\alpha}{\beta}\varphi-\frac{1}{\beta}\xi\right)+\eta\\&=\gamma\varphi+\delta f_{(\xi,\eta)}(\varphi)+\eta,\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(g_{(\xi,\eta)}(\psi))&=\frac{1}{\gamma}\frac{d^2\psi}{dt^2}-\frac{\delta}{\gamma}\frac{d\psi}{dt}-\frac{1}{\gamma}\frac{d\eta}{dt}\\&=\frac{1}{\gamma}\left\{(\alpha+\delta)\frac{d\psi}{dt}-(\alpha\delta-\beta\gamma)\psi+\frac{d\eta}{dt}+\gamma\xi-\alpha\eta\right\}-\frac{\delta}{\gamma}\frac{d\psi}{dt}-\frac{1}{\gamma}\frac{d\eta}{dt}\\&=\alpha\left(\frac{1}{\gamma}\frac{d\psi}{dt}-\frac{\delta}{\gamma}\psi-\frac{1}{\gamma}\eta\right)+\beta\psi+\xi\\&=\alpha g_{(\xi,\eta)}(\psi)+\beta\psi+\xi,\end{aligned}$$

and
$$\gamma g_{(\xi,\eta)}(\psi)+\delta\psi+\eta=\left(\frac{d\psi}{dt}-\delta\psi-\eta\right)+\delta\psi+\eta=\frac{d\psi}{dt}.$$

Now let us introduce the concept of a state point of our system $\{A, B\}$. When the agent C adopts the strategy (ξ, η) in Z, for the action φ (respectively ψ) of the agent A (respectively B) I call the point $(\varphi(t), (f_{(\xi,\eta)}(\varphi))(t))$ (respectively $((g_{(\xi,\eta)}(\psi))(t), \psi(t))$ in $\boldsymbol{R}^2$ the state point of the system $\{A, B\}$ at t.

Lastly we shall explain that the above-mentioned system $\Sigma_{(\xi,\eta)}$ is a model of the disarmament problem between U.S.A. and U.S.S.R. Let the first agent A of the system $\Sigma_{(\xi,\eta)}$ be U.S.A., the second agent B of the system $\Sigma_{(\xi,\eta)}$ be U.S.S.R., and furthermore the third agent C be the group of all the states except U.S.A. and U.S.S.R. Let us understand that for the element φ of X, the set of all actions of U.S.A., $\varphi(t)$ represents the military force at time t ($t\geqq t_0$: present time) and for the element ψ of Y, the set of all actions of U.S.S.R., $\varphi(t)$ represents the military force at time t ($t\geqq t_0$: present time). For each element (ξ, η) of Z ξ (respectively η) represents the strategy against U.S.A. (respectively U.S.S.R.) which all the states except U.S.A. and U.S.S.R. plan about the disarmament probelm between U.S.A. and U.S.S.R. Under the condition that the agent C adopts the strategy (ξ, η), the action of U.S.A. must be in $\Gamma_{C\to A}(\xi, \eta)$ and the action of U.S.S.R. must be in $\Gamma_{C\to B}(\xi, \eta)$. If at the start U.S.A. adopts an action φ in $\Gamma_{C\to A}(\xi, \eta)$, U.S.S.R. must adopt the action $f_{(\xi,\eta)}(\varphi)$ in $\Gamma_{C\to B}(\xi, \eta)$ and if at the start U.S.S.R. adopts an action ψ in $\Gamma_{C\to B}(\xi, \eta)$, U.S.A. must adopt the action $g_{(\xi,\eta)}(\psi)$ in $\Gamma_{C\to A}(\xi, \eta)$.

The rate of increase of military force of U.S.A. (respectively U.S.S.R.) must depend on the military force of U.S.A. and U.S.S.R. and depend on the strategy ξ (respectively η) which the agent C plans. In my model the rate of increase of military force linearly depends on them. Since I am considering that the rate of increase of military force of U.S.A. (respectively U.S.S.R.) increases in proportion to the military force of U.S.S.R. (respectively U.S.A.), the positivity of β and γ is assumed.

After all the above-mentioned model is convenient for the analysis of the control probelm in which the agent C controls the system $\{A, B\}$. In the next section I will analyse this control problem.

4. *The qualitative nature of the control problem*

In this section I will deal with the control problem which has been explained in the previous section. In our control problem the controller is the agent C and the object of control is the state point of the system $\{A, B\}$. The motional equations of the state point of the system $\{A, B\}$ are (3–1) and (x_0, y_0) denotes the state point of the present time. For the simplification of calculation let us assume that the target of the controller is the origin o_2 in $\boldsymbol{R}^2$. Therefore in this model the group of all states except U.S.A. and U.S.S.R. intends the military force of them to vanish at the same time.

Now without loss of generality let us assume that

$$-1 \leqq \xi(t), \eta(t) \leqq 1 \qquad \text{for all} \qquad t \in [t_0, t_1]$$

where $[t_0, t_1]$ is the defining interval of (ξ, η) in Z, namely the control domain of this control problem is a closed interval $[-1, 1]$. And let S be the set of all state points from which the controller can transfer to the tarteg o_2 with some control (ξ, η) in Z. We can consider three types of the set S as follows:

(i) S is a bounded subset of $\boldsymbol{R}^2$,

(ii) S is not a bounded subset of $\boldsymbol{R}^2$ and moreover does not coincide with the whole space $\boldsymbol{R}^2$,

(iii) S coincides with the whole space $\boldsymbol{R}^2$.

The problem which I will take up in this section is which case among three cases above mentioned occurs according to the variation of coefficients of (3–1). Namely considering the correspondence

$$\boldsymbol{R}^2 \times \boldsymbol{R}_+{}^2 \ni ((\alpha, \delta), (\beta, \gamma)) \longrightarrow S \subset \boldsymbol{R}^2$$

the problem is to decide the set of $((\alpha, \delta), (\beta, \gamma)) \ni \boldsymbol{R}^2 \times \boldsymbol{R}_+{}^2$ so that S has type (i), (ii) or (iii) respectively. But using the existence theorem of linear time optimal problem it is necessary only to decide the set whose points can be transferred to the target of our system by the time optimal controls.

Let $M \in \mathfrak{M}_2(\boldsymbol{R}^2)$ be the coefficient matrix of (3–1). The characteristic polynomial $f_M(t)$ and its discriminant D are

$$f_M(t) = t^2 - (\alpha + \delta)t + \alpha\delta - \beta\gamma,$$

and

$$D = (\alpha - \delta)^2 + 4\beta\gamma.$$

Since β and γ are positive real numbers, M has two different real characteristic roots σ and τ, say $\tau < \sigma$.

At first let us consider the characteristic roots σ and τ dividing into five classes, namely

$$
\begin{array}{lllll}
\text{I.} & 0 < \tau < \sigma \Longleftrightarrow \alpha\delta > \beta\gamma & \& & \alpha + \delta > 0, \\
\text{II.} & 0 = \tau < \sigma \Longleftrightarrow \alpha\delta = \beta\gamma & \& & \alpha + \delta > 0, \\
\text{III.} & \tau < 0 < \sigma \Longleftrightarrow \alpha\delta < \beta\gamma, & & \\
\text{IV.} & \tau < 0 = \sigma \Longleftrightarrow \alpha\delta = \beta\gamma & \& & \alpha + \delta < 0, \\
\text{V.} & \tau < \sigma < 0 \Longleftrightarrow \alpha\delta > \beta\gamma & \& & \alpha + \delta < 0.
\end{array}
$$

The characteristic spaces of σ and τ are

$$W_\sigma=\left\{\begin{pmatrix}\beta\lambda\\(\sigma-\alpha)\lambda\end{pmatrix}\in\boldsymbol{R}^2;\ \lambda\in\boldsymbol{R}\right\},$$

and

$$W_\tau=\left\{\begin{pmatrix}\beta\mu\\(\tau-\alpha)\mu\end{pmatrix}\in\boldsymbol{R}^2;\ \mu\in\boldsymbol{R}\right\}.$$

Using the base of these spaces, let us define a nonsingular matrix

$$P=\begin{pmatrix}\beta & \beta\\ \sigma-\alpha & \tau-\alpha\end{pmatrix}.$$

Then we get

$$P^{-1}=\begin{pmatrix}(\alpha-\tau)/\beta(\sigma-\tau) & 1/(\sigma-\tau)\\ (\sigma-\alpha)/\beta(\sigma-\tau) & -1/(\sigma-\tau)\end{pmatrix}$$

and multiplying P^{-1} from the left side of (3–1), we get

$$P^{-1}\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=P^{-1}MP\cdot P^{-1}\begin{pmatrix}x\\y\end{pmatrix}+P^{-1}\begin{pmatrix}\xi\\\eta\end{pmatrix},$$

that is to say

$$\text{(4–1)}\qquad \begin{cases}\dfrac{dx_1}{dt}=\sigma x_1+u_1,\\[2mm] \dfrac{dx_2}{dt}=\tau x_2+u_2,\end{cases}$$

where

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=P^{-1}\begin{pmatrix}x\\y\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}u_1\\u_2\end{pmatrix}=P^{-1}\begin{pmatrix}\xi\\\eta\end{pmatrix}.$$

By the transformation mentioned above, we get a new control domain V, which is the convex closure of four points

$$e^{(1)}=\begin{pmatrix}e_1^{(1)}\\e_2^{(1)}\end{pmatrix}=P^{-1}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}(\alpha-\tau)/\beta(\sigma-\tau)+1/(\sigma-\tau)\\ \{\sigma-(\alpha+\beta)\}/\beta(\sigma-\tau)\end{pmatrix},$$
$$e^{(2)}=\begin{pmatrix}e_1^{(2)}\\e_2^{(2)}\end{pmatrix}=P^{-1}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}\{\tau-(\alpha-\beta)\}/\beta(\sigma-\tau)\\ -(\sigma-\alpha)/\beta(\sigma-\tau)-1/(\sigma-\tau)\end{pmatrix},$$
$$e^{(3)}=\begin{pmatrix}e_1^{(3)}\\e_2^{(3)}\end{pmatrix}=P^{-1}\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}-(\alpha-\tau)/\beta(\sigma-\tau)-1/(\sigma-\tau)\\ -\{\sigma-(\alpha+\beta)\}/\beta(\sigma-\tau)\end{pmatrix},$$
$$e^{(4)}=\begin{pmatrix}e_1^{(4)}\\e_2^{(4)}\end{pmatrix}=P^{-1}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-\{\tau-(\alpha-\beta)\}/\beta(\sigma-\tau)\\ (\sigma-\alpha)/\beta(\sigma-\tau)+1/(\sigma-\tau)\end{pmatrix}.$$

Now it is clear that $e_1^{(1)}>0$, $e_2^{(2)}<0$, $e_1^{(3)}<0$, $e_2^{(4)}>0$, since $f_M(\alpha)=-\beta\gamma<0$ and $\sigma-\tau>0$. Moreover $e^{(1)}+e^{(3)}=0_2$ and $e^{(2)}+e^{(4)}=0_2$. Therefore the sign of $e_2^{(1)}$ and $e_2^{(3)}$ depends on the sign of $\sigma-(\alpha+\beta)$, and the sign of $e_1^{(2)}$ and $e_1^{(4)}$ depends on the sign of $\tau-(\alpha-\beta)$.

Now Hamiltonian function of (4-1) is

$$\begin{aligned}&H(\psi_1,\psi_2;x_1,x_2,u_1,u_2)\\&\quad=\psi_1(\sigma x_1+u_1)+\psi_2(\tau x_2+u_2)\\&\quad=(\psi_1\sigma x_1+\psi_2\tau x_2)+(\psi_1u_1+\psi_2u_2),\end{aligned}$$

and Pontrjagin Maximum Principle is

$$\operatorname*{Max}_{V}\{\psi_1(t)u_1+\psi_2(t)u_2\}=\psi_1(t)u_1(t)+\psi_2(t)u_2(t)$$

for all time t in the control interval, where ψ_1 and ψ_2 are adjoint variables of (4-1) and $\psi_1(t)$ and $\psi_2(t)$ are the solutions of adjoint differential equations

$$\text{(4-2)} \qquad \frac{d\psi_1}{dt}=-\sigma\psi_1, \quad \frac{d\psi_2}{dt}=-\tau\psi_2.$$

But by the theorem of finite number of switching times with respect to linear time optimal problem, the Maximum of $\psi_1(t)u_1+\psi_2(t)u_2$ in V is attained at some extreme point of V, namely $e^{(1)}$, $e^{(2)}$, $e^{(3)}$, or $e^{(4)}$. Therefore we can transform the motional equation (4-1) into the following

$$\text{(4-3)} \qquad \begin{cases} \dfrac{dx_1}{dt}=\sigma x_1+e_1^{(i)} \\ \dfrac{dx_2}{dt}=\tau x_2+e_2^{(i)} \end{cases} \quad i=1, 2, 3, 4.$$

It is well known that i in (4-3) depends on the solution of (4-2). Therefore let us consider the differential equation (4-2). The solution of (4-2) which has the initial condition $\psi_1(t_0)=\psi_1^{(0)}$ and $\psi_2(t_0)=\psi_2^{(0)}$, are clearly

$$\psi_1(t)=\psi_1^{(0)}e^{-\sigma(t-t_0)}, \quad \psi_2(t)=\psi_2^{(0)}e^{-\tau(t-t_0)}.$$

According to five cases I to V of characteristic roots of $f_M(t)$, all the solutions of (4-2) are

FIG. 3.
(Case 1)

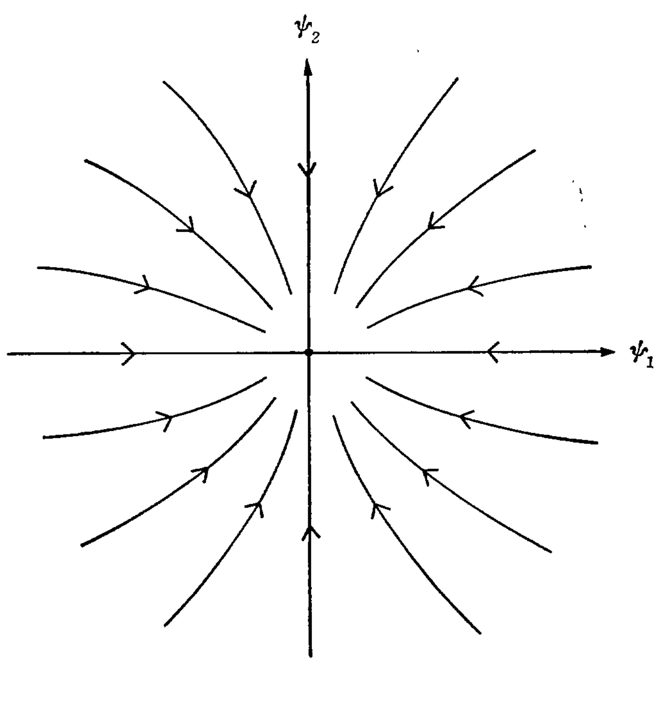


FIG. 4.
(Case II)

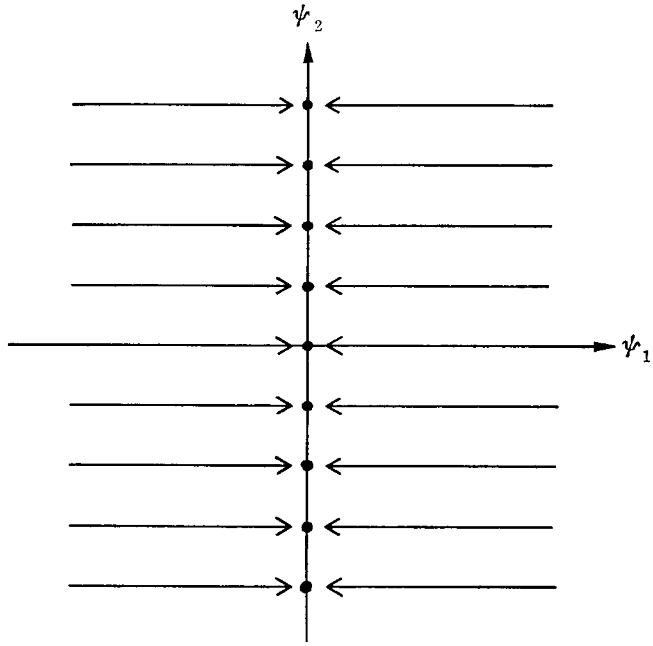


FIG. 5.
(Case III)

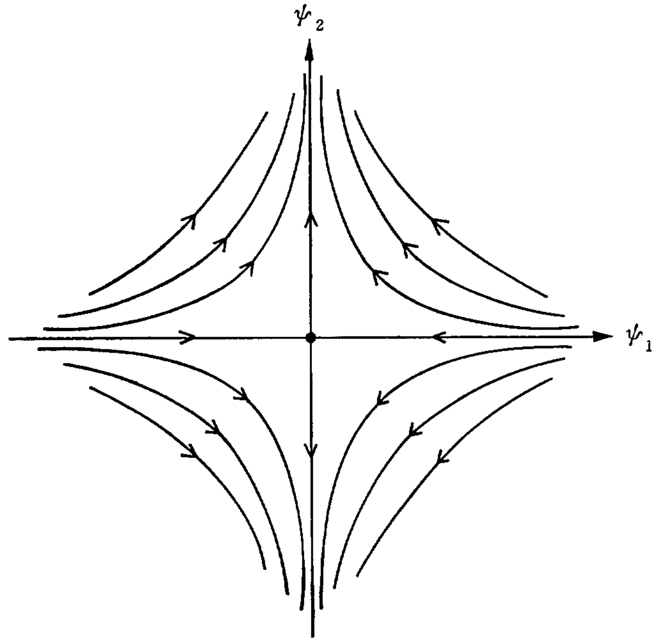


FIG. 6.
(Case IV)

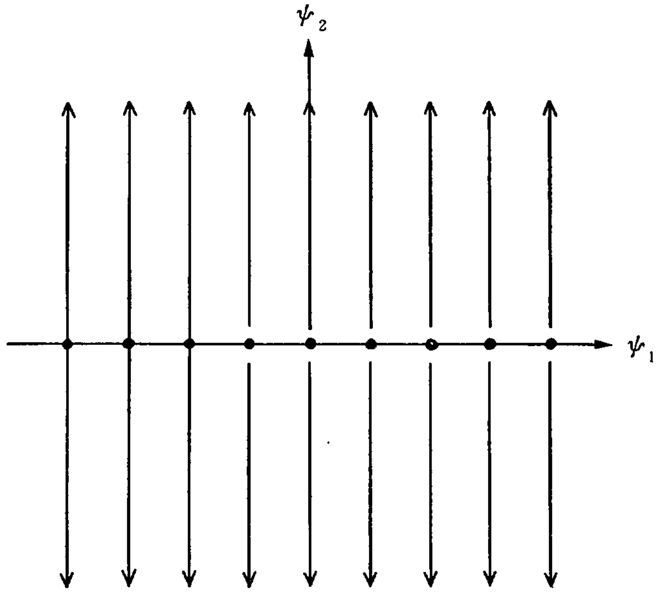
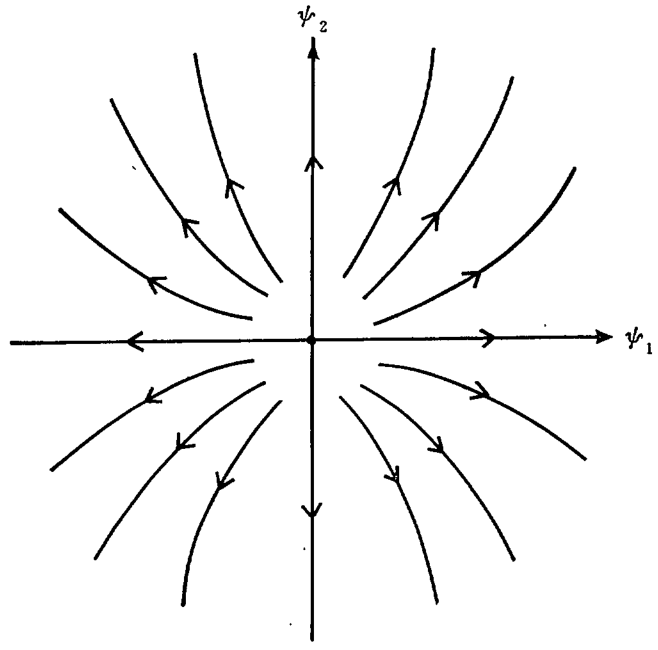


FIG. 7.
(Case V)



drawn as figures 3 to 7. And when $\sigma \neq 0$ and $\tau \neq 0$, that is, in the case of I, III and V we introduce

$$f^{(i)} = \begin{pmatrix} f_1^{(i)} \\ f_2^{(i)} \end{pmatrix} = \begin{pmatrix} -1/\sigma & 0 \\ 0 & -1/\tau \end{pmatrix} e^{(i)}, \quad i=1, 2, 3, 4.$$

Then the motional equation (4-3) is transformed into

$$\text{(4-4)} \qquad \begin{cases} \dfrac{d}{dt}(x_1 - f_1^{(i)}) = \sigma(x_1 - f_1^{(i)}), \\ \dfrac{d}{dt}(x_2 - f_2^{(i)}) = \tau(x_2 - f_2^{(i)}), \end{cases} \qquad i=1, 2, 3, 4.$$

All the solutions of this equation (4-4) are drawn as figures 8 to 10. And we get

$$f^{(1)} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \end{pmatrix} = \begin{pmatrix} -(\alpha-\tau)/\beta\sigma(\sigma-\tau) - 1/\sigma(\sigma-\tau) \\ -\{\sigma-(\alpha+\beta)\}/\beta\tau(\sigma-\tau) \end{pmatrix},$$

$$f^{(2)} = \begin{pmatrix} f_1^{(2)} \\ f_2^{(2)} \end{pmatrix} = \begin{pmatrix} -\{\tau-(\alpha-\beta)\}/\beta\sigma(\sigma-\tau) \\ (\sigma-\alpha)/\beta\tau(\sigma-\tau) + 1/\tau(\sigma-\tau) \end{pmatrix},$$

$$f^{(3)} = \begin{pmatrix} f_1^{(3)} \\ f_2^{(3)} \end{pmatrix} = \begin{pmatrix} (\alpha-\tau)/\beta\sigma(\sigma-\tau) + 1/\sigma(\sigma-\tau) \\ \{\sigma-(\alpha+\beta)\}/\beta\tau(\sigma-\tau) \end{pmatrix},$$

$$f^{(4)} = \begin{pmatrix} f_1^{(4)} \\ f_2^{(4)} \end{pmatrix} = \begin{pmatrix} \{\tau-(\alpha-\beta)\}/\beta\sigma(\sigma-\tau) \\ -(\sigma-\alpha)/\beta\tau(\sigma-\tau) - 1/\tau(\sigma-\tau) \end{pmatrix}.$$

Now it is clear that

FIG. 8.
(Case I)

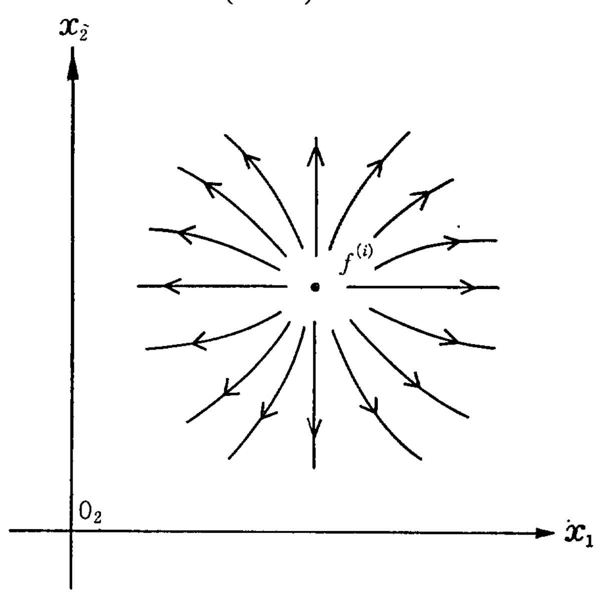


FIG. 9.
(Case III)

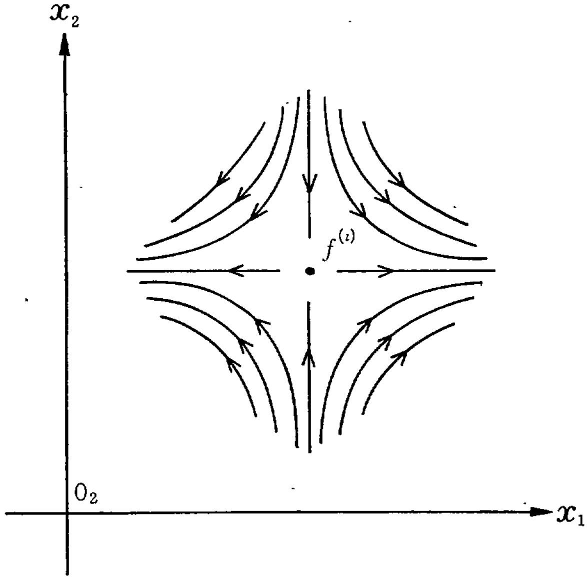
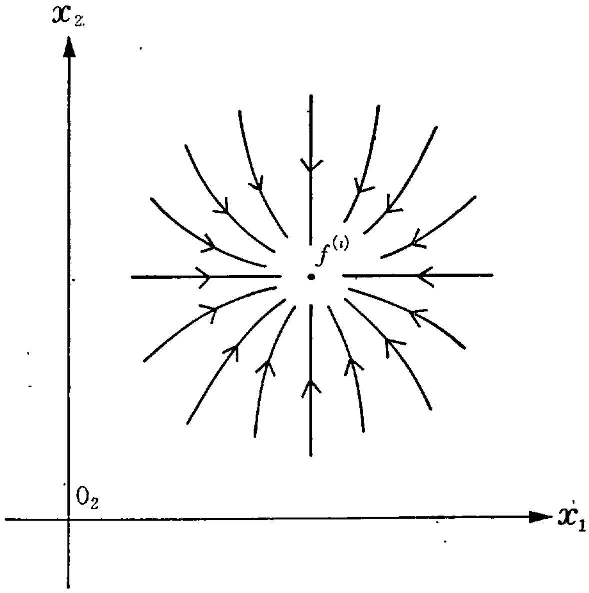


FIG. 10.
(Case V)



case I $\Rightarrow f_1^{(1)}<0, f_2^{(2)}>0, f_1^{(3)}>0, f_2^{(4)}<0,$
case III $\Rightarrow f_1^{(1)}<0, f_2^{(2)}<0, f_1^{(3)}>0, f_2^{(4)}>0,$
case V $\Rightarrow f_1^{(1)}>0, f_2^{(2)}<0, f_1^{(3)}<0, f_2^{(4)}>0,$

$f^{(1)}+f^{(3)}=0_2$ and $f^{(2)}+f^{(4)}=0_2$.

Therefore the sign of $f_2^{(1)}$ and $f_2^{(3)}$ depends on the sign of $\sigma-(\alpha+\beta)$, and the sign of $f_1^{(2)}$ and $f_1^{(4)}$ depends on the sign of $\tau-(\alpha-\beta)$.

Moreover when $\sigma=0$ or $\tau=0$, the motional equation (4-3) is transformed as follows: if case II occurs

$$\text{(4-5)} \qquad \begin{cases} \dfrac{dx_1}{dt}=\sigma x_1+e_1^{(i)}, \\ \dfrac{dx_2}{dt}=e_2^{(i)}, \end{cases} \qquad i=1, 2, 3, 4,$$

and if case IV occurs

$$\text{(4-6)} \qquad \begin{cases} \dfrac{dx_1}{dt}=e_1^{(i)}, \\ \dfrac{dx_2}{dt}=\tau x_2+e_2^{(i)}, \end{cases} \qquad i=1, 2, 3, 4.$$

Accordingly in order to decide the sign of $e_2^{(1)}$, $e_1^{(2)}$, $e_2^{(3)}$, $e_1^{(4)}$ or $f_2^{(1)}$, $f_1^{(2)}$, $f_2^{(3)}$, $f_1^{(4)}$, we must consider the sign of $\sigma-(\alpha+\beta)$ and $\tau-(\alpha-\beta)$. But since $f_M(\alpha)<0$, $\alpha-\beta<\sigma$ and $\tau<\alpha+\beta$. Therefore we get the following nine cases, namely

i : $\alpha-\beta<\tau<\alpha+\beta<\sigma \Leftrightarrow f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)<0$,

ii : $\alpha-\beta<\tau<\alpha+\beta=\sigma \Leftrightarrow f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)=0$ (& $f_M'(\alpha+\beta)>0$),

iii : $\alpha-\beta<\tau<\sigma<\alpha+\beta \Leftrightarrow f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)>0$ (& $f_M'(\alpha-\beta)<0$ & $f_M'(\alpha+\beta)>0$),

iv : $\alpha-\beta=\tau<\alpha+\beta<\sigma \Leftrightarrow f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)<0$,

v : $\alpha-\beta=\tau<\alpha+\beta=\sigma \Leftrightarrow f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)=0$,

vi : $\alpha-\beta=\tau<\sigma<\alpha+\beta \Leftrightarrow f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)>0$ (& $f_M'(\alpha-\beta)<0$),

vii : $\tau<\alpha-\beta<\alpha+\beta<\sigma \Leftrightarrow f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)<0$,

viii: $\tau<\alpha-\beta<\alpha+\beta=\sigma \Leftrightarrow f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)=0$,

ix : $\tau<\alpha-\beta<\sigma<\alpha+\beta \Leftrightarrow f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)>0$,

where the inequalities of ii, iii, and vi in the brackets are unnecessary, since

$$f_M(\alpha+\beta)=0 \Rightarrow f_M'(\alpha+\beta)>0, \quad f_M(\alpha+\beta)>0 \Rightarrow f_M'(\alpha+\beta)>0,$$
$$f_M(\alpha-\beta)>0 \Rightarrow f_M'(\alpha-\beta)<0, \quad f_M(\alpha-\beta)=0 \Rightarrow f_M'(\alpha-\beta)<0.$$

Since

$$f_M(\alpha-\beta)=\beta\{\delta-(\alpha-\beta+\gamma)\}, \quad f_M'(\alpha-\beta)=\alpha-2\beta-\delta,$$
$$f_M(\alpha+\beta)=\beta\{-\delta+(\alpha+\beta-\gamma)\}, \quad f_M'(\alpha+\beta)=\alpha+2\beta-\delta,$$

it is necessary to classify the cases into three groups, namely

$$1: 0<\gamma<\beta, \quad 2: 0<\gamma=\beta, \quad 3: 0<\beta<\gamma.$$

In the first case, $0<\gamma<\beta$,

$$f_M(\alpha-\beta) \leqq 0 \Rightarrow f_M(\alpha+\beta)<0,$$

therefore only five cases i, ii, iii, vi, and ix occur. So we get

1-i : $f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)<0$,
1-ii : $f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)=0$,
1-iii: $f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)>0$,
1-vi : $f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)>0$,
1-ix : $f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)>0$.

In the second case, $0<\gamma=\beta$,

$$f_M(\alpha-\beta)>0 \Rightarrow f_M(\alpha+\beta)<0,$$
$$f_M(\alpha-\beta)=0 \Rightarrow f_M(\alpha+\beta)=0,$$
$$f_M(\alpha-\beta)<0 \Rightarrow f_M(\alpha+\beta)>0.$$

Therefore only three cases i, v, and ix occur. So we get

2-i : $f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)<0$,
2-v : $f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)=0$,
2-ix: $f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)>0$.

In the third case, $0<\beta<\gamma$,

$$f_M(\alpha-\beta)\geqq 0 \Rightarrow f_M(\alpha+\beta)<0$$

therefore only five cases i, iv, vii, viii and ix occur. So we get

3-i : $f_M(\alpha-\beta)>0$ & $f_M(\alpha+\beta)<0$,
3-iv : $f_M(\alpha-\beta)=0$ & $f_M(\alpha+\beta)<0$,
3-vii : $f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)<0$,
3-viii: $f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)=0$,
3-ix : $f_M(\alpha-\beta)<0$ & $f_M(\alpha+\beta)>0$.

Now we get the table of classification to appear in the last part of this paper that shows the sign of $f^{(i)}$, $i=1, 2, 3, 4$ (in the case I, III and V) or $e^{(i)}$, $i=1, 2, 3, 4$ (in the case II and IV).

We can draw the figures of optimal trajectories in 65 cases respcctively. But in order to attract our attention to the qualitative nature of our control problem we draw the figures of optimal trajectories in only five cases, namely I-1-i, II-1-i, III-1-i, IV-1-i and V-1-i. From these five figures from 11 to 14 it is easily seen that

I $\Rightarrow$ S is the case (i),
II, III $\Rightarrow$ S is the case (ii),
IV, V $\Rightarrow$ S is the case (iii).

This is the qualitative nature of our control problem which we have searched for.

ACKNOWLEDGEMENTS

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FIG. 11.
(Case I-1-i)

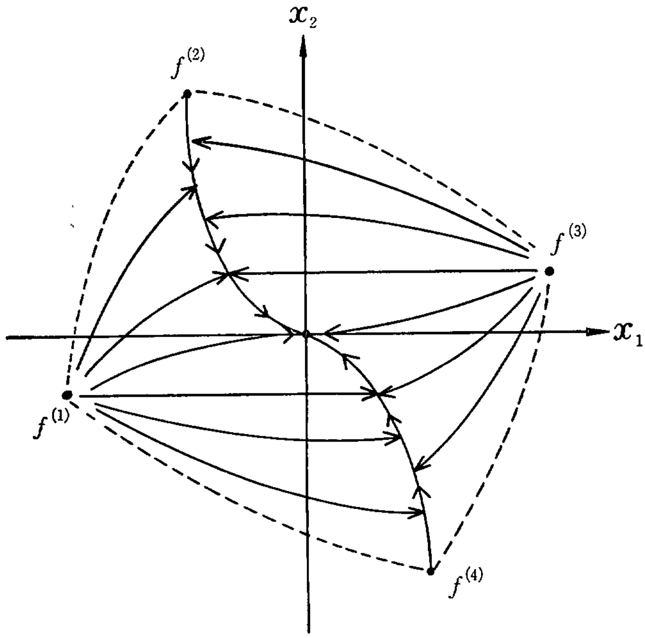


FIG. 12.
(Case II-1-i)

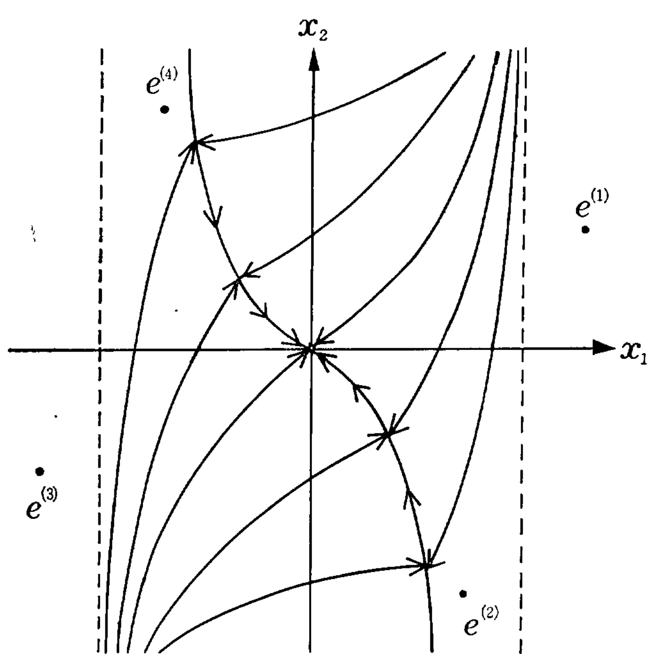


FIG. 13.
(Case III-1-i)

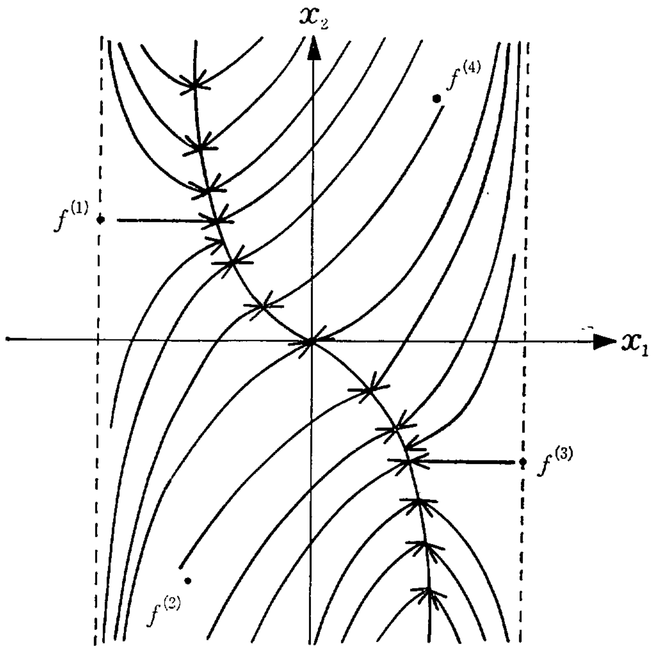


FIG. 14.
(Case IV-1-i)

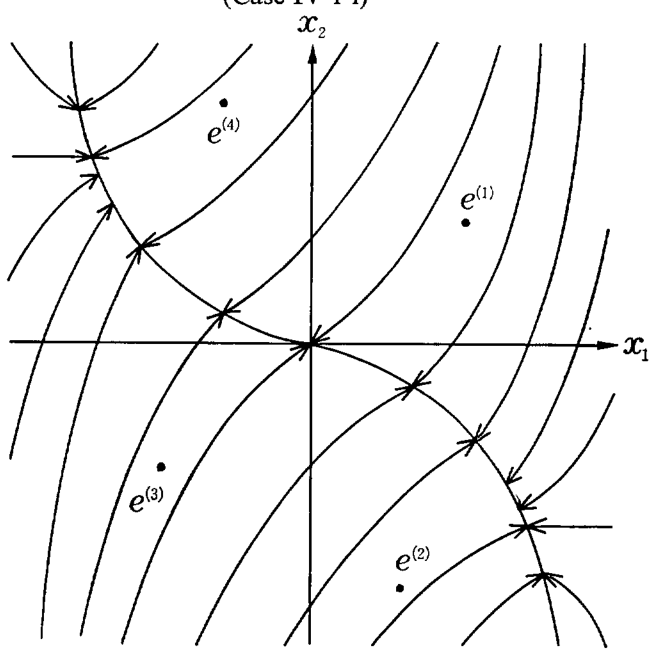
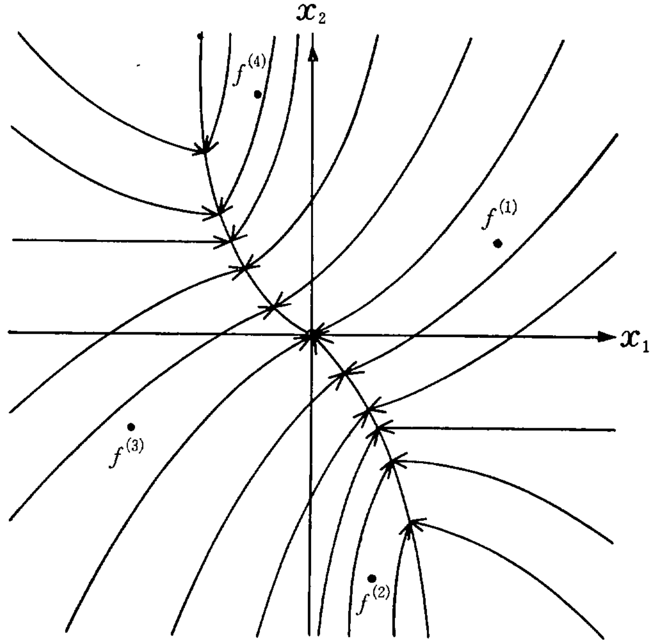


FIG. 15.
(Case V-1-i)



THE CLASSIFICATION TABLE OF THE SIGN OF $f_j^{(i)}$, $i=1, 2, 3, 4, j=1, 2$.

	$f_1^{(1)}$	$f_2^{(1)}$	$f_1^{(2)}$	$f_2^{(2)}$	$f_1^{(3)}$	$f_2^{(3)}$	$f_1^{(4)}$	$f_2^{(4)}$
I-1-i	−	−	−	+	+	+	+	−
I-1-ii	−	O	−	+	+	O	+	−
I-1-iii	−	+	−	+	+	−	+	−
I-1-vi	−	+	O	+	+	−	O	−
I-1-ix	−	+	+	+	+	−	−	−
I-2-i	−	−	−	+	+	+	+	−
I-2-v	−	O	O	+	+	O	O	−
I-2-ix	−	+	+	+	+	−	−	−
I-3-i	−	−	−	+	+	+	+	−
I-3-iv	−	−	O	+	+	+	O	−
I-3-vii	−	−	+	+	+	+	−	−
I-3-viii	−	O	+	+	+	O	−	−
I-3-ix	−	+	+	+	+	−	−	−
III-1-i	−	+	−	−	+	−	+	+
III-1-ii	−	O	−	−	+	O	+	+
III-1-iii	−	−	−	−	+	+	+	+
III-1-vi	−	−	O	−	+	+	O	+
III-1-ix	−	−	+	−	+	+	−	+
III-2-i	−	+	−	−	+	−	+	+
III-2-v	−	O	O	−	+	O	O	+
III-2-ix	−	−	+	−	+	+	−	+

	$f_1^{(1)}$	$f_2^{(1)}$	$f_1^{(2)}$	$f_2^{(2)}$	$f_1^{(3)}$	$f_2^{(3)}$	$f_1^{(4)}$	$f_2^{(4)}$
III-3-i	−	+	−	−	+	−	+	+
III-3-iv	−	+	○	−	+	−	○	+
III-3-vii	−	+	+	−	+	−	−	+
III-3-viii	−	○	+	−	+	○	−	+
III-3-ix	−	−	+	−	+	+	−	+
V-1-i	+	+	+	−	−	−	−	+
V-1-ii	+	○	+	−	−	○	−	+
V-1-iii	+	−	+	−	−	+	−	+
V-1-vi	+	−	○	−	−	+	○	+
V-1-ix	+	−	−	−	−	+	+	+
V-2-i	+	+	+	−	−	−	−	+
V-2-v	+	○	○	−	−	○	○	+
V-2-ix	+	−	−	−	−	+	+	+
V-3-i	+	+	+	−	−	−	−	+
V-3-iv	+	+	○	−	−	−	○	+
V-3-vii	+	+	−	−	−	−	+	+
V-3-viii	+	○	−	−	−	○	+	+
V-3-ix	+	−	−	−	−	+	+	+

THE CLASSIFICATION TABLE OF THE SIGN OF $e_j^{(i)}$, $i=1, 2, 3, 4$, $j=1, 2$.

	$e_1^{(1)}$	$e_2^{(1)}$	$e_1^{(2)}$	$e_2^{(2)}$	$e_1^{(3)}$	$e_2^{(3)}$	$e_1^{(4)}$	$e_2^{(4)}$
II-1-i	+	+	+	−	−	−	−	+
II-1-ii	+	○	+	−	−	○	−	+
II-1-iii	+	−	+	−	−	+	−	+
II-1-iv	+	−	○	−	−	+	○	+
II-1-ix	+	−	−	−	−	+	+	+
II-2-i	+	+	+	−	−	−	−	+
II-2-v	+	○	○	−	−	○	○	+
II-2-ix	+	−	−	−	−	+	+	+
II-3-i	+	+	+	−	−	−	−	+
II-3-iv	+	+	○	−	−	−	○	+
II-3-vii	+	+	−	−	−	−	+	+
II-3-viii	+	○	−	−	−	○	+	+
II-3-ix	+	−	−	−	−	+	+	+
IV-1-i	+	+	+	−	−	−	−	+
IV-1-ii	+	○	+	−	−	○	−	+
IV-1-iii	+	−	+	−	−	+	−	+
IV-1-vi	+	−	○	−	−	+	○	+
IV-1-ix	+	−	−	−	−	+	+	+
IV-2-i	+	+	+	−	−	−	−	+
IV-2-v	+	○	○	−	−	○	○	+
IV-2-ix	+	−	−	−	−	+	+	+
IV-3-i	+	+	+	−	−	−	−	+
IV-3-iv	+	+	○	−	−	−	○	+
IV-3-vii	+	+	−	−	−	−	+	+
IV-3-viii	+	○	−	−	−	○	+	+
IV-3-ix	+	−	−	−	−	+	+	+

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