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ON EMBEDDED COMPLETE MARKETS*

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Abstract

Since the pioneering work of Black and Scholes the assumption of the complete market has been used in the arbitrage pricing theory. The martingale approach by Harrison and Pliska is one of the most powerful tools for determining security prices in the complete market. The existence of such a market, however, may not be guaranteed in the real world, where the martingale measure is not unique. Therefore, it is more natural to assume an incomplete market. Motivated by a non-standard interpretation of the method of least squares, we introduce the concept of an embedded complete market. We will give a new method to determine the option prices. For simplicity, we mainly use a simple discrete time and state model.

Key Words: Incomplete market, method of least squares, martingale measures, and embedded complete markets.
JEL Classification: primary D52, secondary G12

I. *Introduction*

In the last three decades, motivated by the seminal work of Black and Scholes (1973), the arbitrage pricing theory has been a major tool of determining the derivative security prices. The basic assumption of this beautiful method lies partially in the concept of the complete market, where every contingent claim can be reproduced by the self financing portfolio of the underlying basic assets in the market. However, it is not easy to tell if a given market is complete, and we may question if there is a complete market in the real world. A more practical method of pricing derivatives in an incomplete market is needed for the practitioners. Indeed, many authors have addressed such a need [Duffie and Richardson (1991), Shweizer (1992, 1995), and Berstimas, Kogan and Lo (1997), among others.] Our approach is basically the same as these authors to the extent that the method of the least squares is the central tool. We will supplement the previous results by introducing the embedded complete market in an

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incomplete market. A different approach to this problem is discussed by Karatzas, Lehczky, Shreve and Xu (1991), but we do not consider it here.

In this note we mainly assume the discrete time and state model, and we will propose a new method of determining the prices of derivatives in an incomplete market. Our method is reduced to the usual arbitrage pricing theory if the market is complete. Although some generality may be lost by using the discrete models, they help us clarify the basic idea more transparently. The discrete state and time incomplete market considered in this note may be used to analyze the phenomenon when the prediction of future volatility in the stock market and that in the option market diverge (cf. Takahashi (2000)).

If we suppose that the stock and the bond are the only basic assets in the market, the trinomial model provides us the simplest incomplete market. We will first review the method of pricing an option in a complete market in the rest of this section. We will employ single period binomial stock price model.

Suppose the risk free bond $\{(1+r)^t, t=0,1\}$ and the stock $\{S_t, t=0,1\}$ are the only assets in the market, where r is the risk free interest rate and it is nonnegative. Let C_0 be the price of European call option on the stock with the exercise price K and the maturity at $t=1$. If the stock price process follows a Bernoulli one period process, the market is complete and the price of the option can be determined uniquely by the method of Harrison and Kreps (1979), and Harrison and Pliska (1981). To be more specific, at $t=0$ we suppose the stock price is $S_0=S$ for some S, and at day $t=1$, it either goes up to $S_1=uS$ or goes down to $S_1=dS$ with probability q^* or $1-q^*$ respectively. We will also assume that

$$0<d<1+r<u \tag{1}$$

to eliminate the arbitrage opportunity. The value of an option at the maturity $t=1$ equals either $C_1^{(u)}=max\{uS-K,0\}$ or $C_1^{(d)}=max\{dS-K,0\}$. Now, an *equivalent portfolio* (EP for short) to this call option is the one whose value at $t=1$ is the same as the option price. If at $t=0$ the EP price differs from the option price, there is an arbitrage opportunity. Therefore at $t=0$ the option price C_0 should equal the EP price at $t=0$. Suppose the equivalent portfolio consists of B yen of the bond and α unit of the stock, then the following system of linear equations must hold.

$$\alpha(uS)+(1+r)B=C_1^{(u)} \tag{2}$$

$$\alpha(dS)+(1+r)B=C_1^{(d)} \tag{3}$$

If $u\neq d$, then the above equations have the unique root (α, B) and the market becomes complete. Of course, this simply follows from the fact that the number of distinctive equations and unknowns (α, B) are the same. If $u=d$, equations (2) and (3) reduce to a single equation and there is an arbitrage opportunity in the market unless the rate of return for the stock $(u-1=d-1)$ and the bond are both equal to r. If, on the other hand, there are more than two equations; namely the possible states at $t=1$ are more than or equal to 3, we have an incomplete market. We shall discuss the later case more extensively in the next section.

Now, by solving (2) and (3), we have

$$C_0=\frac{1}{1+r}\{qC_1^{(u)}+(1-q)C_1^{(d)}\} \tag{4}$$

where,

$$q = \frac{1+r-d}{u-d} \tag{5}$$

Note that under the condition (1), it follows that $0 < q < 1$, and q can be interpreted as a probability (see equation (6) below). With this in mind, C_0 may be interpreted as the (conditional) expectation of C_1(*given* $S_0 = S$) discounted by the risk free interest rate r, where the expectation is taken under the probability measure $\{q, 1-q\}$, where

$$P\{S_1 = uS\} = q = 1 - P\{S_1 = dS\} \tag{6}$$

It is also known that under this probability, the process $\{ \frac{S_t}{(1+r)^t}, F_t, t = 0,1\}$ is a martingale, where $F_t = \sigma\{S_u, 0 \leq u \leq t\}$ is a sigma algebra of information carried by $\{S_u\}$ up to and including time t. This is the simplest case of the general martingale method proposed by Harrison and Kreps (1979) (also see Harrison and Pliska (1981)), who proved that the necessary and sufficient condition for no arbitrage opportunity in a frictionless market is the existence of an equivalent martingale measure P.

II. *The Simplest Incomplete Market*

Let us consider a problem of pricing derivatives in an incomplete market. We suppose that there are two securities, the stock (risky asset) and the bond (riskless asset), in the market as in Section 1. We will, however, assume that the stock price follows a *trinomial* process, where the present and the maturity time are denoted by $t = 0$ and $t = 1$, respectively. Our problem is to determine at $t = 0$ the price C_0 of the European call option on the stock with exercise price $K(>0)$ and with maturity at $t = 1$. We also assume that the risk free interest rate is $r(\geq 0)$.

Let $S_0 = S$ be the stock price at $t = 0$ as in the previous section. We suppose that, at $t = 1$, the random variable $S_1^{(\cdot)}$ takes one of the following three values; $S_1^{(u)} = uS$, $S_1^{(m)} = mS$, or $S_1^{(d)} = dS$ with probabilities p_u^*, p_m^*, and p_d^* respectively, where $p_y^* > 0 (y = u, m, d)$ and $p_u^* + p_m^* + p_d^* = 1$. The values of $S_1^{(\cdot)}$ are determined by the prediction of the future volatility of the stock return in the market. Also we will assume that

$$d < m < u, \qquad d < 1 + r < u \tag{7}$$

to exclude arbitrage opportunities in the market. We will call the probability measure $\mathrm{P}^* = \{p_u^*, p_m^*, p_d^*\}$ the real probability measure. Note that as far as we are concerned with the pricing of derivatives, the measure P^* may be quite arbitrary except that it assigns positive probabilities to all the states in the space of $S_1^{(\cdot)}$. Also note that at $t = 1$, the value of the option becomes either $C_1^{(u)} = max\{S_1^{(u)} - K, 0\}$, $C_1^{(m)} = max\{S_1^{(m)} - K, 0\}$, or $C_1^{(d)} = max\{S_1^{(d)} - K, 0\}$, in accordance with the realization of the stock price. We will denote the market with this trinomial process by $M = \{S, (dS, mS, uS), r\}$.

Since the necessary and sufficient condition that there is no arbitrage opportunities in the market is the existence of a martingale measure (cf. Harrison and Kreps (1979), and Harrison and Pliska (1981)), our first step is a search of martingale measures. Here, in the following simple lemma, we will give the necessary and sufficient condition for martingale measures.

Lemma 1. *Suppose* $\{S_t,\ t=0,1\}$ *follows a single period trinomial process. Then a probability measure* $\mathrm{P}=\{p_u,\ p_m,\ p_d\}$ *is an equivalent martingale measure for* $\{\dfrac{S_t}{(1+r)^t},\ F_t,\ t=0,1\}$, *if and only if*

$$p_y>0 \qquad \textit{for all } y=u,\ m,\ d \tag{8}$$

and

$$(u-d)p_u+(m-d)p_m=1+r-d \tag{9}$$

Proof. *If* $\{\dfrac{S_t}{(1+r)^t},\ F_t,\ t=0,1\}$ *is a martingale under* P, *we have*

$$\frac{1}{1+r}\ [uS\cdot p_u+mS\cdot p_m+dS\cdot p_u]=E\{\frac{S_1}{(1+r)^1}\ |S_0=S\}=\frac{S_0}{(1+r)^0}=S \tag{10}$$

It follows that $up_u+mp_m+dp_u=(1+r)$, *and* (9) *follows readily. Since* $p_y>0$ *for all* $y=u$, m,d, (8) *is also true. The converse is proved in the same manner.*

Since there are infinitely many $\mathrm{P}=\{p_u,\ p_m,\ p_d\}'$s that satisfy (8) and (9), the martingale measure is not unique in the trinomial model. Therefore, it is not possible to determine the arbitrage free value of C_0 uniquely by the martingale method alone. This can be understood more transparently by the following elementary argument.

There are only two basic assets in this economy: the stock and the riskless bond, and we consider a portfolio PF formed at $t=0$ for $t=1$, which consists of α units of the stock and B units of the risk free bond. The number of values $C_1^{(\cdot)}$ can take is, however, three. Then the necessary condition for the portfolio PF to be an equivalent portfolio to $C_1^{(\cdot)}$ is given by the following three equations,

$$\alpha(uS)+(1+r)B=C_1^{(u)} \tag{11}$$

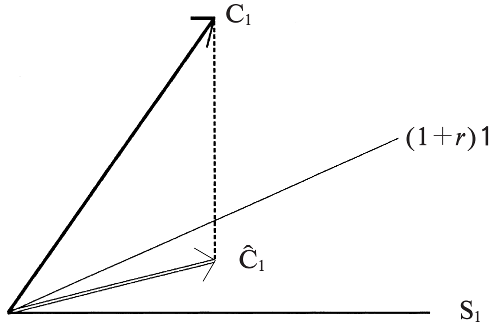
$$\alpha(mS)+(1+r)B=C_1^{(m)} \tag{12}$$

$$\alpha(dS)+(1+r)B=C_1^{(d)} \tag{13}$$

Since, we have three equations and two unknowns, the method of the previous section fails to give the unique price to the option value at $t=0$. Here, the simple geometric argument gives us a clear picture of what is going on. The value vector $\mathrm{C}_1=(C_1^{(u)},\ C_1^{(m)},\ C_1^{(d)})$ does not lay in the space $sp\{\mathrm{S}_1,\ \mathrm{r}\}$ spanned by the vectors $\mathrm{S}_1=(uS,\ mS,\ dS)$ and $\mathrm{r}=(1+r)1=(1+r,1+r,1+r)$. The problem of finding an equivalent portfolio is to express the vector C_1 by the elements of $sp\{\mathrm{S}_1,\ \mathrm{r}\}$ which is to find constants $(\alpha,\ B)$ such that $\mathrm{C}_1=\alpha\mathrm{S}_1+(1+r)B1$. Clearly this is in no way possible in this case.

Although there are numerous ways to determine reasonable price in this circumstance, we will propose a new way to solve the problem. Our method is based on the non-standard interpretation of the method of least squares, and we will review the least squares from the different view point in the next section.

PICTURE 1



III. *The Method of Least Squares and Embedded Complete Markets*

We digress briefly to discuss the classical method of OLS (ordinary least squares). We will keep the notations of the previous sections and consider the regression problem of $C_1 = \alpha S_1 + (1+r)B\mathbf{1} + \varepsilon$ onto $sp\{S_1, r\}$, where ε denotes an error vector. Let $\hat{C}_1$ be the orthogonal projection of C_1 onto the space $sp\{S_1, r\}$ Then, we have,

$$\hat{C}_1 = \hat{\alpha} S_1 + (1+r)\hat{B}\mathbf{1} \tag{14}$$

Here the OLS estimators $(\hat{\alpha}, (1+r)\hat{B})$ of α and $(1+r)B$ are given by

$$\hat{\alpha} = \frac{(uS-\bar{S})C_1^{(u)} + (mS-\bar{S})C_1^{(m)} + (dS-\bar{S})\ C_1^{(d)}}{(uS-\bar{S})^2 + (mS-\bar{S})^2 + (dS-\bar{S})^2}$$

and,

$$(1+r)\hat{B} = \bar{C} - \hat{\alpha}\bar{S}$$

where,

$$\bar{C} = \frac{1}{3}\,(C_1^{(u)} + C_1^{(m)} + C_1^{(d)})$$

and

$$\bar{S} = \frac{1}{3}\,(uS + mS + dS)$$

Then, in view of the modern derivative pricing theory, it is reasonable to define the OLS option price at $t=0$ by,

$$\hat{C}_0^{(OLS)} = \hat{\alpha} S_0 + \hat{B} \tag{15}$$

Now, the following representation of $\hat{\alpha}$ and $(1+r)\hat{B}$ is the key to our analysis (cf. Wu (1986)). It follows from the straightforward algebra that

$$\hat{\alpha}=\frac{(uS-mS)(C_1^{(u)}-C_1^{(m)})+(mS-dS)(C_1^{(m)}-C_1^{(d)})+(dS-uS)(C_1^{(d)}-C_1^{(u)})}{(uS-mS)^2+(mS-dS)^2+(uS-dS)^2}$$

$$=w^{(OLS)}(u,m)\frac{(C_1^{(u)}-C_1^{(m)})}{(uS-mS)}+w^{(OLS)}(m,d)\frac{(C_1^{(m)}-C_1^{(d)})}{(mS-dS)}$$

$$+w^{(OLS)}(d,u)\frac{(C_1^{(d)}-C_1^{(u)})}{(dS-uS)} \tag{16}$$

$$(1+r)\hat{B}=w^{(OLS)}(u,m)\frac{(uC_1^{(m)}-mC_1^{(u)})}{(u-m)}+w^{(OLS)}(m,d)\frac{(mC_1^{(d)}-dC_1^{(m)})}{(m-d)}$$

$$+w^{(OLS)}(d,u)\frac{(dC_1^{(d)}-uC_1^{(u)})}{(d-u)} \tag{17}$$

where, the weights $w^{(OLS)}(x,y)$ are given by

$$w^{(OLS)}(x,y)=\frac{(xS-yS)^2}{(uS-mS)^2+(mS-dS)^2+(uS-dS)^2}$$

$$(x,y)\in\{(u,m),(m,d),(d,u)\} \tag{18}$$

The interpretation of expressions (16) and (17) above are of interest. Note that,

$$\alpha(x,y)=\frac{(C_1^{(x)}-C_1^{(y)})}{(xS-yS)} \text{ and } (1+r)B(x,y)=\frac{(xC_1^{(y)}-yC_1^{(x)})}{(x-y)} \tag{19}$$

are the slope and intercept of the line connecting the points $(xS, C^{(x)})$ and $(yS, C^{(y)})$, $(x,y)\in\{(u,m),(m,d),(d,u)\}$. Therefore, the over all OLS estimator $(\hat{\alpha},(1+r)\hat{B})$ is a weighted average of the slopes and the intercepts of the lines determined by every pair in the sample. Note that in each pair $\{(xS, C^{(x)}),(yS, C^{(y)})\}$, $\alpha(x,y)$ and $B(x,y)$ are determined without error and this can be compared with equations (2) and (3) in the complete market model where the stock takes the values xS and yS at time $t=1$.

Now, we go back to our original pricing problem in the incomplete market $M=\{S,(dS, mS, uS),r\}$. First of all we will introduce a probability mass function g on a set $\Theta=\{(xS, yS), (x,y)\in\{(u,m),(m,d),(d,u)\}\}$,

$$g\{(xS, yS)\}=w^{(OLS)}(x,y), \qquad (x,y)\in\{(u,m),(m,d),(d,u)\} \tag{20}$$

Then, we choose a pair (xS, yS) from the set Θ according to the probability $g\{(xS, yS)\}$. After having chosen a pair, we can form a sub-market $M(x,y)=\{S,(xS, yS), r\}\subset M$. And this is a basic building block of our method. Note that in the sub-market $M(x,y)$,C_1 takes only $(C_1^{(x)}, C_1^{(y)})$ at $t=1$. If the sub-market $M(x,y)$ becomes a complete market, then it is called an *embedded complete market model inside the original incomplete market or* an *embedded complete market* for short. And we will assume every sub-market is complete in the rest of this note. In the sub-market $M(u,m)$, we can evaluate the option price by solving the system of equation $\{(11),(12)\}$. Together with $\{(12),(13)\}$, and $\{(13),(11)\}$, we can altogether obtain three embedded "complete markets". Except for some technical problems which we will discuss later, we obtain martingale measures in each of the three embedded markets as in

Section 1.

From (11) and (12), we have,

$$\begin{aligned}\alpha(u, m)(uS)+(1+r)B(u, m)&=C_1^{(u)}\\ \alpha(u, m)(mS)+(1+r)B(u, m)&=C_1^{(m)}\end{aligned} \tag{21}$$

It follows that the martingale measure in $M(u, m)$ is given by

$$\mathrm{Q}(u, m)=\{p(u, m), q(u, m)=1-p(u, m)\}$$

where,

$$p(u,m)=P^{(u, m)}\{S_1=uS|S_0=S\}=\frac{(1+r)-m}{u-m} \tag{22}$$

Then, at $t=0$ the price of the European call option in $M(m, d)$ is given by

$$C_0^{(u, m)}=\frac{1}{1+r}\{p(u, m)C_1^{(u)}+q(u, m)C_1^{(m)}\} \tag{23}$$

Next, (12) and (13) give us,

$$\begin{aligned}\alpha(m, d)(mS)+(1+r)B(m, d)&=C_1^{(m)}\\ \alpha(m, d)(dS)+(1+r)B(m, d)&=C_1^{(d)}\end{aligned} \tag{24}$$

Then the martingale measure and the option price are

$$\mathrm{Q}^{(m, d)}=\{p^{(m, d)}, q^{(m, d)}=1-p^{(m, d)}\}, \tag{25}$$

$$C_0^{(m, d)}=\frac{1}{1+r}\{p(m, d)C_1^{(m)}+q(m, d)C_1^{(d)}\}$$

where,

$$p(m, d)=P^{(m, d)}\{S_1=mS|S_0=S\}=\frac{(1+r)-d}{m-d} \tag{26}$$

Finally in $M(d, u)$, by (13) and (11)

$$\begin{aligned}\alpha(d, u)(dS)+(1+r)B(d, u)&=C_1^{(d)}\\ \alpha(d, u)(uS)+(1+r)B(d, u)&=C_1^{(u)}\end{aligned} \tag{27}$$

we have

$$\mathrm{Q}^{(d, u)}=\{p(d, u), q(d, u)=1-p(d, u)\} \tag{28}$$

$$C_0^{(d, u)}=\frac{1}{1+r}\{p(d, u)C_1^{(d)}+q(d, u)C_1^{(u)}\}$$

where,

$$p(d, u)=P^{(d, u)}\{S_1=dS|S_0=S\}=\frac{(1+r)-u}{d-u} \tag{29}$$

If $d<m<1+r<u$, then $p^{(m,d)}>1$ and $q^{(m,d)}<0$. It follows that $Q(m,d)$ is not a probability measure. We will, however, consider a linear combination of these martingale measures, so that the resulting measure becomes a probability.

Now, it follows from (15) that, the option price in the original market will be given by,

$$\hat{C}_0^{(OLS)} = \sum_{(x,y)\in\{(u,m),(m,d),(d,u)\}} \{w^{(OLS)}(x,y)\alpha(x,y)S_0 + w^{(OLS)}(x,y)B(x,y)\}$$
$$= \sum_{(x,y)\in\{(u,m),(m,d),(d,u)\}} w^{(OLS)}(x,y)\,C_0^{(x,y)} \tag{30}$$

Namely the option price obtained from the method of least squares is represented by the weighted average of the option prices in the embedded complete market. Since the weight given by OLS is proportional to the volatilities in each sub-market, the above argument suggests us to use the other weight as well. To close this section, we define a new class of pricing options based on the embedded complete sub-market.

Definition 1. (Pricing via Embedded Complete Market) Let $\theta(x,y)$ be any weight attached to the embedded complete sub-market $M(x,y)$, where

$$\sum_{(x,y)\in\{(u,m),(m,d),(d,u)\}} \theta(x,y)=1$$

and,

$$\theta(x,y)>0 \text{ for all } (x,y)\in\{(u,m),(m,d),(d,u)\}$$

Then,

$$\hat{C}_0^{(\Theta)} = \sum_{(x,y)\in\{(u,m),(m,d),(d,u)\}} \theta(x,y)\,C_0^{(x,y)} \tag{31}$$

is *an option price determined by the weight* $\Theta=\{\theta(u,m),\theta(m,d),\theta(d,u)\}$

The weight obtained from OLS is proportional to the volatility of each sub-market. If we use an arbitrary Θ, this may be interpreted as the market's prediction of the distribution of the future volatilities. This observation may be fully utilized in the continuous model. The next question is when our pricing method gives us an arbitrage free price. We will consider this question in the next section.

IV. *Martingale Measures*

We will show that option prices obtained by the OLS method in the previous section is arbitrage free. To start with, we rewrite equation (30). By the straightforward algebra, we have,

$$\hat{C}_0^{(OLS)} = \frac{1}{1+r}\{[w^{(OLS)}(u,m)p(u,m)+w^{(OLS)}(d,u)q(d,u)]C_1^{(u)}$$
$$+[w^{(OLS)}(m,d)p(m,d)+w^{(OLS)}(u,m)q(u,m)]C_1^{(m)}$$
$$+[w^{(OLS)}(d,u)p(d,u)+w^{(OLS)}(m,d)q(m,d)]C_1^{(d)}\} \tag{32}$$

$$= \frac{1}{1+r}\{\hat{p}^{(OLS)}(u)C_1^{(u)} + \hat{p}^{(OLS)}(m)C_1^{(m)} + \hat{p}^{(OLS)}(d)C_1^{(d)}\}$$

where,

$$\begin{aligned}
\hat{p}^{(OLS)}(u) &= w^{(OLS)}(u, m)p(u, m) + w^{(OLS)}(d, u)q(d, u) > 0 \\
\hat{p}^{(OLS)}(m) &= w^{(OLS)}(m, d)p(m, d) + w^{(OLS)}(u, m)q(u, m) > 0 \\
\hat{p}^{(OLS)}(d) &= w^{(OLS)}(d, u)p(d, u) + w^{(OLS)}(m, d)q(m, d) > 0
\end{aligned} \tag{33}$$

It is also shown that

$$\hat{p}^{(OLS)}(u) + \hat{p}^{(OLS)}(m) + \hat{p}^{(OLS)}(d) = 1 \tag{34}$$

Thus, the OLS price can be viewed as the present value of the expected value of C_1 under the probability $\{\hat{p}^{(OLS)}(u), \hat{p}^{(OLS)}(m), \hat{p}^{(OLS)}(d)\}$. We will show that this probability measure is an equivalent martingale measure, so that the OLS price is an arbitrage free price. We will discuss this under arbitrary weight functions.

Note that the martingale measures in sub-market, $Q(u, m)$, $Q(m, d)$, $Q(d, u)$ are neither equivalent to each other, nor they are equivalent to the original probability measure P^*. We can, however, obtain the equivalent martingale measure by considering their strong convex combination. For this purpose, let $\Theta = \{\theta(u, m), \theta(m, d), \theta(d, u)\}$ be as in the previous section. We will define a strong convex combination $Q(\Theta)$ of these measures $\{Q(u, m), Q(m, d), Q(d, u)\}$ where

$$\begin{aligned}
Q(\Theta) &= \theta(u, m)Q(u, m) + \theta(m, d)Q(m, d) + \theta(d, u)Q(d, u) \\
&= \{\hat{p}^{(\Theta)}(u), \hat{p}^{(\Theta)}(m), \hat{p}^{(\Theta)}(d)\}\ say
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\hat{p}^{(\Theta)}(u) &= \theta(u, m)p(u, m) + \theta(d, u)q(d, u) \\
\hat{p}^{(\Theta)}(m) &= \theta(m, d)p(m, d) + \theta(u, m)q(u, m) \\
\hat{p}^{(\Theta)}(d) &= \theta(d, u)p(d, u) + \theta(m, d)q(m, d)
\end{aligned} \tag{36}$$

Then, by choosing appropriate Θ, combined measure $Q(\Theta)$ can be made a probability and is equivalent to Q^*. Moreover, in the next lemma, we will show that as long as it is a probability measure, it is martingale measure.

Lemma 2. *Suppose,* $\theta(x, y) > 0$ and $\sum_{(x, y) \in \{(u, m), (m, d), (d, u)\}} \theta(x, y) = 1$. *If* $Q(\Theta)$ *is a probability measure, then it is a martingale measure for* $\{\frac{S_t}{(1+r)^t}, F_t, t = 0,1\}$.

Proof. *By the straightforward algebra, it follows that,*

$$(u-m)\hat{p}^{(\Theta)}(u) + (m-d)\hat{p}^{(\Theta)}(m) = 1 + r - d \tag{37}$$

holds. The lemma follows readily from Lemma 1.

Note that, for any martingale measure $Q(\Theta) = \{\hat{p}^{(\Theta)}(u), \hat{p}^{(\Theta)}(m), \hat{p}^{(\Theta)}(d)\}$, an arbitrage free price of the option will be given by

$$C_0(\Theta)=(\frac{1}{1+r})[\hat{p}^{(\Theta)}(u)C_1^{(u)}+\hat{p}^{(\Theta)}(m)C_1^{(m)}+\hat{p}^{(\Theta)}(d)C_1^{(d)}] \tag{38}$$

Hence, by taking $\Theta=\{w^{(OLS)}(u, m), w^{(OLS)}(m, d), w^{(OLS)}(d, u)\}$, the OLS price is shown to be an arbitrage free price. The option price is not unique in general, for there are infinitely many weighs Θ that makes $Q(\Theta)$ a probability. The problem of which weight Θ should be selected has been considered by several authors. For example, Miyahara (1996) obtained the supremum or infimum of $C_0(\Theta)$ over the all possible Θ's. Our method is not only compatible with these method, but also gives us more general pricing schemes.

V. *Multi-period Model*

We will extend our trinomial two-period model to the multi-period model. The idea is the same as the extension of the usual two period binomial option price model to the multi-period models. For simplicity, we will consider the Markovian two-period trinomial model. The model may be expressed in the following picture (See Picture 2). In order the model to be Markov, it is necessary that $m^2=ud$ must hold.

PICTURE 2

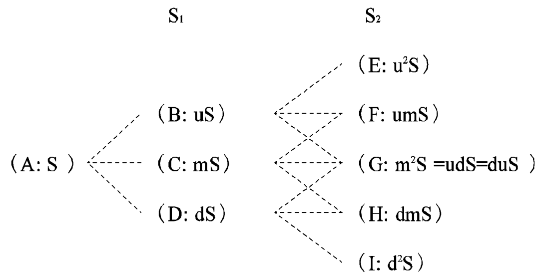


TABLE 1

Node	A	B	C	D	E	F	G	H	I
State	S	uS	mS	dS	u^2S	$umS=muS$	$m^2S=udS=duS$	$dmS=mdS$	d^2S
Probability	1	$\hat{p}^{(w)}(u)$	$\hat{p}^{(w)}(m)$	$\hat{p}^{(w)}(d)$	$\hat{p}^{(w)}(u)^2$	$\hat{p}^{(w)}(u)\hat{p}^{(w)}(m)$	$\hat{p}^{(w)}(m)^2+2\hat{p}^{(w)}(u)\hat{p}^{(w)}(d)$	$2\hat{p}^{(w)}(m)\hat{p}^{(w)}(d)$	$\hat{p}^{(w)}(d)^2$

In each node at $t=1$, we will assign the probability $\{\hat{p}^{(\Theta)}(u), \hat{p}^{(\Theta)}(m), \hat{p}^{(\Theta)}(d)\}$ which is defined in (36). From the node B, for example, the process moves up to the node E with probability $\hat{p}^{(\Theta)}(u)$. Hence the process reaches to the node E with probability $\hat{p}^{(\Theta)}(u)^2$. The node G will be reached either from the nodes B, C, or D, with probabilities $\hat{p}^{(\Theta)}(d)$, $\hat{p}^{(\Theta)}(m)$, and $\hat{p}^{(\Theta)}(u)$ respectively. Hence the probability of getting the state G is given by $\hat{p}^{(\Theta)}(m)^2 + 2\hat{p}^{(\Theta)}(u)\hat{p}^{(\Theta)}(d)$,which complicates obtaining the whole tree structure and as we will see below, makes it difficult to present the general case. We summarize this three-period model in *Picture 2* and *Table 1*.

The general n-period model may be described as follows. The total number of states at the n^{th} stage is given by the number of terms in the expansion of $(u+m+d)^n S = (u+\sqrt{u}\sqrt{d}+d)^n S$, where we have set $m^2 = ud$. The typical state may be expressed as $\sqrt{u}^{2n-k}\sqrt{d}^k$, $k=0,1,\ldots,2n$, and the coefficient of which gives us the number of routs leading to that state. The coefficient is proved to be

$$\sum_{p=0}^{[k/3]} (-1)^p \binom{n}{p} \binom{n+k-3p-1}{n-1} \tag{39}$$

(This formula was personally communicated to the author by Professors Nabeya and Hayakawa, to whom the author expresses his deep appreciation). The associated probability, however, is not easily calculated, for the probability of obtaining each route may differ because $\hat{p}^{(\Theta)}(d)\ \hat{p}^{(\Theta)}(u)$ need not be equal to $\hat{p}^{(\Theta)}(m)^2$. Therefore, the numerical calculation will be necessary to construct the general multi-period model.

VI. *Continuous Time Model*

In this section, we briefly discuss a pricing model in the following incomplete market. We extend the Black-Sholes model to the simple stochastic volatility model. Let $S(t)$ be the stock price process given by the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = S \quad \text{and} \tag{40}$$

$$\sigma \sim g(\sigma) \quad \text{independent of} \quad S(t) \quad \text{for all } t > 0 \tag{41}$$

where, $W(t)$ is Standard Brownian Motion. Our interpretation of this model is that each trader in the market has his/her own prediction on the future volatility. If we fix the value of the volatility, the above extended Black-Sholes model becomes the complete market model and every trader may be able to calculate their option prices from the Black-Scholes formula using their volatility values. We will denote $C_0(\sigma)$ the option price at $t=0$ with volatility σ and we will write its density function by $h_{C_0}(c)$. After having obtained every $C_0(\sigma)$, the market will decide its option price C_0. In view of Definition 1, it may be given by

$$C_0^{(M)} = \int_0^\infty C_0(\sigma)g(\sigma)d\sigma \tag{42}$$

The idea behind equation (42) is the method of least squares. If we use the mean absolute deviations in stead of mean squares, we may use the median of $h_{C_0}(c)$ as the price at $t=0$. On the other hand, if we use a majority rule at $t=0$, we may claim that the option price is given by the mode of $h_{C_0}(c)$. Some empirical studies are now underway to see the validity of our method in a slightly different model. We will address empirical issues in the next paper as well.

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