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A SIMPLIFIED DERIVATION OF ARROW'S IMPOSSIBILITY THEOREM*

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Abstract

I prove Arrow's Impossibility Theorem by using elementary Set Theory. The result of the proof will be available for pedagogical purpose. In the process of the proof, the condition of Independence of Irrelevant Alternatives plays an essential role.

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I. *Introduction*

Many studies have done to prove Impossibility Theorem of Arrow (1963). Fishburn (1970) defined the conditions of the theorem precisely, and proved that all of them are satisfied only when the members of the society is infinite. Sen (1979) proved the theorem by reducing "the decisive group" into one person. He also analyzed each of the conditions of the theorem in the light of welfare economics. Suzumura (1988) gave a smart proof by using contraposition of mathematically inductive method. Takekuma (1997) proved the theorem by the program of Mathematica. A problem of above articles is that they are too difficult for undergraduate students and researchers not being specialized in social choice to understand the proof completely because they require a certain level of knowledge of social choice and mathematical ability. On the other hand, Denicolo (1996) gave an elementary proof. The only problem of his proof is that he assumed the individual preferences to be linear.

This paper intends to prove Arrow's theorem entirely by using only elemental knowledge of the Set Theory. First I define the sets which generate social preferences using the condition of "Independence of Irrelevant Alternatives". At the next step I reduce them. Even undergraduate students can easily understand the whole process of the proof. This paper will be useful from the pedagogical point of view.

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II. *Abstract of Arrow's Impossibility Theorem*

Consider an n-person society ($n \geq 2$). Let $N = \{1, 2, \ldots, n\}$ be a set of individuals. $X = \{x, y, \ldots\}$ denotes the set of social alternatives. There are at least 3 alternatives and the set of individuals is finite. Each individual preference (IP) on X is rational (complete, reflexive and transitive). A profile is a list of IPs. For example, in the case of $n = 2$ and $m = 3$, the number of profiles is $13^2 = 169$. The social welfare function (SWF) is a rule that assigns a social preference (SP) to all logically possible profiles of IPs. The SP is also assumed to be rational. Arrow required that the SWF be consistent with the following four rules. Rule 1: The SWF is defined for all logically possible profiles of IPs, i.e., the domain is unrestricted. Rule 2: The SWF satisfies the Pareto Principle; for any $x, y \in X$, if all members of the society prefer x to y, then the society also prefers x to y. Rule 3: The SWF satisfies the condition of Independence of Irrelevant Alternatives. Hence, for any $x, y \in X$, the SP on x and y are determined from the IPs only on x and y. Rule 4: Nondictatorship; there is no dictator in the society. The dictator is defined as person d; for any $x, y \in X$, whenever person d prefers x to y, the society prefers x to y. "Arrow's Impossibility Theorem" is expressed as follows: There is no SWF that satisfies Rule 1, Rule 2, Rule 3 and Rule 4.

III. *Proof*

1. Formal preparation

A. Process of the proof

I treat only the SWF that satisfies Rule 1, Rule 2 and Rule 3. I consider sets of individuals that generate SP under such rules, and verify that no SWF satisfies Rule 4 (Nondictatorship). First, I verify the above results in the case of 3 alternatives and n ($n \geq 2$) individuals. Next, I expand the results into the case of m alternatives.

B. Definition of symbols

In this paper, I use the following notations for simplicity:
"$x > (\geq =)\ y$" implies that x is preferred (preferred or indifferent, indifferent) to y, and
"$x < (\leq)\ y$" implies that y is preferred (preferred or indifferent) to x.
Especially, I use the following notation "$\triangledown$".
"$A^* \triangledown x > y$" implies that all members of Set A^* prefer x to y.
"$S \triangledown x > y$" implies that the society prefers x to y.

C. Definition of sets that generate social preferences

Using the rules of "Independence of Irrelevant Alternatives", I would like to define following non-empty sets $A^*, \ldots, F^*$ that generate social preferences regarding only two alternatives.[1]

[1] Non-emptiness is guaranteed by the Pareto Principle.

$A^*\nabla x > y$	$\rightarrow$	$S\nabla x > y$	(1.a)
$B^*\nabla y > z$	$\rightarrow$	$S\nabla y > z$	(1.b)
$C^*\nabla z > x$	$\rightarrow$	$S\nabla z > x$	(1.c)
$D^*\nabla x < y$	$\rightarrow$	$S\nabla x < y$	(1.d)
$E^*\nabla y < z$	$\rightarrow$	$S\nabla y < z$	(1.e)
$F^*\nabla z < x$	$\rightarrow$	$S\nabla z < x$	(1.f)

Next, I define A_G as the group of sets that satisfy condition (1.a). The number of elements of A_G is finite. In the same way, I define $B_G, \ldots, F_G$. I define A as a set composed of the minimum number among A_G . Likewise, I define Sets $B, \ldots, F$.

2. Proof

Step 1. $A = B = C$ (for any A, B, C).[2]
See Appendix.
Step 2. A $(=B=C)$ consists of only one person (named i) (for any A, B, C).
See Appendix.
Step 3. $D = E = F$ (for any D, E, F).
The proof is similar to that of Step 1.
Step 4. D $(=E=F)$ consists of only one person (named j) (for any D, E, F).
The proof is similar to that of Step 2.
Step 5. For i and j mentioned above, $i=j$. (Hence, person $i(=j)$ is a dictator.)
(Proof) Assume $i \neq j$. There exist the following preferences: $i\nabla x > y$, $j\nabla x < y$. Then, the SP is determined as follows: $S\nabla x > y$, $S\nabla x < y$. This SP is irrational. Hence, $i=j$.
Step 6. In the case of m alternatives, the SWF that satisfies Rule 1, Rule 2 and Rule 3 must be dictatorial.
(Proof) See Appendix.

Appendix

(1) Proof of Step 1

A. I prove that for any A, B, C, $A \cap B \neq \phi$, $B \cap C \neq \phi$, $C \cap A \neq \phi$.

(Proof) Assume $A \cap B = \phi$. Consider the following IPs (in the case of $n=2$, $\overline{A \cup B}$ is empty): $A\nabla z > x > y$, $B\nabla y > z > x$, $\overline{A \cup B}\nabla z > x$. Then, the SP is determined as follows: $S\nabla x > y$, $S\nabla y > z$ and $S\nabla z > x$. This is irrational. Therefore, the assumption is false, and $A \cap B \neq \phi$. The proof of $B \cap C \neq \phi$ and $C \cap A \neq \phi$ are similar.

B. I prove that for any A, B, C, $A=B=C$.

(Proof) First, I prove $A = B = (A \cap B)$.

[2] I consider the possibility of the existence of multiple sets that satisfy the requirements of Set A in A_G. As to $B, \ldots F$, the situations are similar.

I divide set A and set B as follows:

$$A=(A\cap B)\cup\{A-(A\cap B)\},\ B=(A\cap B)\cup\{B-(A\cap B)\}.$$

where $\{A-(A\cap B)\}\cap\{B-(A\cap B)\}=\phi$.

1) Suppose $\{A-(A\cap B)\}\neq\phi$ and $\{B-(A\cap B)\}\neq\phi$.

Consider the following IPs: $(A\cap B)\triangledown x>y>z$, $\{A-(A\cap B)\}\triangledown x>y$, $\{B-(A\cap B)\}\triangledown y>z$. Then, SP is determined as follows: $S\triangledown x>y$ (definition of Set A), $S\triangledown y>z$ (definition of Set B), $S\triangledown x>z$ (transitivity of the SP). Once IPs on x and z of all the members are determined, SP on x and z is also determined (Rule3). As for the members not belonging to $(A\cap B)$, IPs on x and z are not restricted at all, so all logically possible IPs on x and z can be realized (Rule 1). Now $(A\cap B)\triangledown x>z$ and $S\triangledown x>z$ are realized. Hence, it follows that

$$(A\cap B)\triangledown x>z \quad\rightarrow\quad S\triangledown x>z. \tag{A.1}$$

Next, consider the following IPs: $(A\cap B)\triangledown x>y>z$, $\{A-(A\cap B)\}\triangledown y>x$, $\{B-(A\cap B)\}\triangledown z>y$. Then, there exists the following SP: $S\triangledown y\geq x$ (definition of Set A), $S\triangledown z\geq y$ (definition of Set B) and $S\triangledown z\geq x$ (transitivity of the SP). This result contradicts Eq. (A.1).

2) Suppose $\{A-(A\cap B)\}\neq\phi$ and $\{B-(A\cap B)\}=\phi$.

Consider the following IPs: $(A\cap B)\triangledown x>y>z$, $\{A-(A\cap B)\}\triangledown x>y$. It follows that

$$(A\cap B)\triangledown x>z \quad\rightarrow\quad S\triangledown x>z. \tag{A.2}$$

Next, consider the following IPs: $(A\cap B)\triangledown x>z>y$, $\{A-(A\cap B)\}\triangledown z>y>x$. Then, there exists the following SP: $S\triangledown y\geq x$ (definition of Set A), $S\triangledown z>y$ (Pareto Principle) and $S\triangledown z>x$ (transitivity of the SP). This result contradicts Eq. (A.2).

3) Suppose $\{A-(A\cap B)\}=\phi$ and $\{B-(A\cap B)\}\neq\phi$.

The proof is similar to that of 2).

From the results of 1), 2) and 3), $\{A-(A\cap B)\}=\{B-(A\cap B)\}=\phi$. Hence, $A=B=(A\cap B)$.

By the same method, $A=B=C$ can be easily proven.

(2) Proof of Step 2

(Proof) Assume that A $(=B=C)$ consists of complexed members $\{1,\ldots r\}$ $(r\geq 2)$.

Consider the following IPs: {person 1}$\triangledown x>y>z$, {person 2,…, person r}$\triangledown x>y>z$. Then, the SP is determined as follows: $S\triangledown x>y$ and $S\triangledown y>z$. It follows that

$$A\triangledown x>z \quad\rightarrow\quad S\triangledown x>z. \tag{A.3}$$

Next, consider the following IPs: {person 1}$\triangledown x>z>y$, {person 2,…, person r}$\triangledown y>x>z$. Then, there exists the following SP: $S\triangledown y\geq x$ (definition of Set A), $S\triangledown z\geq y$ (definition of Set B) and $S\triangledown z\geq x$ (transitivity of the SP). This result contradicts Eq. (A.3).

(3) Proof of Step 6

(Proof) In the case of k $(k\geq 3)$ alternatives, consider that Rules 1-3 are satisfied and a dictator exists (named i). In the case of $k+1$ alternatives, assume that Rules 1-3 are satisfied and person i does not have dictatorial power on the $k+1$ alternative (w). So, without losing generality, we can assume that there exist the following preferences: {person i}$\bigtriangledown x > w$, $S\bigtriangledown w \geq x$. Then, consider the following IPs: {person i}$\bigtriangledown x > y > w$, {others}$\bigtriangledown y > w$. The SP is determined as follows: $S\bigtriangledown x > y$ (person i has dictatorial power on x and y), $S\bigtriangledown y > w$ (Pareto Principle), $S\bigtriangledown w \geq x$ (person i does not have dictatorial power on w) and $S\bigtriangledown y > x$ (transitivity of the SP). This SP is irrational. Therefore, person i should be a dictator. Hence, in the case of m alternatives (m is any natural number), a dictator exists.

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