EQUILIBRIUM PRICING AND ADVERTISING STRATEGY IN OLIGOPOLY

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1. *Introduction*

In this paper we shall analyse some dynamic aspects of advertising expenditures in oligopoly. It is often said that the present demand for the product advertised is affected by the present and past advertising expenditures. K.J. Arrow and M. Nerlove call the latter amounts goodwill which affects the present demand and they denote it by $A(t)$ in the context of a monopoly firm.[1] There it is supposed that the price of a unit of goodwill is \$1, so that a dollar of current advertising outlays increases goodwill by a like amount.

Additional advertising outlays increase goodwill by means of attracting more customers to the product or brand advertised. But it depreciates over time by means of other firms' advertising campaigns drawing customers away from the product initially considered and a tendency for the preferences of customers to their old pattern in the time when the advertising campaigns were not carried out. So the relationship between the current advertising outlays $a(t)$ and the present goodwill $A(t)$ will be written as follows:

$$\dot{A}(t)=a(t)-\delta A(t),$$

where δ is a proportional depreciation rate of goodwill.[2]

In the model of Arrow and Nerlove, the relationship between those amounts is considered in the context of a monopoly firm. Though the effect of other firms' advertising expenditures on the goodwill of the firm in question is included in the term δ, it is not implicitly introduced in their model. So in this paper we shall introduce this effect into our model in the more explicit form and analyse the equilibrium strategy (defined below) in oligopoly.

2. *The Model*

Assume that there are n firms in the differentiated oligopoly in which the cross-elasticities of demand among their products are not so large. Further assume that there is the relationship between the current advertising expenditures of all firms and the i-th firm's goodwill, i.e.,

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1 K.J. Arrow and M. Nerlove, [2].

2 In this paper we ignore the diffusion process of information among the potential customers that is analysed by S. Glaister, [6], J.P. Gould, [7], S. Ozga, [11] and G.J. Stigler [14].

$$\dot{A}_i(t)=\sum_{j=1}^{n} w_{ij}a_j(t)-\delta A_i(t),$$
$$i=1,\ 2,\ \ldots,\ n. \tag{1}$$

where $A_i\ (t)$ and $a_j\ (t)$ are the i-th firm's goodwill and the j-th firm's current advertising expenditures at time t respectively, and δ is a proportional depreciation rate of goodwill (common to all firms). Concerning the coefficient w_{ij}'s, we assume as follows:

$$w_{ii}=1,\quad w_{ij}<0\ (i\neq j),\quad i,\ j=1,\ 2,\ \ldots.,\ n. \tag{2}$$
$$\sum_{j\neq i}|w_{ij}|<w_{ii}=1,\quad i=1,\ 2,\ \ldots,\ n.$$

The negativity of $w_{ij}\ (j\neq i)$ expresses the reductive effect of the j-th firm's current advertising expenditures on the i-th firm's goodwill. And the last expression in (2) means that the total effect of all other firm's advertising expenditures on the i-th firm's goodwill is less than what the i-th firm's advertising expenditures have on it, if all of the firms spend money on advertisement by the same amounts. One firm's advertising policy has influence on every other firm's goodwill through the terms w_{ij}'s.

Here let's define other notations as follows:

$p_i(t)$ .... A price that the i-th firm sets at time t.

$\boldsymbol{p}(t)$ .... A vector in n-dimensional Euclidean space whose i-th component is $p_i(t)$.

$\bar{\boldsymbol{p}}_i(t)$ .... A vector in $(n-1)$-dimensional Euclidean space that is obtained by removing the i-th component from the vector $\boldsymbol{p}(t)$ defined above.

$a_i(t)$ .... Current advertising expenditures that the i-th firm spends at time t.

$\boldsymbol{a}(t)$ .... A vector in n-dimensional Euclidean space whose i-th component is $a_i(t)$.

$A_i(t)$ ...... A level of the i-th firm's goodwill at time t.

$A(t)$ .... A vector in n-dimensional Euclidean space whose i-th component is $A_i(t)$.

Further let's define the sets, $\mathscr{P}_i$, $\mathscr{a}_i$ and $\mathscr{A}_i$ as follows:

$$\mathscr{P}_i=\{p_i(t)|t\geq 0\},\quad \mathscr{a}_i=\{a_i(t)|t\geq 0\}\quad \text{and}$$
$$\mathscr{A}_i=\{A_i(t)|t\geq 0\},\ i=1,\ 2,\ \ldots.,\ n. \tag{3}$$

where the sets $\mathscr{P}_i$ and $\mathscr{a}_i$ show the whole of pricing and advertising strategies respectively that the i-th firm can take over time, and the set $\mathscr{A}_i$ is the whole of values that the level of the i-th firm's goodwill is attainable over time.

Assumption 1. We assume the sets defined in (3) take the following forms respectively,

$$\mathscr{P}_i=[0,\ p_i^m],\quad \mathscr{a}_i=[0,\ \infty],\ \text{and}\ \mathscr{A}_i=[0,\ A_i^m]$$
$$i=1,\ 2,\ \ldots.,\ n. \tag{4}$$

where p_i^m and A_i^m are finite positive large numbers respectively.

The i-th firm's demand at time t depends on its own price $p_i(t)$, the prices $\bar{p}_i(t)$ that the other firms set, and the level of its own goodwill $A_i(t)$. So its demand $D_i(t)$ at time t is written:

$$D_i(t)=D_i(p_i(t),\ \bar{p}_i(t),\ A_i(t)),\ i=1,\ 2,\ \dots,\ n.$$

If each firm's production cost is dependent upon only its production rates, the total production cost function is written $C_i(D_i(t))$. Then the gross profit function for the i-th firm $R_i(t)$ is defined as follows;

$$R_i(t)=p_i(t)D_i(p_i(t),\ \bar{p}_i(t),\ A_i(t))-C_i(D_i(p_i(t),\ \bar{p}_i(t),\ A_i(t)))\quad i=1,\ 2,\ \dots,\ n.$$

Assume that the function R_i ($i=1,\ 2,\dots,\ n.$) has the following properties.

Assumption 2. (1) The function R_i ($i=1, 2,\dots, n.$) is continuously twice differentiable in $\mathscr{P}\times\mathscr{A}_i$, where $\mathscr{P}=\mathscr{P}_1\times\mathscr{P}_2\times\dots\times\mathscr{P}_n$. (2) It is strictly concave with respect to p_i and A_i. (3) The function $R_i-(r+\delta)\ A_i$ ($i=1,\ 2,\dots,\ n.$) has an inner maximum in $\mathscr{P}_i\times\mathscr{A}_i$.

The net profit for the i-th firm at time t is the gross profit $R_i(t)$ net of its advertising expenditures $a_i\ (t)$. Selecting a suitable pair of functions $(p_i(t),\ a_i(t))$ in $\mathscr{P}_i\times\mathscr{A}_i$, the i-th firm is supposed to make efforts to maximize the following cash flows for a given other firms' pricing and advertising strategies and for a given initial condition of its own goodwill.

$$\int_0^\infty(R_i(p_i(t),\ \bar{p}_i(t),\ A_i(t))-a_i(t))e^{-rt}dt$$
$$i=1,\ 2,\ \dots,\ n.$$

where r is a rate of interest (common to all firms).

Definition 1. If there exists a pair of vector functions $(\boldsymbol{a}^*(t),\ \boldsymbol{p}^*(t))$ in $\mathscr{A}\times\mathscr{P}$ that satisfies the following inequalies, subject to the constraints (1), (2) and (4), and the given initial condition A^0,

$$\int_0^\infty(R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))-a_i^*(t))e^{-rt}dt$$
$$\geq\int_0^\infty(R_i(p_i(t),\ \bar{p}_i^*(t),\ A_i(t))-a_i(t))e^{-rt}dt$$
$$\text{for all } a_i(t)\in\mathscr{A}_i,\ p_i(t)\in\mathscr{P}_i \text{ and all } i\text{'s.} \tag{5}$$

then we call it an equilibrium (advertising and pricing) strategy in the system,[3] where $\mathscr{A}=\mathscr{A}_1\times\mathscr{A}_2\times\dots\times\mathscr{A}_m$ and $\mathscr{P}=\mathscr{P}_1\times\mathscr{P}_2\times\dots\times\mathscr{P}_n$, and $A^*(t)$ and $A(t)$ are trajectories corresponding to the functions $a^*(t)$ and $a(t)$, respectively.

In the next section, we shall show the existence of an equilibrium strategy in the system and analyse the natures of it.

[3] The concept of equilibrium strategy just defined is clearly a dynamic extension of the Cournot point [4] or the Nash's equilibrium point [8] that is defined in the static contexts.

3. *Equilibrium Strategy*

Reformulating the problem in the last section, it is written as follows;

$$\max_{\{p_i(t),\ a_i(t)\}} \int_0^\infty (R_i(p_i(t),\ \bar{p}_i^*(t),\ A_i(t)) - a_i(t)) e^{-rt} dt$$

subject to,

$$\dot{A}_i(t) = \sum_{j \neq i} w_{ij} a_j^*(t) + a_i(t) - \delta A_i(t),\quad i=1,\ 2,\ \ldots,\ n.$$

and $\quad a_i(t) \geq 0,\quad i=1,\ 2,\ \ldots,\ n.$

If we assume the existence of an equilibrium strategy $(\boldsymbol{a}^*(t),\ \boldsymbol{p}^*(t))$ and the corresponding trajectory $\boldsymbol{A}^*(t)$, then we can obtain a necessay condition that an equilibrium strategy should satisfy by making use of optimal control theory.

Let L_i be the Lagrangean for the i-th firm, and write it as follows:

$$L_i = (R_i(p_i(t),\ \bar{p}_i^*(t),\ A_i(t)) - a_i(t)) e^{-rt} + \sum_{k=1}^{n} \lambda_{ik}(t) \Big(\sum_{j \neq i} w_{kj} a_j^*(t) + w_{ki} a_i(t) - \delta A_k(t)\Big) + \mu_i(t) a_i(t),\quad i=1,\ 2,\ \ldots,\ n.$$

Then an equilibrium strategy $(\boldsymbol{a}^*(t),\ \boldsymbol{p}^*(t))$ and the corresponding trajectory $A^*(t)$ satisfy the following relations:

$$\partial L_i / \partial p_i(t) = (\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t)) / \partial p_i(t)) e^{-rt} = 0,\quad i=1,\ 2,\ \ldots,\ n. \tag{6}$$

$$\partial L_i / \partial a_i(t) = -e^{-rt} + \sum_{k=1}^{n} \lambda_{ik}(t) w_{ki} + \mu_i(t) = 0$$
$$\mu_i(t) a_i^*(t) = 0,\quad \mu_i(t) \geq 0,\quad i=1,\ 2,\ \ldots,\ n. \tag{7}$$

$$\dot{\lambda}_{ii}(t) = -\partial L_i / \partial A_i(t) = -[\{\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t)) / \partial A_i(t)\} e^{-rt} - \delta \lambda_{ii}(t)],\quad i=1,\ 2,\ \ldots,\ n. \tag{8}$$

$$\dot{\lambda}_{ij}(t) = -\partial L_i / \partial A_j(t) = \delta \lambda_{ij}(t)\quad j \neq i;\ i,\ j=1,\ 2,\ \ldots,\ n. \tag{9}$$

with the traversarity conditions,

$$\lim_{t \to \infty} \lambda_{ij}(t) e^{-rt} = 0,\ i,\ j=1,\ 2,\ \ldots,\ n. \tag{10}$$

From (9) and (10), $\lambda_{ij}(t)=0$, for all $t \in [0,\ \infty)$, $j \neq i\quad i,\ j=1,\ 2,\ \ldots,\ n$.. In addition, replacing $\lambda_{ii}(t)$ and $\mu_i(t)$ with $\lambda_i(t)e^{-rt}$ and $\mu_i(t)e^{-rt}$ in (7) and (8), the system of (6), (7), (8) and (9) is reduced to much simpler form, i.e.,

$$\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t)) / \partial p_i(t) = 0,\quad i=1,\ 2,\ \ldots,\ n. \tag{6)$'$(}$$

$$\lambda_i(t) + \mu_i(t) = 1,\ \mu_i(t) a_i^*(t) = 0,\ \mu_i(t) \geq 0\quad i=1,\ 2,\ \ldots,\ n. \tag{7)$'$(}$$

$$\dot{\lambda}_i(t)=(r+\delta)\lambda_i(t)-\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))/A_i(t)$$
$$i=1,\ 2,\ \ldots,\ n. \tag{8'}$$

First of all, we shall show the following proposition.

Proposition 1. If Assumption 1 and 2 are true, then there exists a price vector $\boldsymbol{p}^+ \in \mathscr{P}=\mathscr{P}_1\times\mathscr{P}_2\times\ldots\times\mathscr{P}_n$ for all $A\in\mathscr{A}=\mathscr{A}_1\times\mathscr{A}_2\times\ldots.\times\mathscr{A}_n$ that satisfies the relations,

$$\partial R_i(p_i^\dagger,\ \boldsymbol{p}_i^\dagger,\ A_i)/\partial p_i=0,$$
$$i=1,\ 2,\ \ldots,\ n. \tag{11}$$

(proof)

The function R_i is continous in the compact and convex set $\mathscr{P}$ (A. 1 and A. 2(1)) and strictly concave with respect to p_i (A. 2(2)). So there exists a price vector $\boldsymbol{p}^+$ in $\mathscr{P}$ satisfying the following inequalities,[4]

$$R_i(p_i^\dagger,\ \bar{\boldsymbol{p}}_i^\dagger,\ A_i)\geq R_i(p_i,\ \bar{\boldsymbol{p}}_i^\dagger,\ A_i)$$
for all p_i in $\mathscr{P}_i$ and all i's.

Further, the function R_i is continuously differentiable and has an inner maximum in $\mathscr{P}_i$ (A. 2(1) and A. 2(3)). It reaches its maximum value at $p_i^\dagger$ for a given $(\bar{p}_i^\dagger, A_i)$. Therefore, its first partial derivative becomes zero at the point $(\boldsymbol{p}^+, A_i)$. Or Proposition 1 is true. (q.e.d.).

From Proposition 1, we know that an equilibrium pricing strategy corresponding to the state variable $\boldsymbol{A}(t)$ always exists under our Assumption 1 and 2. Here we assume the followings with regard to the vector $\boldsymbol{p}^+$ that satisfies the relation (11).

Assumption 3. A price vector that satisfies the relation (11) is a continuously differentiable function of $\boldsymbol{A}$.

According to this Assumption, we can write the price vector $\boldsymbol{p}$ that satisfies the relation (11) as follows,

$$\boldsymbol{p}=\boldsymbol{p}(A),\ (p\colon \mathscr{A}\rightarrow\mathscr{P}) \tag{12}$$

Using the expression (12), we can define a function N_i as follows,

$$N_i(A(t))=\partial R_i(p_i(A(t)),\ \bar{p}_i(A(t)),\ A_i(t))/\partial A_i(t)$$
$$i=1,\ 2,\ \ldots.,\ n. \tag{13}$$

The function N_i is interpreted as the marginal revenue of goodwill A_i. And the sum of

[4] This is an application of J.F. Nash's theorem [8]. With regard to its proof, for example, see K. Okuguchi [10].

a rate of interest r and a depreciation rate of goodwill δ, i.e., $(r+\delta)$ is interpreted as the marginal cost of goodwill for all firms. Let's define a stationary state of the system by using the above terms.

Definition 2. If there exists a state in which the marginal revenue of goodwill for each firm is equal to the common marginal cost of it, then we call it a stationary state of the system. In notation, if there exists a vector A^* that satisfies the following relations simultaneously,

$$N_i(A^*)=r+\delta, \quad i=1, 2, \ldots, n. \tag{14}$$

then it is a stationary state of the system.

In order to show the existence of a stationary state, we need an additional assumption with regard to the function N_i's. This assumption is not necessarily followed from the preceding ones which do not refer to the relative values of $\partial^2 R_i / \partial p_i^2$, $\partial^2 R_i / \partial p_j \partial p_i$, and $\partial^2 R_i / \partial A^2{}_i$. The following assumption is stated in the inclusive form without specifying the above values.

Assumption 4. The function N_i's satisfy the following condition:

$$\partial N_i(A)/\partial A_i<0, \text{ for all } A\in\mathscr{A},$$
$$i=1, 2, \ldots, n.$$

Proposition 2. If Assumption 1, 2, 3 and 4 are true, then there exists a goodwill vector A^* that satisfies the following equations.

$$N_i(A^*)=r+\delta, \quad i=1, 2, \ldots, n.$$

In other words, there exists at least a stationary state of the system.

(Proof)

The proof of Proposition 2 is attainable following the same procedures as those of Proposition 1. In this time we have only to use $R_i(p(A), A_i)-(r+\delta)A_i$ instead of $R_i(p_i, \bar{p}_i, A_i)$ in the proof of Proposition 1. So we shall omit the details.

(q.e.d.)

Assumption 1 to 4 assure us the existence of a stationary state, but they do not necessarily assure the uniqueness of it. If there are many stationary states in the system, we cannot know where the system starting from an arbitrary initial point A^0 in $\mathscr{A}$ will go forward after a long time elapses. Therefore, in order to know a clear-cut time path that the system will proceed along, we have to assume the followings.

Assumption 5. There exists a unique goodwill vector A^* satisfying the equations (14). In other words, there exists an only stationary state in the system.

Based on the above preparations, let's make clear the characters of an equilibrium strategy. We have already shown that an equilibrium pricing strategy would be determined if the corresponding trajectory $A^*(t)$ were predetermined. So the next step to be taken is to know the characters of an equilibrium advertising strategy for a given initial condition of goodwill $A(0)=A^0$ with aid of the function N_i's.

From (7)$'$, we know that $\lambda_i(t)\leq 1$ is always true for all i's and that

$$a_i^*(t)=0, \text{ if } \lambda_i(t)<1, \quad i=1, 2, \ldots, n.$$
$$a_i^*(t)\geq 0, \text{ if } \lambda_i(t)=1, \quad i=1, 2, \ldots, n. \tag{15}$$

where the asterrisk means an equilibrium strategy as before. And applying the elementary calculus to (8)$'$, we have the following relations, i.e.,

$$(\lambda_i(t)-1)\cdot e^{-(r+\delta)t}=\int_t^\infty (N_i(A^*(t))-(r+\delta))e^{-(r+\delta)\tau}d\tau$$
$$i=1, 2, \ldots, n.$$

Therefore,

$$\lambda_i(t)\leqq 1\Leftrightarrow\int_t^\infty (N_i(A^*(\tau))-(r+\delta))e^{-(r+\delta)\tau}d\tau\leqq 0$$
$$i=1, 2, \ldots, n. \tag{16}$$

Now let's permit jump possibilities in the advertising strategy at time 0, as Arrow and Nerlove do in their paper.[5] To do so, it is necessary that we should correct the coefficient w_{ij}'s in (1) and (2) a little. Namely, they are replaced with the following function $w_{ij}(t)$'s.

$$w_{ii}(t)=1, \quad t\geq 0, \quad i=1, 2, \ldots, n.$$
$$w_{ij}(t)=\begin{cases} 0, & t=0, \\ w_{ij} \text{ (constant)}<0, & t>0, \end{cases}$$
$$j\neq i; \quad i, \ j=1, 2, \ldots, n. \tag{17}$$

and,

$$\sum_{j\neq i}|w_{ij}|<w_{ii}=1, \quad i=1, 2, \ldots, n.$$

So the relation in (1) is also corrected as follows,

$$\dot{A}_i(t)=\sum_{j=1}^n w_{ij}(t)a_j(t)-\delta A_i(t)$$
$$i=1, 2, \ldots, n. \tag{1$'$}$$

Though this correction does not bring about any alteration to (1) and (2) except the one at time 0, it enables us to escape the troublesome problem that comes about by permitting jump possibilities.

Putting together the results of Proposition 1 and 2, and the relations (15), (16) and (17), an equilibrium pricing and advertising strategy for any firm is stated in the following proposition.

Proposition 3. If assumptions A.1—3. are satisfied, then an equilibrium pricing strategy for the i-th firm at time t is to set its price on the level in which the following relation is satisfied,

$$\partial R_i(p_i(t), \bar{p}_i^*(t), A_i^*(t))/\partial p_i(t)=0,$$
$$i=1, 2, \ldots, n.$$

where the equilibrium trajectory $A^*(t)$ will be determined corresponding to the equilibrium advertising strategy $a^*(t)$ stated later on. And further if assumptions A. 1—5. and the following conditions are satisfied,

[5] K.J. Arrow and M. Nerlove, [2].

$$a_i^*(t)\geq 0,\quad t\geq 0,\quad \text{and for all } i\text{'s.} \tag{18}$$

then an equilibrium advertising strategy for the i-th firm is determined as follows for a given initial condition,

1°. If $N_i(A^0)>r+\delta$, then $a_i^*(0)=+\infty$, and $a_i^*(t):\{N_i(A^*(t))=r+\delta$, for $t>0\}$[6]
2°. If $N_i(A^0)=r+\delta$, then $a^*(t):\{N_i(A^*(t))=r+\delta$, for $t\geq 0\}$.
3°. If $N_i(A^0)<r+\delta$, then $a_i^*(t)=0$, for $0\leq t<t_i$, and $a_i^*(t):\{N_i(A^*)(t))=r+\delta$, for $t\geq t_i\}$, where t_i is defined as follows,

$$t_i=\inf\{t|N_i(A^*(t))=r+\delta\}$$
$$i=1,\ 2,\ \ldots,\ n. \tag{19}$$

(proof)

It is enough to show that the equilibrium pricing and advertising strategy for any firm is not dominated by any admissible strategy of it when the other firms' equilibrium strategies are given. Namely, we have only to show that the following inequality holds for all i's.

$$\int_0^\infty (R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))-a_i^*(t))e^{-rt}dt$$
$$\geq \int_0^\infty (R_i(p_i(t),\ \bar{p}_i^*(t),\ A_i(t))-a_i(t))e^{-rt}dt \tag{20}$$

for all $p_i(t)\in\mathscr{P}_i$, $a_i(t)\in_{\alpha i}$, $i=1,\ 2,\ \ldots,\ n$.

Let,

$$V_i=(\text{the LHS of (20)})-(\text{the RHS of (20)})$$
$$i=1,\ 2,\ \ldots.,\ n.$$

Since R_i is a strictly concave function with respect to p_i and A_i.

$$V_i>\int_0^\infty [\{\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))/\partial p_i\}(p_i^*(t)-p_i(t))$$
$$+\{R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))/\partial A_i\}(A_i^*(t)-A_i(t))]e^{-rt}dt$$
$$-\int_0^\infty (a_i^*(t)-a_i(t))e^{-rt}dt,\ \text{for all } i\text{'s}.$$

Since,

$$\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))/\partial p_i=0,$$

and

$$\partial R_i(p_i^*(t),\ \bar{p}_i^*(t),\ A_i^*(t))/\partial A_i=N_i(A^*(t)),\ \text{for all } i\text{'s.}$$

$$V_i>\int_0^\infty N_i(A^*(t))(A_i^*(t)-A_i(t))e^{-rt}dt$$
$$-\int_0^\infty (a_i^*(t)-a_i(t))e^{-rt}dt,\quad \text{for all } i\text{'s.}$$

From (1)′, (17) and the definition of the equilibrium advertising strategy for the i-th firm, the following two equations are derived.

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[6] The expression $a_i^*(t):\{N_i(A^*(t))=r+\delta,\ t\geq t_i\}$, means that the i-th firm takes an advertising strategy after time t_i so as to keep the relation $N_i(A^*(t))=r+\delta$.

$$\sum_{j\neq i} w_{ij}(t)a_j^*(t)+a_i(t)=\dot{A}_i(t)+\delta A_i(t)$$
$$\sum_{j\neq i} w_{ij}(t)a_j^*(t)+a_i^*(t)=\dot{A}_i^*(t)+\delta A_i^*(t)$$
$$i=1,\ 2,\ \ldots,\ n.$$

So that,

$$a_i^*(t)-a_i(t)=\dot{A}_i^*(t)-\dot{A}_i(t)+\delta(A_i^*(t)-A_i(t))$$
$$i=1,\ 2,\ \ldots,\ n.$$

Substituting this relation into the last inequality and applying integration by parts to it,

$$V_i>\int_0^\infty (N_i(A^*(t))-(r+\delta))(A_i^*(t)-A_i(t))e^{-rt}dt$$
$$i=1,\ 2,\ \ldots,\ n. \qquad (21)$$

In all three cases of Proposition 3, it will be shown that the RHS of (21) are non-negative. First, for the i-th firm whose initial condition is expressed by 1^o,

$$\int_{0+}^\infty (N_i(A^*(t))-(r+\delta))(A_i^*(t)-A_i(t))e^{-rt}dt=0.$$

and since we have not put any upper limit on the advertising *expenditure* for every firm at each time, it is clear that an advertising expenditure for the i-th firm at time 0 is plus infinite in the equilibrium strategy when $N_i(A^0)>r+\delta$, i.e., $A_i^0<A_i^*(0)$.

In case of 2^o, the RHS of (21) is clearly zero.

Lastly, in case of 3^o the RHS of (21) is divided into two parts, i.e.,

The RHS of (21)

$$=\int_{0+}^{t_i} (N_i(A^*(t))-(r+\delta))(A_i^*(t))-A_i(t))e^{-rt}dt$$
$$+\int_{t_i}^\infty (N_i(A^*(t))-(r+\delta))(A_i^*(t)-A_i(t))e^{-rt}dt.$$

where t_i is defined in (19).

Since $N_i(A^*(t))<r+\delta$, and $A_i^*(t)<A_i(t)$, for $0\leq t<t_i$, and $N_i(A^*(t))=r+\delta$, for $t\geq t_i$, so that the RHS of (21) is non-negative. (q.e.d.)

In this section we have put forward our analysis, based on the rather general assumptions with regard to the gross profit function R_i's and the marginal revenue function of goodwill N_i's derived from the formers. But the preceding five assumptions do not assure us the truth of the condition in the premise of Proposition 3, i.e., $a_i^*(t)\geq 0$, for $t\geq 0$ and all i's. Non-negativity of $a_i^*(t)$'s are dependent on the nature of the matrices $\mathbf{N}$ and $\mathbf{W}$, where $\mathbf{N}$ and $\mathbf{W}$ are $n\times n$ matrices respectively, and

$$\mathbf{N}=[\partial N_i(A^*(t))/\partial A_j] \text{ and } \mathbf{W}=[w_{ij}]$$
$$i,\ j=1,\ 2,\ \ldots,\ n.$$

If we assume that the k $(0\leq k\leq n)$ firms reach successively a position till time t in which the marginal revenue of goodwill for each of those firms is equal to its marginal cost of goodwill, (here it is supposed that those k firms are from the first to the k-th one without loss of generality.), then the following equation holds at time t (>0),

$$(\mathbf{N}_{kk}\mathbf{W}_{kk}+\mathbf{N}_{k,n-k}\mathbf{W}_{n-k,k})\boldsymbol{a}^{k*}(t)=\delta(N_{kk}A^{k*}(t)+\mathbf{N}_{k,n-k}A^{(n-k)*}(t))$$
$$k=1,\ 2,\ \ldots\ldots,\ n. \qquad (22)^{7}$$

where the matrices $\mathbf{N}$, $\mathbf{W}$ and the vector $\mathbf{a}^*(t)$, $\mathbf{A}^*(t)$ are partitioned into as follows, respectively,

$$\mathbf{N}=\begin{pmatrix}\mathbf{N}_{kk} & \mathbf{N}_{k,\,n-k}\\ \mathbf{N}_{n-k,\,k} & \mathbf{N}_{n-k,\,n-k}\end{pmatrix},\qquad W=\begin{pmatrix}W_{kk} & W_{k,\,n-k}\\ W_{n-k,\,k} & W_{n-k,\,n-k}\end{pmatrix}$$
$$\boldsymbol{a}^*(t)=\begin{pmatrix}\mathbf{a}^{k*}(t)\\ \mathbf{a}^{(n-k)*}(t)\end{pmatrix}\qquad \text{and } A^*(t)=\begin{pmatrix}\mathbf{A}^{k*}(t)\\ \mathbf{A}^{(n-k)*}(t)\end{pmatrix}$$
$$k=1,\ 2,\ \ldots,\ n.$$

$\mathbf{N}_{kk}$, $\mathbf{W}_{kk}$: $k\times k$,
$\mathbf{N}_{k,\,n-k}$, $\mathbf{W}_{k,\,n-k}$: $k\times(n-k)$
$\mathbf{N}_{n-k,\,k}$, $\mathbf{W}_{n-k,\,k}$: $(n-k)\times k$
$\mathbf{N}_{n-k,\,n-k}$, $\mathbf{W}_{n-k,\,n-k}$: $(n-k)\times(n-k)$
$\boldsymbol{a}^{k*}(t)$, $A^{k*}(t)$: $k\times 1$
$\boldsymbol{a}^{(n-k)*}(t)$, $\mathbf{A}^{(n-k)*}(t)$: $(n-k)\times 1$.

Therefore, the non-negativity of $\mathbf{a}^*(t)$, for all $t\geq 0$, is not assured without knowing the nature of the matrix $\mathbf{N}$. So in the next section, we shall analyse more concrete cases by specifying the price-setting behaviors of oligopolists or adopting the concrete demand and cost function of them, so that we can examine whether or not our Assumption 3, 4, and 5, and the non-negativity assumption of the current advertising expenditures for all firms in the equilibrium strategy will hold.

4. *Simple Cases*

(1) Fixed Price

Many empirical studies of advertising expenditures in oligopoly point out that prices have been fixed for a long time, although advertising activities by oligopolists are prevailing in the aggressive forms.[8] In other words, an oligopolist uses advertisement rather than price-cutting as a competitive weapon to his opponents. W.J. Baumol says,

[7] According to Proposition 3, the i-th firm that has reached the position mentioned above goes on taking an advertising move thereafter so as to keep the relation, $N_i(A^*(t))=r+\delta$, $(i=1, 2, \ldots, n.)$. Differentiating the both sides of this equation with respect to time t, we have the following equation,

$$\sum_{j=1}^{n} N_{ij}(A^*(t))\dot{A}_j^*(t)=0,\quad i=1,\ 2,\ \ldots\ldots,\ k.$$

Substituting $\dot{A}_j^*(t)=\sum_{l=1}^{n} w_{jl}a_l^*(t)-\delta A_j^*(t)$, $t>0$, $(j=1,\ 2,\ \ldots\ldots,\ n.)$ into the above equation, then we have

$$\sum_{j=1}^{n}\sum_{l=1}^{n} N_{ij}(A^*(t))w_{jl}a_l^*(t)=\delta\sum_{j=1}^{n} N_{ij}(A^*(t))A_j^*(t)$$
$$i=1,\ 2,\ \ldots\ldots,\ k.$$

Since $a_l^*(t)=0$, for $t>0$, and $l=k+1,\ k+2,\ \ldots\ldots,\ n.$,

$$\sum_{j=1}^{k}\sum_{l=1}^{k} N_{ij}(A^*(t))w_{jl}a_l^*(t)+\sum_{j=k+1}^{n}\sum_{l=1}^{k} N_{ij}(A^*(t))w_{jl}a_l^*(t)$$
$$=\delta\Big(\sum_{j=1}^{k} N_{ij}(A^*(t))A_j^*(t)+\sum_{j=k+1}^{n} N_{ij}(A^*(t))A_j^*(t)\Big)$$
$$i=1,\ 2,\ \ldots,\ k.$$

[8] For example, M.A. Alemson [1], P. Doyle [5] and L.G. Telser [15].

"It would appear then that the large firm's competitive effort has been channeled away from price policy and into advertising, service, and product characteristic modification."[9]

He explains this phenomenon from the point of view of his sales maximisation hypothesis. But here let's admit this phenomenon a fact as such, without inspecting further how, why and on what level a common price is fixed in oligopoly. Taking this fact into account, we have not to treat price as an instrumental variable but as a given one that is common to every oligopolist and pre-determined by price leadership or cartel, etc.

Let

$$p_i^*(t)=p^0, \text{ for all } i\text{'s and } t\geq 0.$$

where p^0 is supposed to be a pre-determined common and fixed price in oligopoly.

Since the price p^0 is given exogenously in this case, the equations in (6)′ are redundant, consequently Proposition 1 is irrelevant. Only Assumption 1 and 2 assure us the validity of Proposition 2, 3 and the non-negativity of $\boldsymbol{a}^*(t)$ without Assumption 3, 4 and 5. First, the price p^0 is determined independently of the goodwill vector $A^*(t)$, so Assumption 3 is followed instantly.

Secondly, Assumption 4 is followed from Assumption 2(2) and the definition of the function N_i's in (13), i.e.,

$$N_i(A(t))=\partial R_i(P^0,\ A_i(t))/\partial A_i,\ i=1,\ 2,\ \ldots,\ n.$$

where P^0 is a vector in n-dimensional Euclidean space whose all components are p^0. So,

$$\partial N_i(A(t))/\partial A_i=\partial^2 R_i(P^0,\ A_i(t))/\partial A_i^2<0, \quad i=1,\ 2,\ \ldots,\ n. \tag{23}$$

From Assumption 2(2), the RHS of the last expression is negative. Following the same procedures as before, we can show,

$$\partial N_i(A(t))/\partial A_j=\partial^2 R_i(P^0,\ A_i(t))/\partial A_j\partial A_i=0, \quad i\neq j\ ;\ i,\ j=1,\ 2,\ \ldots,\ n. \tag{24}$$

Thirdly, since $(R_i(P^0,\ A_i(t))-(r+\delta)A_i)$ is a function of A_i, a vector A^* that satisfies the following equations is unique according to Assumption 2(2) and 2(3),

$$\partial R_i(P^0,\ A_i^*)/\partial A_i=N_i(A^*)=r+\delta \quad i=1,\ 2,\ \ldots,\ n.$$

Lastly, the non-negativity of $a^*(t)$ is shown in the following Proposition.

Proposition 4. If we assume the existence of a price vector P^0 mentioned above and our Assumption 1 and 2 are true, then the non-negativity assumption of $a_i^*(t)$, $i=1,\ 2,\ \ldots,\ n.$, $t\geq 0$, in the premise of Proposition 3 will hold.

(proof)

[9] W.J. Baumol [3], p. 73.

As already shown in (23) and (24), the matrices N and N_{kk} ($k=1, 2, \ldots, n.$) in (22) are diagonal and the matrices $N_{n-k,\,k}$ ($k=1, 2, \ldots, n.$) consisting of the off-diagonal elements of the matrix N are zero matrices respectively in this case. Therefore, the equation (22) is reduced to the following form,

$$\mathbf{W}_{kk}\boldsymbol{a}^{k*}(t)=A^{k*}(t),\ k=1, 2, \ldots, n.$$

Setting $w_{ii}+\sum_{j\neq i} w_{ij}=c_i(>0),\ i=1, 2, \ldots, k.$,
(from (17)), then the equation $\mathbf{W}_{kk}\ \mathbf{x}=\mathbf{c}$, where $\mathbf{c}$ is a k-dimensional positive vector whose i-th element is c_i, has the solution $x_i=1,\ i=1, 2, \ldots, k.$. So the matrices $\mathbf{W}_{kk}$ ($k=1, 2, \ldots, n.$) satisfy the Hawkins-Simon condition and have their non-negative inverse matrices respectively, i.e.,[10]

$$\boldsymbol{a}^{k*}(t)=\delta \mathbf{W}_{kk}{}^{-1}A^{k*}(t)\geq \mathbf{0},\quad k=1, 2, \ldots, n. \tag{25}$$

(q.e.d.)

The existence of the practice of price-fixing in oligopoly makes much simpler the informational structures for each oligopolist concerning his opponents' responses to his current activities. This fact reflects in the matrix N which becomes diagonal in this case. Further, it enables us to make forward our analysis with only Assumption 1 and 2.[11]

(2) Log linear demand functions and constant marginal production costs.

Here we shall specify the demand and production cost functions in the preceding sections. As to the demand functions, we shall consider the following ones,

$$D_i(t)=\eta_i p_i(t)^{-\alpha_i}\cdot A_i(t)^{\beta_i}\prod_{j\neq i}\left(\frac{p_j(t)}{p_i(t)}\right)^{\varepsilon_{ij}}$$
$$i=1, 2, \ldots, n.$$

where the coefficients η_i, α_i, β_i, and ε_{ij}'s are positive, and η_i is a constant number, α_i and β_i are the elasticity of demand D_i for the i-th firm with respect to the price p_i and goodwill A_i respectively, and the coefficient ε_{ij} is the cross-elasticity of demand D_i between the products of the i-th and j-th firms. In addition to the positivity of these coefficients, we suppose the following relations between them,

$$\alpha_i+\varepsilon_i>1, \text{ where } \varepsilon_i=\sum_{j=1}^{n}\varepsilon_{ij}$$

and $$1>\beta_i>0,\quad i=1, 2, \ldots, n. \tag{26}$$

With regard to the production cost function, we shall adopt the constant marginal cost hypothesis that is supported by many empirical studies. So the gross profit function for the i-th firm is written,

$$R_i(t)=\eta_i(p_i(t)-c_i)p_i(t)^{-(\alpha_i+\varepsilon_i)}A_i(t)^{\beta_i}\prod_{j\neq i}p_j(t)^{\varepsilon_{ij}}-F_i,$$
$$i=1, 2, \ldots, n. \tag{27}$$

[10] See H. Nikaido, [9], pp. 13-19, and pp. 107-113.

[11] With respect to more detailed treatments of this problem, see [13]. In [13], the jump possibilities of the advertising strategy at time 0 are not permitted, i.e., the i-th firm's advertising expenditures at time t have the following constraint,

$$0\leqslant a_i(t)\leqslant m_i,\quad 0\leqslant t<\infty,\quad i=1, 2, \ldots, n.$$

where c_i and F_i mean a marginal and fixed cost in production for the i-th firm respectively.

In this case, the equilibrium pricing strategy for the i-th firm is easily calculated from the equation $\partial R_i/\partial p_i=0$, $(i=1, 2, \ldots, n.)$ as shown in Proposition 3 in the last section, i.e.,

$$p_i^*=(\alpha_i+\varepsilon_i)c_i/(\alpha_i-\varepsilon_i-1)$$
$$i=1, 2, \ldots, n. \qquad (28)$$

Since

$$\partial^2 R_i(p^*, A_i(t))/\partial p^2_i=-\eta_i(\alpha_i+\varepsilon_i)c_i p_i^{*-(\alpha_i+\varepsilon_i)}A_i(t)^{\beta_i}\times\prod_{j\neq i}p^{*\varepsilon_{ij}}<0,$$
$$i=1, 2, \ldots, n.$$

the function R_i is maximized at p_i^*, for a given $(\bar{p}_i^*, A_i(t))$, where a vector $\bar{p}_i^*$ is given by (28).

As the price p_i^* is determined independently of the level of goodwill $A_i(t)$'s, Assumption 3 is clearly satisfied and the function N_i defined in (13) is written as follows:

$$N_i(A(t))=l_iA_i(t)^{\beta_i-1}, \quad i=1, 2, \ldots, n.$$

where $l_i=\eta_i(p_i^*-c_i)p_i^{*-(\alpha_i+\varepsilon_i)}\prod_{j\neq i}p_j^{*\varepsilon_{ij}}$

$$i=1, 2, \ldots, n.$$

Therefore,

$$\partial N_i(A(t))/\partial A_i=l_i\beta_i(\beta_i-1)A_i(t)^{(\beta_i-2)}<0,$$
$$i=1, 2, \ldots, n.$$
$$\partial N_iA(t))/\partial A_j=0, \quad j\neq i\ ;\ i, j,=1, 2, \ldots, n.$$

So Assumdtion 4 and the non-negativity assumption of $a_i^*(t)$'s are satisfied. And A_i^* that satisfies the relation

$$N_i(A^*)=r+\delta, \quad i=1, 2, \ldots, n.$$

is equal to $((r)+\delta)/l_i\beta_i)^{-1/(1-\beta_i)}$, $i=1, 2, \ldots, n.$

The above values, of course, are unique, so Assumption 5 is satisfied, too.

As examined above, all our assumptions in the previous section are satisfied except the concavity of R_i's in Assumption 2 (2) when we assume log linear demand functions and constant marginal production cost functions. This assumption was concerned with Proposition 3. But we can show the validity of the proposition without Assumption 2(2).

Let $\bar{P}_i^*=\prod_{j\neq i}p_i^{*\varepsilon_{ij}}$ and $U_i=R_i(p_i^*, \bar{p}_i^*, A_i^*(t))$

$-R_i(p_i, \bar{p}_i^*, A_i(t))$, then,

$$U_i=\eta_i(p_i^*-c_i)p_i^{*-(\alpha_i+\varepsilon_i)}\bar{P}_i^*A_i^*(t)^{\beta_i}-\eta_i(p_i(t)-c_i)p_i(t)\bar{P}_i^*A_i(t)^{\beta_i}$$
$$\geq\eta_i(p_i^*-c_i)p_i^{*-(\alpha_i+\varepsilon_i)}\bar{p}_i^*(A_i^*(t)^{\beta_i}-A_i(t)^{\beta_i}).$$

Since $0<\beta_i<1$, the function $A_i^{\beta_i}$ is strictly concave with respect to A_i. So,

$$U_i>\eta_i\beta_i(p_i^*-c_i)p_i^{*-(\alpha_i+\varepsilon_i)}\bar{p}_i^*A_i^*(t)^{(\beta_i-1)}(A_i^*(t)-A_i(t))$$
$$=N_i(A^*(t))(A_i^*(t)-A_i(t))$$

Namely,

$$R_i(p_i^*(t), p_i^*(t), A_i^*(t))-R_i(p_i(t), p_i^*(t), A_i(t))$$
$$>N_i(A_i^*(t))(A_i^*(t)-A_i(t)), \text{ for all } p_i(t),\in\mathscr{P}_i,$$

$A_i(t)\in\mathscr{A}_i$, $i=1, 2, \ldots, n.$

The last inequalities show the validity of (21). Therefore, Proposition 3 is proved in this case, too. Of course, when the price is fixed by some collusive agreements among oligopolists, the above is also true as already investigated.

Let's take the case of duopoly in order to illustrate the equilibrium pricing and advertising strategy. The equilibrium pricing strategy for the 1-st and 2-nd firm are given

by (28) where $i=1, 2$.. In this case, we have four cases with regard to the initial conditions for two firms, i.e.,
(1) $A_1^0 \geq A_1^*$, $A_2^0 \geq A_2^*$, (2) $A_1^0 \geq A_1^*$, $A_2^0 < A_2^*$, (3) $A_1^0 < A_1^*$, $A_2^0 \geq A_1^*$,
(4) $A_1^0 < A_1^*$, $A_2^0 < A_2^*$, where A_i^0 and A_i^* are the levels of goodwill in the initial and stationary state for the i-th firm ($i=1, 2$.) respectively.

In the first case, with regard to time t_i defined in (19), we shall assume $0 \leq t_1 \leq t_2$ without loss of generality, where t_i is calculated as follows,

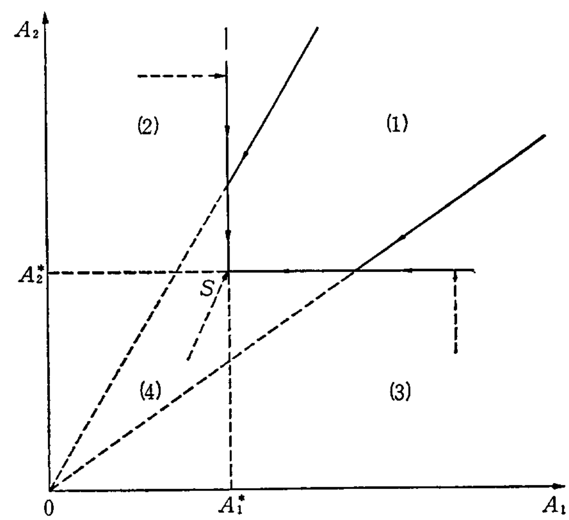
$$t_i = \frac{1}{\delta} \log (A_i^0 / A_i^*), \quad i=1, 2. \tag{29}$$

and,

$$A_i^* = ((r+\delta)/l_i \beta_i)^{-1/(1-\beta_i)}, \quad i=1, 2. \tag{30}$$

The equilibrium advertising strategy for the first firm is determined according to Proposition 3-3° as follows:

FIG 1. EQUILIBRIUM TRAJECTORIES



S: The Stationary State.

$$a_1^*(t) = \begin{cases} 0, & \text{for } 0 \leq t < t_1, \\ \delta A_1^*, & \text{for } t_1 \leq t < t_2 \\ \delta(A_1^* - w_{12} A_2^*)/(1 - w_{12} w_{21}), & \text{for } t \geq t_2 \end{cases} \tag{31}$$

And that for the second firm is,

$$a_2^*(t) = \begin{cases} 0, & \text{for } 0 \leq t < t_2 \\ \delta(A_2^* - w_{21} A_1^*)/(1 - w_{12} w_{21}), & \text{for } t \geq t_2 \end{cases} \tag{32}$$

where t_i and A_i^* are given by (29) and (30) respectively.

In case of (2), (3), and (4), we have only to set $t_1 = 0_+$, $t_2 = 0_+$, and $t_1 = t_2 = 0_+$ in the expressions (31) and (32) respectively.

An equilibrium trajectory that starts from an arbitrary initial point in the $\mathscr{A}_1 \times \mathscr{A}_2$ plane is depicted in Figure 1 where the regions (1), (2), (3) and (4) correspond to the previous cases, and a real line shows an equilibrium trajectory with no jump in its process, and a dotted line shows the one with a jump at time 0. When the initial condition is given in the region (1), the equilibrium trajectory goes forward to the original point (0, 0), hits the vertical line $A_1 = A_1^*$ at time t_1 (or the horizontal line $A_2 = A_2^*$ at time t_2) later, and proceeds the line until it reaches the stationary state $S = (A_1^*, A_2^*)$. In case of (2) ((3)), the trajectory on that hits the horizontal line $A_2 = A_2^*$ (the vertical line $A_1 = A_1^*$) on the moment, and proceeds on that line until it reaches the point S. Lastly, in case of (4), it reaches the point S on the moment.

5. *Summary*

In this paper we analyse the equilibrium pricing and advertising strategy in oligopoly by extending the concept of goodwill that was used by Arrow and Nerlove. The concept of equilibrium strategy in the meaning of this paper gives a dynamic content to the Cournot point whose original nature is static.

We show that the equilibrium advertising strategy for each oligopolist is determined in reference to his marginal revenue and marginal cost of goodwill at time 0, and that his equilibrium pricing strategy is determined by the vector function $A^*(t)$ which is dependent on the equilibrium advertising strategy of the system as a whole. Since his equilibrium pricing strategy depends on $A^*(t)$, each oligopolist has to take account of all reactions of his opponents to his activities. As a result, the informational structures that he faces when making decisions become so complicated that he cannot reach the stationary state without some additional informations such as our A. 3—5 and the non-negativity assumption of $\boldsymbol{a}^*(t)$.

But as a matter of fact, the informational structures might be less complicated than in the preceding argument. For example, the practice of price fixing might be prevailing as frequently observed in the oligopolistic industries, or each oligopolist's demand function might take a multicative form whose validity waits for more empirical researches together with his marginal production being constant. We have analysed these cases in Section 4.

In this paper we permit jump possibilities of the advertising strategies at time 0 which are unrealistic assumption. They may be removed if we set an upper limit on the current advertising expenditures[12] or introduce a non-linear advertising cost function for each oligopolist as Arrow and Nerlove suggested.[13] In the context of monopoly, these problems are dissolved in the comparatively easy ways.[14] But their treatment in oligopoly are much more difficult than in monopoly. Further, in this paper we ignore an important factor, i.e., the diffusion process of information among potential customers with regard to the products that has an effect on the formation of goodwill for each oligopolist. This factor has to be considered in our framework, too.

[12] See A. Sakuma [13].

[13] K.J. Arrow and M. Nerlove [2], p. 141.

[14] See J.P. Gould [7] with regard to a non-linear advertising cost function, and S. Perrakis and I. Sahin [12] with regard to an upper limit of current advertising expenditures.

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