ON THE TIME OPTIMAL CONTROL OF SIMON-HOMANS SYSTEM TO ITS DISSOLUTION†

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1. *Introduction*

How does a social group behave as a system? It is possible that there exist many solutions for this problem according to the kind of the system. Namely many kinds of solutions shall be shown to us according as the system is an enterprise, a political party, a government or a state. On the other hand I make no question of the fact that we have something in common among a good many kinds of solutions. Namely we can consider the common behavior of systems which can be an enterprise, a political party, a government or a state. This common behavior of systems is the very problem that I will take up in this note.

H. A. Simon showed a concrete model of system in his article [1]. He constructed the model on the basis of G. H. Homans' book [2], and he considered that his model systematized a substantial number of the important empirical relationships that had been observed in the behavior of human groups. I, also, consider that his concrete model of system represents some substantial parts of the common behavior of systems described above.

For a short account of description let us call this model Simon-Homans system. The behavior of Simon-Homans system is characterized by four variables, which are all functions of time:

$I(t)$ = the intensity of interaction among the members,
$F(t)$ = the level of friendliness among the members,
$A(t)$ = the amount of activity carried on by members within the system,
$E(t)$ = the amount of activity imposed on the system by the external environment (the "external system").

The detailed explanation on Simon-Homans system shall be described in the next section. But here let us use these four variables for the explanation of the purpose of this note. Let us consider that $I(t)$, $F(t)$ and $A(t)$ are state variables of Simon-Homans system, namely ($I(t)$, $F(t)$, $A(t)$) is a phase point of the state space of Simon-Homans system. And let us consider that $E(t)$ is a control function from the external system. Let us put a definition on the dissolution of Simon-Homans system. Namely if and only if all of values of state variables, $I(t)$, $F(t)$ and $A(t)$ vanish, the state of the system will be called DISSOLUTION. Now the purpose

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of this note is to decide the region of state space whose point we can transfer to the dissolution by the optimal control $E(t)$ in the meaning of minimum time. Using some control function $E(t)$, perhaps we might transfer to the dissolution the phase points outside of our region which we are going to decide in this note, but the time interval neccessary to transfer the phase points to the dissolution is not minimum. We are going to decide the region of phase space which is constructed by points transferable to the dissolution, by some external pressure $E(t)$, with minimum time interval.

In section 2, we shall describe the detailed explanation on Simon-Homans system according to H. A. Simon [1] and in section 3 we shall decide the region described above.

2. *Simon-Homans System*

Let us assume Simon-Homans system constructed by n members which are indicated by numbers from 1 to n. For i and j such that $1 \leqq i \neq j \leqq n$, let $I_{ij}(t)$ represent the intensity of interactions at time t of the i^{th} member of the system with the j^{th} member. Then we can define the intensity of interaction $I(t)$ at time t of the system as

$$I(t) = \sum_{1 \leqq i \neq j \leqq n} I_{ij}(t).$$

Similarly for i and j such that $1 \leqq i \neq j \leqq n$, let $F_{ij}(t)$ represent the level of friendliness at time t of the i^{th} member of the system with j^{th} member. Then we can define the level of friendliness $F(t)$ at time t of the system as

$$F(t) = \sum_{1 \leqq i \neq j \leqq n} F_{ij}(t).$$

For each $i=1, 2, \cdots, n$, let $A_i(t)$ represent the amount of activity carried on by the i^{th} member of the system and we can define the amount of activity $A(t)$ at time t carried on by the system as

$$A(t) = \sum_{1 \leqq i \leqq n} A_i(t).$$

Finally $E(t)$ is defined as the total amount of external pressures at time t. In section 3 $E(t)$ shall be construed in the meaning of a control function of the system.

H. A. Simon put the postulates in his article [1] which can be represented by the following three equations, where a_1, a_2, b, c_1, c_2, β and γ are positive real constants. Namely

$$(1) \qquad I(t) = a_1F(t)+a_2A(t),$$

$$(2) \qquad \frac{dF}{dt}(t) = b(I(t)-\beta F(t)),$$

$$(3) \qquad \frac{dA}{dt}(t) = c_1(F(t)-\gamma A(t))-c_2(E(t)-A(t)).$$

In these equations $F(t)$ and $A(t)$, accordingly $I(t)$ too, are assumed to be differentiable and $E(t)$ is assumed to be arcwise continuous.

For these equations the explanations by H. A. Simon are as follows. First, for the equation (1), the intensity of interaction depends upon, and increases with, the level of friendliness and the amount of activity carried on within the system. In other words we postulate that the interaction is produced on the one hand by friendliness, and on the other hand by requirements of the activity pattern; and that these two causes of communication are additive in their effect. Secondly, for the equation (2), the level of system friendliness will increase if the actual level of interaction is higher than that "appropriate" to the existing level of friendliness. That is, if a system of members with little friendliness is induced to interact a great deal, the friendliness will grow; while, if a system with a great deal of friendliness seldom interacts, the friendliness will weaken. Finally, for the equation (3), the amount of activity carried on by the system will tend to increase if the actual level of friendliness is higher than that "appropriate" to the existing amount of activity, and if the amount of activity imposed externally on the system is higher than the existing amount of activity.

On seven real positive constants in the equality from (1) to (3), the explanations by H. A. Simon are as follows. First, in the differential equation (2), $\beta F(t)$ may be regarded as the amount of interaction "appropriate" to the level, $F(t)$, of friendliness. For if $I(t)=\beta F(t)$, then $F(t)$ will have no tendency either to increase or to decrease. The reciprocal of the coefficient β, that is, $1/\beta$, might be called the "congeniality coefficient" since it measures the amount of friendliness that will be generated per unit of interaction. Secondly, from equation (1), $a_1F(t)$ may be regarded as the amount of interaction generated by the level, $F(t)$, of friendliness in the absence of any group activity. That is, if $A(t)=0$, then $I(t)=a_1F(t)$. Further, the coefficient a_2 measures the amount of interaction generated per unit of system activity in the absence of friendliness. Hence a_1 and a_2 might be called "coefficients of interdependence." Finally, from the differential equation (3) the reciprocal of the coefficient γ measures the amount of activity that is generated per unit of friendliness, in the absence of external pressure. We may call $1/\gamma$ a coefficient of "spontaneity." The remaining coefficients, b, c_1 and c_2, determine how rapidly the system will adjust itself if it starts out from a position of disequilibrium.

The above is the gist of explanation for Simon-Homans system by H. A. Simon [1]. In the next section we shall decide the controllability region of this system.

3. *The region controllable to the dissolution*

Using (1) let us eliminate $I(t)$ from (2) and (3) and replace $E(t)$ with palameter E. Then we obtain linear differential equation with constant coefficients;

$$(4) \qquad \begin{cases} \dfrac{dF}{dt}(t) = (a_1-\beta)bF + a_2bA , \\ \dfrac{dA}{dt}(t) = c_1F-(c_1\gamma+c_2)A+c_2E, \end{cases}$$

where for simplicity the region of palameter E is a closed interval $[-1, 1]$. Let us denote the coefficient matrix of (4) by M and then we obtain

$$\frac{d}{dt}\begin{pmatrix} F(t) \\ A(t) \end{pmatrix} = M\begin{pmatrix} F \\ A \end{pmatrix}+\begin{pmatrix} 0 \\ c_2 \end{pmatrix}\dot{E}, \quad \text{and} \quad M = \begin{pmatrix} (a_1-\beta)b & a_2b \\ c_1 & -(c_1\gamma+c_2) \end{pmatrix}.$$

The characteristic polynomial $f_M(x)$ of M is

$$\begin{aligned} f_M(x) &= \det(xE_2-M) \\ &= x^2+\{(c_1\gamma+c_2)-(a_1-\beta)b\}x-b\{(a_1-\beta)(c_1\gamma+c_2)+a_2c_1\}, \end{aligned}$$

and the discriminant D of $f_M(x)$ is positive, since

$$\begin{aligned} D &= \{(c_1\gamma+c_2)-(a_1-\beta)b\}^2+4b\{(a_1-\beta)(c_1\gamma+c_2)+a_2c_1\} \\ &= \{(c_1\gamma+c_2)+(a_1-\beta)b\}^2+4a_2bc_1, \end{aligned}$$

and a_2, b and c_1 are positive real constants by our assumption. Therefore eigenvalues of M are two different real numbers. Let us denote them by ξ and η, namely

$$\xi = \frac{1}{2}\{-(c_1\gamma+c_2)+(a_1-\beta)b+\sqrt{\{(c_1\gamma+c_2)+(a_1-\beta)b\}^2+4a_2bc_1}\},$$

and
$$\eta = \frac{1}{2}\{-(c_1\gamma+c_2)+(a_1-\beta)b-\sqrt{\{(c_1\gamma+c_2)+(a_1-\beta)b\}^2+4a_2bc_1}\}.$$

So we can consider five cases: (i) $0<\eta<\xi$, (ii) $\eta<\xi<0$, (iii) $\eta<0<\xi$, (iv) $0=\eta<\xi$, (v) $\eta<\xi=0$. Let us consider only one case (i) in this note. Now the necessary and sufficient condition that two eigenvalues of M are two positive real numbers, is by the relation of root and coefficient of $f_M(x)$

$$(c_1\gamma+c_2)-(a_1-\beta)b>0, \quad \text{and} \quad -b\{(a_1-\beta)(c_1\gamma+c_2)+a_2c_1\}>0,$$

which are equivalent to the following two inequalities,

$$\begin{cases} a_1<\beta, \\ (\beta-a_1)(c_1\gamma+c_2)>a_2c_1. \end{cases}$$

In Simon [1], the above two inequalities were dealed with a necessary and sufficient condition for stability of the system.

Now we seek two eigenspaces in a wide sense W_ξ and W_η. Solving linear equations:

$$\begin{cases} (a_1-\beta)x+ a_2by = \xi x, \\ c_1x-(c_1\gamma+c_2)y = \xi y, \end{cases} \quad \text{and} \quad \begin{cases} (a_1-\beta)x+ a_2by = \eta x, \\ c_1x-(c_1\gamma+c_2)y = \eta y, \end{cases}$$

we obtain that

$$W_\xi = \left\{\begin{pmatrix} \frac{c_1\gamma+c_2+\xi}{c_1}\cdot\lambda \\ \lambda \end{pmatrix} \in \boldsymbol{R}^2;\ \lambda \in \boldsymbol{R}\right\}, \qquad W_\eta = \left\{\begin{pmatrix} \frac{c_1\gamma+c_2+\eta}{c_1}\cdot\mu \\ \mu \end{pmatrix} \in \boldsymbol{R}^2;\ \lambda \in \boldsymbol{R}\right\}.$$

Then using an element of W_ξ with $\lambda=1$ and an element of W_η with $\mu=1$, square matrix P with degree 2 is defined as follows;

$$P=\begin{bmatrix} \dfrac{c_1\gamma+c_2+\xi}{c_1} & \dfrac{c_1\gamma+c_2+\eta}{c_1} \\ 1 & 1 \end{bmatrix}.$$

Then P is a non-singular matrix and we get

$$P^{-1}MP=\begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix}, \qquad P^{-1}=\begin{bmatrix} \dfrac{c_1}{\xi-\eta} & -\dfrac{c_1\gamma+c_2+\eta}{\xi-\eta} \\ -\dfrac{c_1}{\xi-\eta} & \dfrac{c_1\gamma+c_2+\xi}{\xi-\eta} \end{bmatrix}.$$

Multiplying the left side of (5) by P^{-1}, we get

$$P^{-1}\frac{d}{dt}\begin{pmatrix} F \\ A \end{pmatrix}=P^{-1}MP\cdot P^{-1}\begin{pmatrix} F \\ A \end{pmatrix}+P^{-1}\begin{pmatrix} 0 \\ c_2 \end{pmatrix}E.$$

Accordingly we introduce two new variables x_1 and x_2 in place of F and A as follows;

$$(6) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=P^{-1}\begin{pmatrix} F \\ A \end{pmatrix}, \qquad \text{i.e.} \quad \begin{cases} x_1=\dfrac{c_1}{\xi-\eta}F-\dfrac{c_1\gamma+c_2+\eta}{\xi-\eta}A, \\ x_2=-\dfrac{c_1}{\xi-\eta}F+\dfrac{c_1\gamma+c_2+\xi}{\xi-\eta}A, \end{cases}$$

and two new control variables v_1 and v_2 in place of E as follows;

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=P^{-1}\begin{pmatrix} 0 \\ c_2 \end{pmatrix}E, \qquad \text{i.e.} \quad \begin{cases} v_1=-\dfrac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}\cdot E, \\ v_2=\dfrac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta}\cdot E. \end{cases}$$

Then system (5) becomes

$$(7) \qquad \begin{cases} \dfrac{dx_1}{dt}=\xi x_1+v_1, \\ \dfrac{dx_2}{dt}=\eta x_2+v_2. \end{cases}$$

According to new control variables v_1 and v_2, the control domain of the system (7) becomes V, where

$$V=\left\{(1-\lambda)\begin{bmatrix} -\dfrac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta} \\ \dfrac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta} \end{bmatrix}+\lambda\begin{bmatrix} \dfrac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta} \\ -\dfrac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta} \end{bmatrix}\in \boldsymbol{R}^2; 0\leqq\lambda\leqq 1\right\}.$$

Defining two points $A^{(1)}$, $A^{(2)}$ as follows;

$$(8)\quad A^{(1)}=\begin{pmatrix}A_1^{(1)}\\A_2^{(1)}\end{pmatrix}\begin{bmatrix}-\dfrac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}\\\dfrac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta}\end{bmatrix},\qquad A^{(2)}=\begin{pmatrix}A_1^{(2)}\\A_2^{(2)}\end{pmatrix}=\begin{bmatrix}\dfrac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}\\-\dfrac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta}\end{bmatrix},$$

V is a segment which is pictured in Figure 1. So the condition of general position with respect to V is fulfilled. Accordingly we can use the general theory of optimal control.

Now using adjoint variables ψ_1 and ψ_2 of the system (7), we get the Hamiltonian of system (8) as follows;

$$\begin{aligned}&H(\psi_1(t),\ \psi_2(t);\ x_1(t),\ x_2(t);\ v_1,\ v_2)\\&=\psi_1(t)(\xi x_1(t)+v_1)+\psi_2(t)(\eta x_2(t)+v_2)\\&=\{\xi\psi_1(t)x_1(t)+\eta\psi_2(t)x_2(t)\}+\{\psi_1(t)v_1+\psi_2(t)v_2\}.\end{aligned}$$

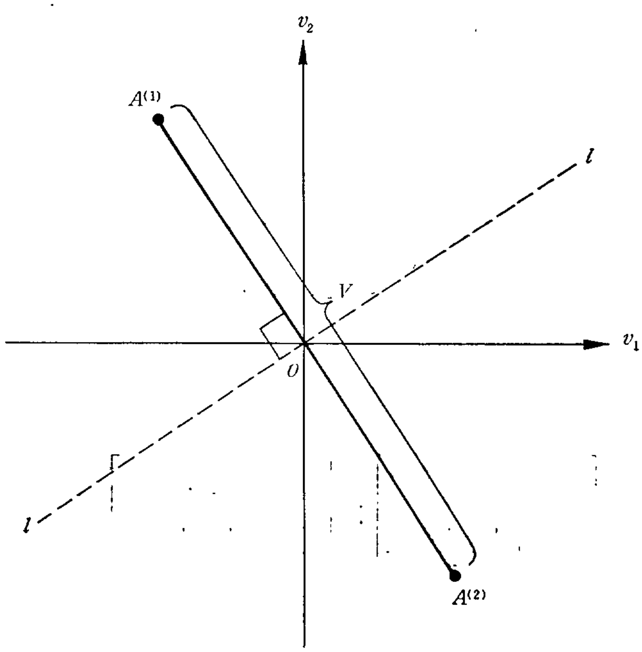
By Pontrjagin's Maximum Principle, if $v_1(t)$ and $v_2(t)$ are the optimal control of the system (8) transfering initial phase point $x^{(0)}$ to the origin O_2 with minimum time interval, $v_1(t)$ and $v_2(t)$ must satisfy the following condition:

$$\underset{(v_1,v_2)\in V}{\mathrm{Max}}\ H(\psi_1(t),\ \psi_2(t);\ x_1(t),\ x_2(t);\ v_1,\ v_2)\doteq H(\psi_1(t),\ \psi_2(t);\ x_1(t),\ x_2(t);\ v_1(t),\ v_2(t))$$

at all time t. And the adjoint differential equation of (7) is

$$(9)\qquad \frac{d\psi_1}{dt}=-\xi\psi_1,\qquad \frac{d\psi_2}{dt}=-\eta\psi_2$$

FIG. 1



and its general solution is

$$\phi_1(t) = K_1 e^{-\xi t}, \qquad \phi_2(t) = K_2 e^{-\eta t}, \tag{10}$$

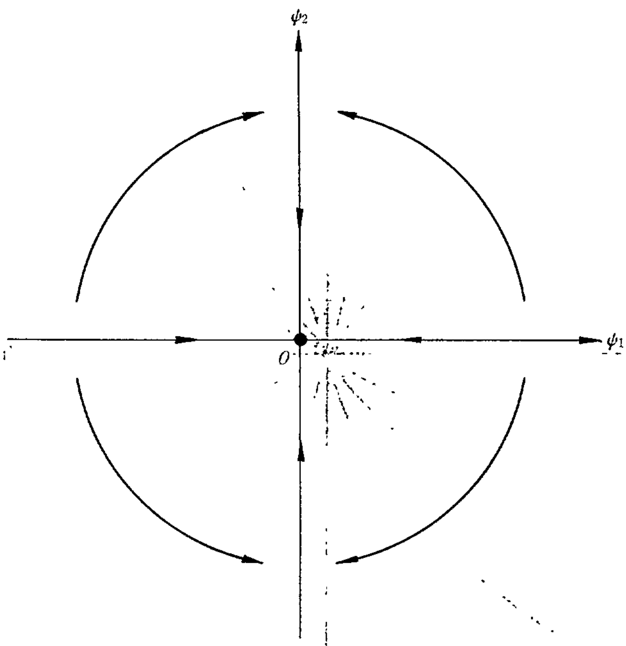
where K_1 and k_2 are integral constants. Let us picture the integral curves for all initial conditions. First, by (10) if $\phi_1(t^*)>0$ and $\phi_2(t^*)>0$ for some time t^*, $\phi_1(t)>0$ and $\phi_2(t)>0$ for all time t. Similarly if $\phi_1(t^*)<0$ and $\phi_2(t^*)>0$ for some time t^*, $\phi_1(t)<0$ and $\phi_2(t)>0$ for all time t, and if $\phi_1(t^*)<0$ and $\phi_2(t^*)<0$ for some time t^*, $\phi_1(t)<0$ and $\phi_2(t)<0$ for all time t, and if $\phi_1(t^*)>0$ and $\phi_2(t^*)<0$ for some time t^*, $\phi_1(t)>0$ and $\phi_2(t)<0$ for all time t. Furthermore if $\phi_1(t^*)=0$ for some time t^*, $\phi_2(t)=0$ for all time t and if $\phi_2(t^*)=0$ for some time t^*, $\phi_2(t)=0$ for all time t. Secondly, since we get by (9)

$$\frac{d}{dt}\left(\frac{\phi_2}{\phi_1}\right) = \frac{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}{\phi_1{}^2} = (\xi-\eta)\frac{\phi_2}{\phi_1}, \tag{11}$$

with the increase in time t the tangent of the segment, which is constructed by combining the origin with the point $(\phi_1(t), \phi_2(t))$, increases when $\phi_1(t)$ and $\phi_2(t)$ have the same signature and decreases when $\phi_1(t)$ and $\phi_2(t)$ have the different signature. Therefore the point $(\phi_1(t), \phi_2(t))$ turns as in Figure 2. Since by (10) both functions ϕ_1 and ϕ_2 are increasing functions, with the increase in time t the point on axis must move toward the origin as in Figure 2. Thirdly, since we get by (9)

$$\frac{d\phi_2}{d\phi_1} = \frac{\dot{\phi}_2}{\dot{\phi}_1} = \frac{\eta}{\xi}\frac{\phi_2}{\phi_1}, \tag{12}$$

FIG. 2



with the increase in ψ_1, ψ_2 increases when $\psi_1(t)$ and $\psi_2(t)$ have the same signature and decreases when $\psi_1(t)$ and $\psi_2(t)$ have the different signature. Finally, since we get by (9), (11) and (12)

$$\frac{d^2\psi_2}{d\psi_1{}^2}=\frac{d}{d\psi_1}\left(\frac{d\psi_2}{d\psi_1}\right)=\frac{dt}{d\psi_1}\cdot\frac{d}{dt}\left(\frac{\eta}{\xi}\frac{\psi_2}{\psi_1}\right)=\frac{\frac{\eta}{\xi}(\xi-\eta)\frac{\psi_2}{\psi_1}}{-\xi\psi_1}=-\frac{\eta(\xi-\eta)}{\xi^2}\cdot\frac{\psi_2}{\psi_1{}^2},$$

the integral curve of (9) is convex when $\psi_2(t)>0$, and concave when $\psi_2(t)<0$. Synthesizing all things described above we can picture the integral curves for all initial conditions as Figure 3.

In Figure 1 let l be a line which is orthogonal to the segment $A^{(1)}A^{(2)}$. Then if we lie Figure 1 upon Figure 2, we come to a conclusion that any one of integral curves cuts the line l in one time at most. The reason is as follows; let α be the tangent of line l and let some integral curve $(\psi_1(t), \psi_2(t))$ cut the line l in two times, then for some different time t_1 and t_2

$$\psi_2(t_1)=\alpha\psi_1(t_1), \qquad \psi_2(t_2)=\alpha\psi_1(t_2),$$

therefore by (10),

$$K_2e^{-\eta t_1}=\alpha K_1e^{-\xi t_1} \quad\text{and}\quad K_2e^{-\eta t_2}=\alpha K_1e^{-\xi t_2}, \qquad \text{then}\quad e^{(\xi-\eta)t_1}=\alpha K_1/K_2=e^{(\xi-\eta)t_2}$$

and then $t_1=t_2$, this is a contradiction. Accordingly the optimal control transfering some initial phase point to the origin with minimum time interval is a step function which takes at most two value $A^{(1)}$ or $A^{(2)}$. Namely it is impossible that the optimal control could have any type without four types of Figure 4.

FIG. 3

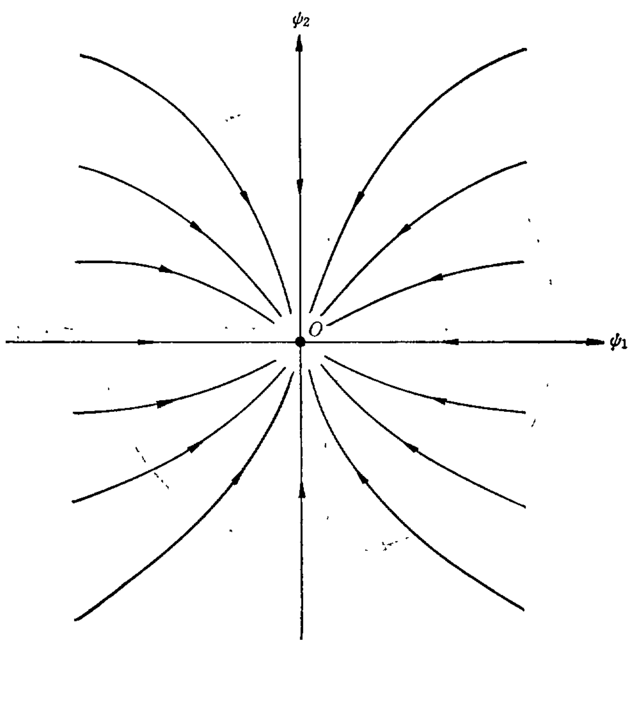
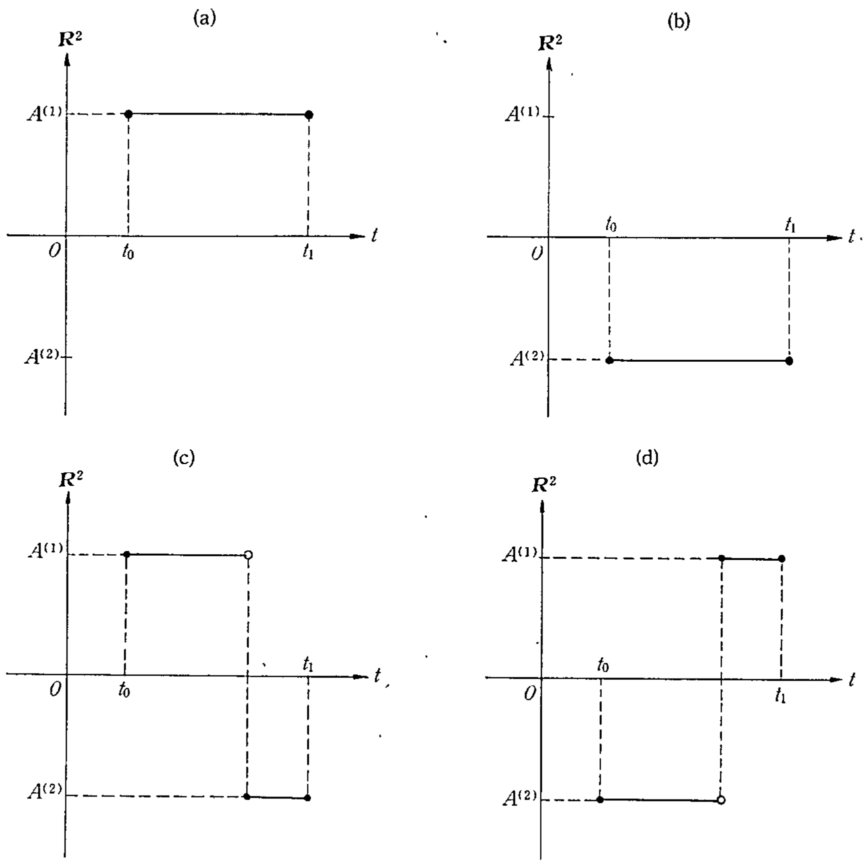


FIG. 4



Now let us solve the differential equation (9) in the time interval in which the value of control function is a constant $A^{(i)}$. We can get an optimal trajectory by combining the above solutions for each type of control function from (a) to (d). Let us define $B^{(1)}$ and $B^{(2)}$ by the following relation:

$$B^{(i)} = \begin{pmatrix} B_1^{(i)} \\ B_2^{(i)} \end{pmatrix} = \begin{bmatrix} -\dfrac{1}{\xi} & 0 \\ 0 & -\dfrac{1}{\eta} \end{bmatrix} A^{(i)}, \ i = 1, 2, \tag{13}$$

and then it is clear that $B_1^{(1)}>0$, $B_2^{(1)}<0$ and $B_1^{(2)}<0$, $B_2^{(2)}>0$. Using $B^{(i)}$ we can rewrite the equation (9) as follows;

$$\begin{cases} \dfrac{d}{dt}(x_1 - B_1^{(i)}) = \xi(x_1 - B_1^{(i)}), \\ \dfrac{d}{dt}(x_2 - B_2^{(i)}) = \eta(x_2 - B_2^{(i)}). \end{cases} \tag{14$_i$}$$

As the solution of the above differential equation we obtain

$$(15)_i \qquad \begin{cases} x_1 = B_1^{(i)} + L_1^{(i)} e^{\xi t}, \\ x_2 = B_2^{(i)} + L_2^{(i)} e^{\eta t}, \end{cases}$$

where $L_1^{(i)}$ and $L_2^{(i)}$ are integral constants. All integral curves of $(14)_1$ for every initial phase point are pictured in Figure 5 and all integral curves of $(14)_2$ for every initial phase point are pictured in Figure 6.

Now let us decide the optimal trajectory with respect to each optimal control which has one of four kinds of type from (a) to (d) in Figure 4. First if an optimal control has type (a), a phase point (x_1, x_2) moves along the trajectory in Figure 5. In the trajectories of Figure 5 the one which passes the origin is only $B^{(1)}O$. Therefore the initial phase point, which we can transfer to the origin by the control of type (a) with minimum time interval, must belong to the arc $B^{(1)}O$ in Figure 5. Similarly the initial phase point, which we can transfer to the origin by the control of type (b) with minimum time interval, must belong to the arc $B^{(2)}O$ in Figure 6. Let us seek for the equations which indicate $B^{(1)}O$ and $B^{(2)}O$. If $(15)_1$ arrives at the origin at some time t^*, we get

$$(16) \qquad \begin{cases} 0 = B_1^{(1)} + L_1^{(1)} e^{\xi t^*}, \\ 0 = B_2^{(1)} + L_2^{(1)} e^{\eta t^*}. \end{cases}$$

By the signature of $B_1^{(1)}$ and $B_2^{(1)}$, it must be $L_1^{(1)} < 0$ and $L_2^{(1)} > 0$, and therefore

$$(17) \qquad \frac{(-B_2^{(1)})^{1/\eta}}{(B_1^{(1)})^{1/\xi}} = \frac{(L_2^{(1)})^{1/\eta}}{(-L_1^{(1)})^{1/\xi}}.$$

FIG. 5

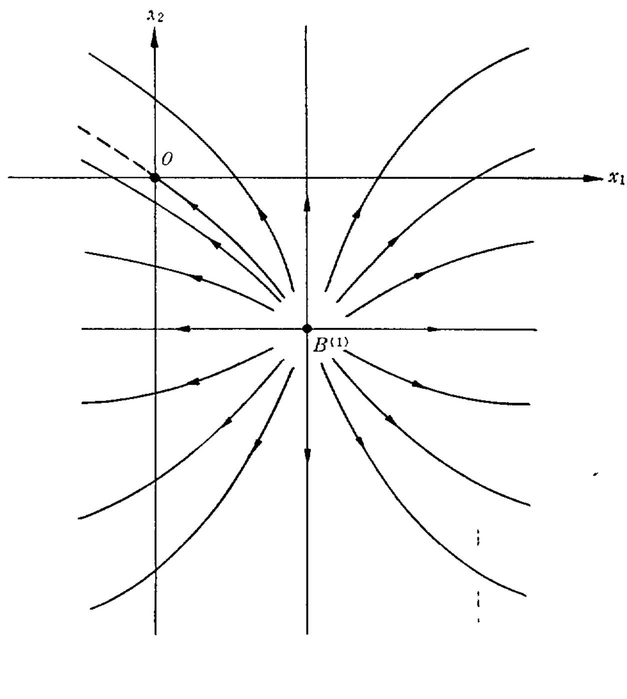
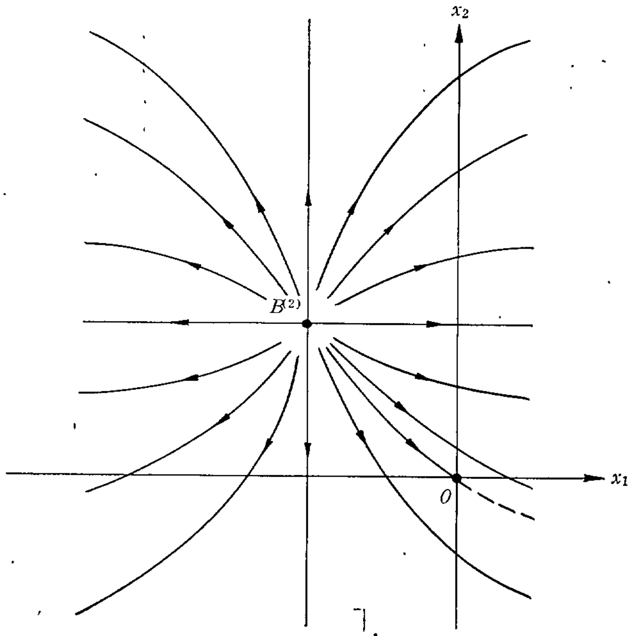


FIG. 6



Conversely, it is clear that (17) concludes (16) for $t^* = \log (B_1^{(1)}/(-L_1^{(1)}))^{1/\xi} = \log ((-B_2^{(1)})/L_2^{(1)})^{1/\eta}$. Therefore the necessary and sufficient condition that the trajectory $(15)_1$ arrives at the origin at some time, is (17). Similarly the necessary and sufficient condition that the trajectory $(15)_2$ arrives at the origin at some time is

$$\text{(18)} \qquad \frac{(B_2^{(2)})^{1/\eta}}{(-B_1^{(2)})^{1/\xi}} = \frac{(-L_2^{(2)})^{1/\eta}}{(L_1^{(2)})^{1/\xi}},$$

where $L_2^{(2)} < 0$ and $L_1^{(2)} > 0$.

By $(15)_1$ and (16), we get

$$\left(\frac{B_1^{(1)} - x_1}{-L_1^{(1)}}\right)^{1/\xi} = \left(\frac{x_2 - B_2^{(1)}}{L_2^{(1)}}\right)^{1/\eta}$$

and

$$\frac{(x_2 - B_2^{(1)})^{1/\eta}}{(B_1^{(1)} - x_1)^{1/\xi}} = \frac{(L_2^{(1)})^{1/\eta}}{(-L_1^{(1)})^{1/\xi}} = \frac{(-B_2^{(1)})^{1/\eta}}{(B_2^{(1)})^{1/\xi}}.$$

But by (8) and (13) we get

$$\frac{(-B_2^{(1)})^{1/\eta}}{(B_1^{(1)})^{1/\xi}} = \frac{\left(\frac{1}{\eta} A_2^{(1)}\right)^{1/\eta}}{\left(-\frac{1}{\xi} A_1^{(1)}\right)^{1/\xi}} = \frac{\left\{\frac{1}{\eta} \cdot \frac{(c_1\gamma + c_2 + \xi)c_2}{\xi - \eta}\right\}^{1/\eta}}{\left\{\frac{1}{\xi} \cdot \frac{(c_1\gamma + c_2 + \eta)c_2}{\xi - \eta}\right\}^{1/\xi}}$$
$$= \frac{\{\xi(\xi - \eta)\}^{1/\xi} \{(c_1\gamma + c_2 + \xi)c_2\}^{1/\eta}}{\{\eta(\xi - \eta)\}^{1/\eta} \{(c_1\gamma + c_2 + \eta)c_2\}^{1/\xi}}.$$

On the other side we get

$$(19)\qquad \frac{(x_2-B_2^{(1)})^{1/\eta}}{(B_1^{(1)}-x_1)^{1/\xi}}=\frac{\left\{x_2+\frac{1}{\eta}\cdot\frac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta}\right\}^{1/\eta}}{\left\{\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}-x_1\right\}^{1/\xi}}$$
$$=\frac{\{\xi(\xi-\eta)\}^{1/\xi}\{\xi(\xi-\eta)x_2+(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{\eta(\xi-\eta)\}^{1/\eta}\{(c_1\gamma+c_2+\eta)c_2-\xi(\xi-\eta)x_1\}^{1/\xi}}.$$

Accordingly we get

$$\frac{\{\xi(\xi-\eta)x_2+(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{(c_1\gamma+c_2+\eta)c_2-\xi(\xi-\eta)x_1\}^{1/\xi}}=\frac{\{(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{(c_1\gamma+c_2+\eta)c_2\}^{1/\xi}}.$$

Therefore we get

$$(20)\qquad x_2=\frac{(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[\left\{1-\frac{\xi(\xi-\eta)}{(c_1\gamma+c_2+\eta)c_2}\cdot x_1\right\}^{\eta/\xi}-1\right],$$
$$0\leqq x_1<\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}.$$

This is the equation of arc $B^{(1)}O$. Similarly we can obtain the equation of arc $B^{(2)}O$ as follows;

$$(21)\qquad x_2=\frac{(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[1-\left\{\frac{\xi(\xi-\eta)}{(c_1\gamma+c_2+\eta)c_2}\cdot x_1+1\right\}^{\eta/\xi}\right],$$
$$-\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}<x_1\leqq 0.$$

Secondly let us consider the trajectory controlled by the function which has a type (c) in Figure 4. In this case the optimal trajectory moves along the curve in Figure 5 at first and then moves along the curve in Figure 6 and ultimately arrives at the origin. Accordingly we must discover the set of curves in Figure 5 which intersect with the arc $B^{(2)}O$. It is clear that these curves exist between the arc $B^{(1)}O$ and the arc $B^{(1)}B^{(2)}$ in Figure 7. Therefore let us consider the arc $B^{(1)}B^{(2)}$. If $(15)_1$ arrives at $B^{(2)}$ at some time t^*, we get

$$(22)\qquad \begin{cases}B_1^{(2)}=B_1^{(1)}+L_1^{(1)}e^{\xi t^*},\\ B_2^{(2)}=B_2^{(1)}+L_2^{(1)}e^{\eta t^*},\end{cases}$$

and

$$(23)\qquad \frac{(B_2^{(2)}-B_2^{(1)})^{1/\eta}}{(B_1^{(1)}-B_1^{(2)})^{1/\xi}}=\frac{(L_2^{(1)})^{1/\eta}}{(-L_1^{(1)})^{1/\xi}}.$$

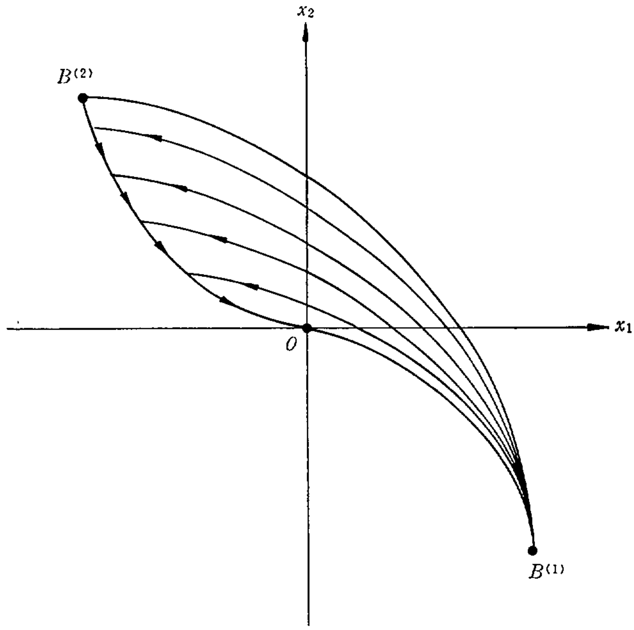
Conversely, it is clear that (23) concludes (22) for $t^*=\log((B_1^{(1)}-B_1^{(2)})/(-L_1^{(1)}))^{1/\xi}=\log((B_2^{(2)}-B_2^{(1)})/L_2^{(1)})^{1/\eta}$. Therefore the necessary and sufficient condition that the trajectory $(15)_1$ arrives at $B^{(2)}$ at some time, is (23). By $(15)_1$ and (23), we get

$$\left(\frac{B_1^{(1)}-x_1}{-L_1^{(1)}}\right)^{1/\xi}=\left(\frac{x_2-B_2^{(1)}}{L_2^{(1)}}\right)^{1/\eta},$$

and

$$\frac{(x_2-B_2^{(1)})^{1/\eta}}{(B_1^{(1)}-x_1)^{1/\xi}}=\frac{(L_2^{(1)})^{1/\eta}}{(-L_1^{(1)})^{1/\xi}}=\frac{(B_2^{(2)}-B_2^{(1)})^{1/\eta}}{(B_1^{(1)}-B_1^{(2)})^{1/\xi}}.$$

FIG. 7



But by (8) and (13), we get

$$\frac{(B_2^{(2)}-B_2^{(1)})^{1/\eta}}{(B_1^{(1)}-B_1^{(2)})^{1/\xi}}=\frac{\left\{2\cdot\frac{1}{\eta}\cdot\frac{(c_1\gamma+c_2+\xi)c_2}{\xi-\eta}\right\}^{1/\eta}}{\left\{2\cdot\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}\right\}^{1/\xi}}=\frac{\{\xi(\xi-\eta)\}^{1/\xi}}{\{\eta(\xi-\eta)\}^{1/\eta}}\cdot\frac{\{2(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{2(c_1\gamma+c_2+\eta)c_2\}^{1/\xi}}.$$

Accordingly by (19), we get

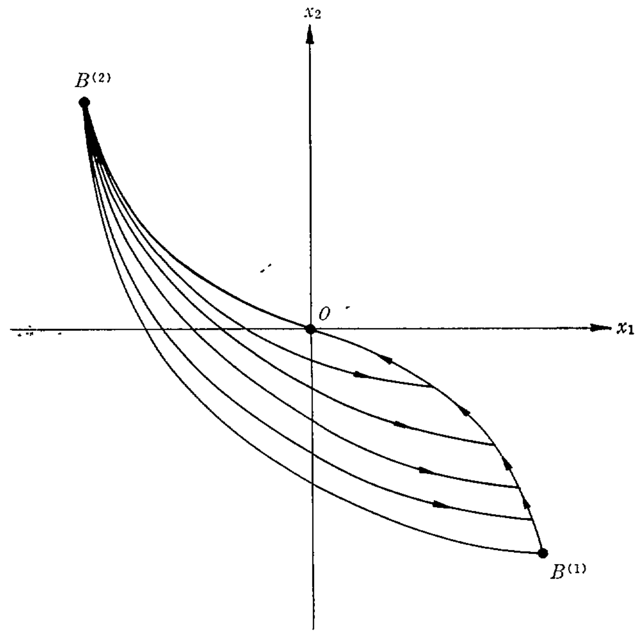
$$\frac{\{\eta(\xi-\eta)x_2+(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{(c_1\gamma+c_2+\eta)c_2-\xi(\xi-\eta)x_1\}^{1/\xi}}=\frac{\{2(c_1\gamma+c_2+\xi)c_2\}^{1/\eta}}{\{2(c_1\gamma+c_2+\eta)c_2\}^{1/\xi}}.$$

Therefore we get

$$\text{(24)}\qquad x_2=\frac{2(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[\left\{\frac{1}{2}-\frac{\xi(\xi-\eta)}{2(c_1\gamma+c_2+\eta)c_2}\cdot x_1\right\}^{\eta/\xi}-\frac{1}{2}\right],$$
$$-\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}\leqq x_1<\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}.$$

Finally let us consider the trajectory controlled by the function which has a type (a) in Figure 4. In this case the optimal trajectory moves along the curve in Figure 6 at first and then move along the curve in Figure 5 and ultimately arrives at the origin. Accordingly we must discover the set of curves in Figure 6 which intersect with the arc $B^{(1)}O$. It is clear that these curves exist between the arc $B^{(2)}O$ and the arc $B^{(2)}B^{(1)}$ in Figure 8. Therefore let us

FIG. 8



consider the arc $B^{(2)}B^{(1)}$. The process of computation is the same as the case in which the type of control function is (c) in Figure 4. Therefore let us denote the conclusion; namely

$$(25)\qquad x_2=\frac{2(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[\frac{1}{2}-\left\{\frac{\xi(\xi-\eta)}{2(c_1\gamma+c_2+\eta)c_2}\cdot x_1+\frac{1}{2}\right\}^{\eta/\xi}\right],$$
$$-\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}<x_1\leqq\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}.$$

(24) and (25) decide the region whose points can be transfered according to the equation of motion of the system by some optimal control. Namely

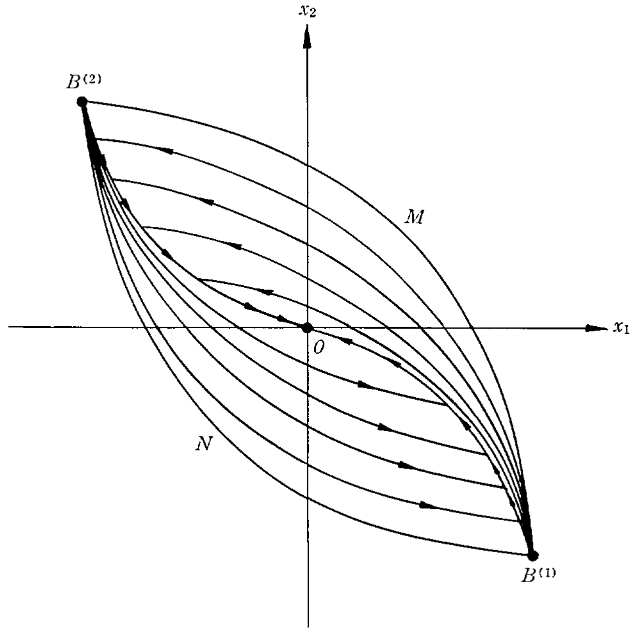
$$(26)\qquad \left\{(x_1,x_2)\in \boldsymbol{R}^2;\ \frac{2(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[\frac{1}{2}-\left\{\frac{\xi(\xi-\eta)}{2(c_1\gamma+c_2+\eta)c_2}\cdot x_1+\frac{1}{2}\right\}^{\eta/\xi}\right]\right.$$
$$<x_2<\frac{2(c_1\gamma+c_2+\xi)c_2}{\eta(\xi-\eta)}\cdot\left[\left\{\frac{1}{2}-\frac{\xi(\xi-\eta)}{2(c_1\gamma+c_2+\eta)c_2}\cdot x_1\right\}^{\eta/\xi}-\frac{1}{2}\right],$$
$$\text{and}\quad \left.-\frac{1}{\xi}\cdot\frac{(c_1\gamma+c_2+\eta)c_2}{\xi-\eta}<x_1<\frac{1}{\xi}\cdot\frac{(c_1\gamma+c^2+\eta)c_2}{\xi-\eta}\right\}.$$

Figure 9, which is obtained by combining Figure 7 with Figure 8, is a rough map of this region. In Figure 9 the arc $B^{(1)}MB^{(2)}$ is indicated by (24) and the arc $B^{(2)}NB^{(1)}$ is indicated by (25).

We can solve the synthesis problem for a control problem formulated by (7). Namely synthesis function $v(x_1, x_2)$ is defined as follows;

$$v(x_1,x_2)=\begin{cases}A^{(1)}; & \text{if}\quad (x_1,x_2)\in B^{(1)}MB^{(2)}O,\\ A^{(2)}; & \text{if}\quad (x_1,x_2)\in B^{(1)}NB^{(2)}O.\end{cases}$$

FIG. 9



Finally let us rewrite the region (26) using the original variables F and A in Simon-Homans system.

$$\begin{aligned}\Big\{(A, F)\in R^2;\ &\frac{1}{2}-\left\{\frac{c_1\xi}{2(c_1\gamma+c_2+\eta)c_2}\cdot F-\frac{\xi}{2c_1}\cdot A+\frac{1}{2}\right\}^{\eta/\xi}\\ &<-\frac{c_1\eta}{2(c_1\gamma+c_2+\xi)c_2}\cdot F+\frac{\eta}{2c_2}\cdot A\\ &<\left\{\frac{1}{2}-\frac{c_1\xi}{2(c_1\gamma+c_2+\xi)c_2}\cdot F+\frac{\xi}{2c_2}\cdot A\right\}^{\eta/\xi}-\frac{1}{2},\quad\text{and}\\ &-\frac{c_2}{\xi}<\frac{c_1}{c_1\gamma+c_2+\eta}\cdot F-A<\frac{c_2}{\xi}\Big\}.\end{aligned}$$

REFERENCES

[1] H. A. Simon, "A Formal Theory of Interaction in Social Groups," *American Sociological Review*, April, 1952.. Vol. 17, No:2.
[2] G. C. Homans, *The Human Group*, New York, Harpers, 1950.
[3] L. S. Pontrjagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Process*, John Wiley and Sons 1962.

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