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EQUILIBRIA IN ECONOMIES WITH VECTOR LATTICES AND NON-ORDERED PREFERENCES*

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Abstract

In a seminal paper, Peleg-Yaari (1970) provides a theoretical foundation for the intertemporal resource allocation problem over discrete time. The authors illustrate the existence of a competitive equilibrium in an economy with countably many periods. In this paper, we present a far-reaching generalization that includes non-ordered preferences, externalities, no free disposal and infinite dimensional spaces with empty interiority. Furthermore, we extend the methodology that is developed in Florenzano (1991). For future applications of the model, the existence of equilibrium under weaker conditions is important. (86 words)

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I. *Introduction*

In Peleg-Yaari (1970), the authors deal with an infinite horizon economy whose commodity space is $\mathbb{R}^{\infty}$. The striking feature of the paper is that the equilibrium price functional does not assign finite valuation to every commodity in the commodity space. Thus the equilibrium price functional is not an element of the topological dual of the commodity space. Aliprantis-Brown-Burkinshaw (1987) extend Peleg-Yaari (1970) to an economy whose commodity space is a vector lattice or Riesz space, but their primary concerns are the existence of Edgeworth equilibria and the equivalence of approximate quasi-equilibrium, extended Walrasian equilibrium and Edgeworth equilibrium. Following Peleg-Yaari (1970), Besada-Estevez-Herves (1988) are concerned with a countably many periods exchange economy where they construct the price space as a set of price functionals which give finite valuation to the total endowment. The commodity space is defined as a set of commodities whose valuation with respect to the price space is finite. They are then able to make apparent the dual relation of commodity and price. Florenzano (1991) generalizes Besada-Estevez-Herves (1988) by assuming that the

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economy can have a different Riesz space as its commodity space in differing periods. The author first confines herself to a reduced economy on the principal ideal, where, by Mas-Colell's (1986) uniform properness, there exists a quasi-equilibrium for the reduced economy. Using the Riesz decomposition theorem and the fact that the price space is a subspace of the order continuous dual of the commodity space, she proves that the quasi-equilibrium for the reduced economy is a quasi-equilibrium for the entire economy. Since she assumes that preferences are monotonic, transitive, and complete, she can apply Mas-Colell's (1986) existence theorem to the economy defined on the principal ideal.

In this paper, we extend Florenzano (1991) to an economy with non-ordered preferences, externalities and no free-disposal. Specifically, we consider an infinite horizon exchange economy with a finite number of agents. In each period, we allow for differing vector lattice in the representation of the commodity space. Furthermore, we represent preferences by a correspondence which is assumed to be convex, open with respect to the relevant topology, and affected by other agents' consumption. With the assumption of non-ordered preferences, we cannot appeal to Mas-Colell's theorem as Florenzano (1991). Instead, we will appeal to the properness of preferences which is applicable to non-ordered preferences. Similar to Florenzano (1991), we begin with a reduced economy on the principal ideal where we consider finite dimensional subeconomies. In each subeconomy, we utilize Shafer (1976) to obtain a competitive equilibrium and then, form a convergent net of the equilibria. Following Bewley (1972), we take the limit and find an allocation and price functional which is an element of the topological dual space of the principal ideal. However, the norm-topology on the principal ideal is finer than the topology on the entire space. To make the price functional of the reduced economy continuous with respect to the entire economy's topology, we shall use the properness of preference as modeled in Podczeck (1996). We then prove the existence of the equilibrium.

It is well known that infinite dimensional general equilibrium theory suffers from some technical difficulties.[1] Empty-interior points in the better-than-set is one such difficulty. In the finite dimensional model, the price supportability of individual preferred sets is a result of the separating hyperplane theorem. The infinite dimensional version of the theorem requires not only convexity but also interior points in the preferred sets. Since Mas-Colell (1986), several types of properness of preferences have been introduced into the literature to guarantee non-empty interior points. Podczeck (1996) obtains the existence of a competitive equilibrium with no free-disposal and non-ordered preferences. He introduces E-proper preferences and uses the properness to prove the existence result without the assumption that the principal ideal is dense in the entire space. Furthermore, he shows the same result in the case where the ideal is dense, using F-properness as first used by Yannelis-Zame (1986). Unlike other papers, he does not assume uniform properness. Instead he requires preferences be proper at individually rational and Pareto efficient allocations. We will appeal to Podczeck (1996) to obtain the existence of competitive equilibria, but we assume preferences are proper at individually rational points since we allow externalities.

The paper is organized as follows. Section 2 is devoted to mathematical definitions. In section 3, we present a model and some immediate results regarding the properties of the

[1] The main difficulties are as follows: attainable sets may not be compact; preferred sets may not be supportable by prices; budget(or wealth) may not be jointly continuous as a function of prices and quantities. For details, readers are referred to Mas-Colell-Zame (1991).

commodity and price spaces. Section 4 contains results on the existence of a competitive equilibrium. Proofs of our results are provided in Section 5.

II. *Definitions*

A partially ordered vector space E is said to be a *Riesz space or vector lattice* if for any x, $y \in E$, the supremum $x \vee y$ and the minimum $x \wedge y$ of the set $\{x, y\}$ exist. We denote by E^+ the positive cone of E. For $x, y \in E$, we say $x \geq y$ if $x - y \in E^+$ and $|x| = x \vee (-x)$.

A subset A of E is called a *solid set* if $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of E is called an *ideal*. Let $x, y \in E$ satisfy $x \leq y$. Then the set $[x, y] = \{z \in E: x \leq z \leq y\}$ is called an *order interval* of E. Subsets of order intervals of E are referred to as *order bounded* sets. For an element $u \geq 0$, there exists a smallest (with respect to inclusion) ideal of E that contains u. This ideal is called the ideal generated by u and is the set $A_u = \{x \in E: \exists \lambda > 0 \text{ with } |x| \leq \lambda u\}$. Any ideal of the form A_u is referred to as a *principal ideal*.

A linear functional $f: E \to \mathbb{R}$ is said to be *order bounded* if it maps order bounded subsets of E onto order bounded subsets of $\mathbb{R}$. The set of all order bounded linear functionals of E is called the *order dual* of E and denoted by $E^{\sim}$. A Riesz space E is said to be *Dedekind complete* if every nonempty subset that is bounded from above has a supremum.

A *Riesz dual system* $<E, E'>$ is a dual system such that: (i)E is a Riesz space; (ii)E' is an ideal of the order dual of $E^{\sim}$; (iii)$<x, x^*> = x^* \cdot x$ holds for all $x \in E$ and all $x^* \in E'$. A Riesz dual system $<E, E'>$ is symmetric whenever E is an ideal of E'', the topological dual of E'.

A seminorm ρ on a Riesz space E is said to be a *Riesz seminorm* if $|u| \leq |v|$ in E implies $\rho(u) \leq \rho(v)$. A complete normed Riesz space is called a *Banach lattice*.

III. *The Model*

We consider an infinite horizon exchange economy. Following Florenzano (1991), we first model the commodity and price spaces.

1. Commodity and Price Spaces

Time is discrete and indexed by $t = 1, 2, \cdots$. At each period t, we have a symmetric Riesz dual system $<E_t, E_t'>$.[2] For $x_t \in E_t$ and $p_t \in E_t'$, we shall denote the evaluation as $p_t \cdot x_t$. The symmetric Riesz dual system $<E_t, E_t'>$ can take any space above at each period. Thus different period can have various symmetric Riesz dual systems. There are m agents indexed by $i \in I = \{1, 2, \cdots, m\}$. Each of them has an endowment $\omega^i = (\omega_t^i) \in \Pi_{t=1}^{\infty} E_t^+$. Let $\omega = \sum_{i=1}^m \omega^i \in \Pi_{t=1}^{\infty} E_t^+$ be the aggregate endowment.

In the spirit of Peleg-Yaari (1970), the price space is a set of price functionals which

[2] The following are some examples of symmetric Riesz dual systems: $<\mathbb{R}^n, \mathbb{R}^n>$; $<l_p, l_q>$, $(1 \leq p, q \leq \infty)$, $1/p + 1/q = 1$; $<L_p, L_q>$, $(1 \leq p, q \leq \infty)$, $1/p + 1/q = 1$; $<c_0, l_1>$; $<\mathbb{R}^{\infty}, \mathbb{R}'^{\infty}>$; $<ca(\Omega), ca'(\Omega)>$, where $c_0 = \{x \in \mathbb{R}^{\mathbb{N}}: \lim_{n \to \infty} x_n = 0\}$ and $ca(\Omega)$ is the collection of all signed measures of bounded variation.

provide the finite valuation to the total endowment. Our price space is given by

$$H=\{p\in\Pi_{t=1}^{\infty}E_t' : \sum_{t=1}^{\infty}|p_t|\cdot\omega_t<+\infty\}$$

where $\omega_t=\sum_{i=1}^{m}\omega_t^i$. The commodity space is given by

$$\Lambda(H)=\{x\in\Pi_{t=1}^{\infty}E_t\colon \sum_{t=1}^{\infty}|p_t|\cdot|x_t|<+\infty, \forall p\in H\}.$$

Thus the commodity space rests on the price space. We say that for $x, y\in\Lambda(H)$, $x\geq y$ if $x-y\in\Pi_{t=1}^{\infty}E_t^+$ and also define an order on H in a similar way. Obviously, $\Lambda(H)$ and H are vector lattices. We consider topologies on commodity and price spaces. Topology τ on $\Lambda(H)$ is defined by a Riesz seminorm

$$\rho_p(x)=\sum_{t=1}^{\infty}|p_t|\cdot|x_t|, \text{ where } p\in H \text{ and } x\in\Lambda(H)$$

and τ' on H is defined by a Riesz seminorm

$$\rho_x(x)=\sum_{t=1}^{\infty}|p_t|\cdot|x_t|, \text{ where } x\in\Lambda(H) \text{ and } p\in H.$$

Thus we are given two topological spaces $(\Lambda(H), \tau)$ and (H, τ'). For $x\in\Lambda(H)$, $p\in H$, the bilinear map is defined by $<x, p>=p\cdot x=\sum_{t=1}^{\infty}p_t\cdot x_t$. Given two topological spaces with their bilinear map, it is natural to ask whether they are a dual system. In this regard, we first deal with the compactness of order intervals in the commodity space in Proposition 1. Then by taking advantage of Proposition 1, we shall show that $<\Lambda(H), H>$ is a symmetric Riesz dual system. The following two propositions are due to Florenzano (1991). Let $\sigma(\Lambda(H), H)$ be the weak topology on $\Lambda(H)$ and $\sigma(H, \Lambda(H))$ is similarly defined.

Proposition 1 *Every order interval of* $\Lambda(H)$ *is* $\sigma(\Lambda(H), H)$*-compact and every order interval of* H *is* $\sigma(H, \Lambda(H))$*-compact.*

Proof. See Florenzano (1991).

Proposition 1 shows that every order interval is compact in a weaker topology, which is sufficient for the existence of an equilibrium if the preferences defined in the next section are continuous with respect to the same weaker topology.

In the next section, we will study a reduced economy as described in the introduction. For this we need a principal ideal generated by the aggregate endowment ω. We will investigate some properties of the ideal.

Suppose $\omega\in\Lambda(H)^+$. Let A_ω be the principal ideal of $\Lambda(H)$ generated by ω;

$$A_\omega=\{x\in\Lambda(H)\colon \exists\lambda>0 \text{ such that } |x|\leq\lambda\omega\}.$$

On A_ω, a lattice norm is defined by

$$\|x\|_\infty=\inf\{\lambda>0\colon |x|\leq\lambda\omega\}, \text{ where } x\in A_\omega.$$

A_ω with the topology τ_∞ generated by the $\|\cdot\|_\infty$-norm is locally convex-solid Riesz space.[3] Let A'_ω be a τ_∞-topological dual of A_ω. The following proposition shows that our commodity and price spaces are a symmetric Riesz dual system.

Proposition 2 *$<\Lambda(H), H>$ is a symmetric Riesz dual system. The topologies τ and τ' are Hausdorff locally convex-solid and consistent with the duality. $\Lambda(H)$ is Dedekind complete and $H \subset A'_\omega$.*

Proof. See Florenzano (1991) and Theorem 2.1 in Aliprantis-Brown-Burkinshaw (1987).

Proposition 2 shows that H is the τ-dual of $\Lambda(H)$ and $\Lambda(H)$ is the τ'-dual of H. The topological dual space H is also the subspace of the order dual space of $\Lambda(H)$ so that $H \subset A'_\omega$. This shows that $p \in A'_\omega$ is not always continuous with respect to the topology τ.

Since $\Lambda(H)$ is Dedekind-complete, so is A_ω. Then $(A_\omega, \|\cdot\|_\infty)$ is a Banach lattice.[4] Thus A'_ω is the same as the order dual of A_ω, $A_\omega^\sim$. On the other hand, for each $x \in A_\omega$ we have $|x| \leq \|x\|_\infty \cdot \omega$. Then for all $x \in A_\omega$, $\rho_p(x) \leq \rho_p(\omega)\|x\|_\infty$ holds. Thus the Riesz seminorm $\rho_p(\cdot)$ on $\Lambda(H)$ is τ_∞-continuous on A_ω. In particular, the restrictions of the functionals of H to A_ω belongs to A'_ω. Clearly, the positive cone $A_\omega^+ = A_\omega \cap \Lambda(H)^+$ is τ_∞-closed in A_ω and ω is a τ_∞-interior point of A_ω^+.

2. Economy

An exchange economy $\mathcal{E}$ is an m-tuple $(X_i, P_i, \omega^i)_{i=1}^m$ where X_i is a consumption set, and P_i: $X \to 2^{X_i}$ is a preference map ($X = \Pi_{i=1}^m X_i$). We assume $X_i = \Lambda(H)^+, \forall i \in I$. It is clear that $\Lambda(H)^+$ is convex. As assumed in the previous section, agent i's initial endowment ω^i is in E_t^+ so that $\omega^i \in \Lambda(H)^+$. The aggregate endowment $\omega = \sum_{i=1}^m \omega^i$ is, therefore, in the $\Lambda(H)^+$. We assume $\omega \neq 0$. An *allocation* is $x \in X$. An allocation x is *feasible* if $\sum_{i=1}^m x^i = \omega$ and it is *individually rational* if $\omega^i \notin P_i(x), \forall i \in I$. We make use of the following assumptions;

A.1 $x^i \notin P_i(x^1, \cdots, x^m)$ for each agent i and each point $(x^1, \cdots, x^m) \in X$.

A.2 $P_i(x^1, \cdots, x^m)$ is convex for all $x \in X$.

A.3 P_i has an open graph in $X \times X_i$ where X is endowed with the product topology and with each constituent set being endowed with $\sigma(\Lambda(H), H)$ topology and X_i with τ topology.

A.4 $P_i\,(x) \cap A_\omega \neq \emptyset$ for all $x \in [0, \omega]^m$ and for all $i \in I$.

A.1 shows that preferences are irreflexive. By A.2 the convexity of preferences is assumed. A.3 is about the continuity of preferences. The continuity shows that preferences is myopic. A. 4 implies that nonsatiation holds on the feasible sets in A_ω.

An *equilibrium* for an exchange economy $\mathcal{E}$ is an $(m+1)$-tuple $(x^1, \cdots, x^m; p)$ where $x^i \in X_i$ and $p \in H \setminus \{0\}$ such that

[3] See Aliprantis-Brown-Burkinshaw (1987) p.1125.

[4] See Aliprantis-Brown-Burkinshaw (1987) p.1125.

(i) $\sum_{i=1}^{m} x^i = \sum_{i=1}^{m} \omega^i$,

(ii) $p \cdot x^i = p \cdot \omega^i$, $\forall i \in I$,

(iii) $y^i \in P_i(x)$ implies $p \cdot y^i > p \cdot \omega^i$.

A *quasi-equilibrium* is a $(x, p) \in (X \times H)$ such that (i), (ii) above and the following hold: (iii') if $y^i \in P_i(x)$, then $p \cdot y^i \geq p \cdot \omega^i$. A quasi-equilibrium (x, p) is said to be *non-trivial* if there is $i \in I$, such that inf $\{p \cdot x \colon x \in \Lambda(H)^+\} < p \cdot \omega^i$.

IV. *Results*

We shall show the existence of a competitive equilibrium for ε. Infinite dimensional economic models have a well-known empty interior point problem. To avoid such problem, we limit ourselves to a principal ideal by defining a reduced economy. Note that A_ω contains all the feasible allocations. Even though A_ω is infinite dimensional, its positive cone A_ω^+ has non-empty τ_∞-interior points so that we can apply the Separating Hyperplane Theorem to obtain an equilibrium price functional. Thus we first consider a reduced economy defined on the principal ideal for the reduced economy.

Let $\mathcal{E}^{A_\omega}$ be the economy $((X_i \cap A_\omega), P_i^{A_\omega}, \omega^i)_{i=1}^m$ where $P_i^{A_\omega}$ is the restriction of P_i to $\Pi_{i=1}^m (X_i \cap A_\omega)$ and such that for any point x in the domain $P_i^{A_\omega}(x) = P_i(x) \cap A_\omega$. Let $\sigma(\Lambda(H), H)^{A_\omega}$ be the weak topology $\sigma(\Lambda(H), H)$ restricted to A_ω.

Proposition 3 *Suppose $\mathcal{E}^{A_\omega}$ satisfies A.1-A.4. Then there exist an individually rational allocation $\bar{x}$ and a linear functional $\bar{p} \colon A_\omega \to \mathbb{R}$, $\bar{p} \neq 0$ with $\bar{p} \cdot \omega > 0$, inf $\{\bar{p} \cdot z \colon z \in A_\omega^+\} < \bar{p} \cdot \omega^i$ for some $i \in I$, and for $y \in P_i(\bar{x}) \cap A_\omega$, $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i$ $\forall i \in I$.*

Proof. See Section 5.

We are now in a position to consider an equilibrium for the entire economy. Using $(\bar{x}, \bar{p})$ of the reduced economy, we are going to derive an equilibrium. As discussed in the introduction, $\bar{p}$ is not τ-continuous on A_ω. To make it τ-continuous, we appeal to properness of preference. We will employ F-properness used first by Yannelis-Zame (1986) and E-properness suggested by Podczeck (1996). Definitions of each type of properness are slightly modified to accommodate externalities and are given below.

Definition 1 *Let $x = (x^1, \cdots, x^m) \in X$. We say that P_i is F-proper at x^i if there exists a vector $v_i \in \Lambda(H)$ and τ-neighborhood U_i of zero such that*

1. $x^i + v_i \in X_i$;

2. if $u \in U_i$, then $x^i + \alpha v_i - \alpha u_i \in \Lambda(H)^+$ implies that $x^i + \alpha v_i - \alpha u_i \in P_i(x)$ for every real number $\alpha > 0$ which is sufficiently small.

The economic meaning of F-properness is as follows. An agent whose consumption bundle x^i gives up α portion of any sufficiently small u_i for an additional increment of v_i

measured by α, and the resulting bundle belongs to the better-than set. In this sense v_i is *extremely desirable* as pointed out by Yannelis-Zame (1986).

Definition 2 *Let* $x=(x^1,\cdots,x^m)\in X$ *and* K *be a linear subspace of* $\Lambda(H)$ *with* $x^i\in K$. P_i *is E-proper at* x^i *relative to* K *if there is some* $v_i\in X_i$, *some* τ*-neighborhood of* U_i *of zero, and some* $A_i\subset K$ *which is radial at* x^i *(in* K*) such that*

1. $x^i+\alpha v_i\in P_i(x)$ *for every sufficiently small real number* $\alpha>0$;

2. if $\tilde{x}^i\in A_i\cap X_i$ *and* $\tilde{x}^i\notin P_i(x)$, *then* $u_i\in U_i$ *implies* $\tilde{x}^i-\alpha v_i+\alpha u_i\notin P_i(x)$ *for every real number* $\alpha>0$.

The meaning of E-properness at x^i relative to K is as follows. The commodity bundle v_i is desirable in the sense that adding any sufficiently small amounts of this bundle results in a bundle in the better-than set of x^i. The set A_i, radial at x^i, reflects the idea of a set of sufficiently close points of x^i. Now consider an agent who starts at a consumption bundle $\tilde{x}^i$ in K which is not in the better-than set of x^i but sufficiently close to x^i. If we take αv_i out of $\tilde{x}^i$ and substitute some amount αu_i of any other sufficiently small commodity bundle u_i, then the results of the substitution cannot lie in the better-than set of x^i. It is worth noting that we do not assume uniform properness. Instead we apply properness of preferences only to individually rational allocations.[5]

The following lemma is important in our proof of the theorem. It is from the separation theorem. Podczeck (1996) uses a similar lemma to extend the price functional on the principal ideal to the entire economy. But we take the lemma from Deghdak and Florenzano (1999) which is easier to apply to our proof of the existence theorem than Podczeck's.

Lemma 1 *Let* (Y, τ) *be an ordered topological vector space and* M *be a vector subspace of* Y. *Let* Z *be an* τ*-open and convex subset of* Y *such that* $Z\cap M_+\neq\emptyset$. *Let* $x\in clZ\cap M_+$ *(*clZ *denotes the* τ*-closure of* Z*) and* p *be a linear functional on* M. *Suppose* $p\cdot x\leq p\cdot x'$, $\forall x'\in Z\cap M_+$. *Then there exists a* τ*-continuous linear functional* π *on* Y *such that* $\pi\leq p$ *on* M *and* $p\cdot x=\pi\cdot x\leq\pi\cdot x'$, $\forall x'\in Z$.

Proof. See Lemma 6.1 in Deghdak and Florenzano (1999).

The proof of following theorem is a simple application of Lemma 4 in Podczeck (1996). We shall show that for $(\bar{x},\bar{p})$, competitive equilibrium for $\mathcal{E}^{A_\omega}$, we can extend $\bar{p}$ to a continuous linear functional on $\Lambda(H)$.

Theorem 1 *Let* $\mathcal{E}$ *be an exchange economy and satisfy A.1-A.4. suppose either*

- A_ω *is* τ*-dense in* $\Lambda(H)$ *and if* $x=(x^1,\cdots,x^m)\in X$ *is individually rational, then* $P_i(x)$ *is F-proper at* x^i, $\forall i\in I$.

  or,

- *if* $x=(x^1,\cdots,x^m)\in X$ *is individually rational, then for each* $i\in I$ P_i *is E-proper at* x^i *relative to* A_ω *with a properness vector* v_i *satisfying* $x^i+v_i\in A_\omega^+$.

Then there exist a $\bar{x}\in X$ *and a* τ*-continuous price functional* $\bar{\pi}$ *such that* $(\bar{x}, \bar{\pi})$ *is a non-trivial*

[5] For detailed discussion of properness, readers are refereed to Podczeck (1996).

quasi-equilibrium of $\mathcal{E}$.

Proof. See Section 5.

We shall say that the economy $\mathcal{E}$ is *irreducible* if x is any feasible allocation and if I_1 and I_2 is a non-trivial partition of I, then there exists an allocation $\tilde{x}$ such that $\tilde{x}^i \in P_i(x)$ for all $i \in I_1$ with $\sum_{I_1} \tilde{x}^i = \sum_{I_1} x^i + \sum_{I_2} (\omega^i - \tilde{x}^i)$.

Proposition 4 *Suppose that* $\mathcal{E}$ *satisfies A.1-A.4, and is irreducible. Then there exist an equilibrium* $(\bar{x}, \bar{\pi}) \in (\Lambda(H))^m \times H$, $(\bar{\pi} \neq 0)$.

Proof. See Section 5.

V. *Proofs*

Proof of Proposition 3. Let $\mathcal{F}$ be the set of all finite dimensional subspaces of A_ω containing ω^i $\forall i \in I$. $\mathcal{F}$ will be directed by incursion. From A.3 and A.4, there exists an $F_1 \in \mathcal{F}$ such that for $x \in [0, \omega]^m$, $P_i(x) \cap F_1 \neq \emptyset$ for all $i \in I$. Let $\omega^i_\nu = (1-\nu)\omega^i + \nu \dfrac{1}{m}\omega$, where $0 < \nu < 1$. Since ω is a τ_∞-interior point of A^+_ω, so is $\dfrac{1}{m}\omega$ and, then, ω^i is also in A^+_ω. Since A^+_ω is convex, ω^i_ν is a τ_∞-interior point of A^+_ω. We take any $F \in \mathcal{F}$ with $F_1 \subset F$. Let $\mathcal{E}^{A_\omega F}$ be the economy $(X_i \cap F, P_i^{A_\omega F}, \omega^i_\nu)_{i=1}^m$ where $P_i^{A_\omega F}$ is the restriction of P_i to $\Pi_{i=1}^m (X_i \cap F)$. It is easy to verify that this restricted economy satisfies the assumptions of Theorem 2 along with remark 3 in Shafer (1976). By letting ν go to zero and taking limits, we obtain a quasi-equilibrium $(x^{iF}, \cdots, x^{mF}, p^F)$. Since ω is a τ_∞-interior point of A^+_ω and also lies in F, it belongs to F^+. With the fact of $p^F \neq 0$, we have $p^F \cdot \omega > 0$ so that we can normalize p^F to have norm one and to be $p^F \cdot \omega = 1$. We appeal to the Hahn-Banach Theorem to extend p^F to $\bar{p}^F$ in A'_ω, the τ_∞-dual of A_ω, which agrees with p^F on F and has norm one. We now have a net $(x^{1F}, \cdots, x^{mF}, \bar{p}^F) \in [0, \omega]^m \times A'_\omega$ which satisfies the following:

(i) $\sum_{i=1}^m x^{iF} = \omega$;

(ii) $\omega^i \notin P_i(x^F)$, $\forall i \in I$;

(iii) if $y \in P_i(x^F) \cap F$, then $\bar{p}^F \cdot y \geq \bar{p}^F \cdot \omega^i$, $\forall i \in I$.

The order interval $[0, \omega]$ is $\sigma(\Lambda(H), H)^{A_\omega}$-compact and contains x^{iF} for all i and for all $F \in \mathcal{F}$ with $F_1 \subset F$. For all $i \in I$, x^{iF} has a subnet which $\sigma(\Lambda(H), H)$-converges to some $\bar{x}^i \in [0, \omega]$. We obtain $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^m)$ in $[0, \omega]^m$. Since for all F, $\sum_{i=1}^m x^{iF} = \sum_{i=1}^m \omega^i$ and x^{iF} converges to $\bar{x}^{iF}$ in the topology $\sigma(\Lambda(H), H)$, we can conclude $\sum_{i=1}^m \bar{x}^i = \sum_{i=1}^m \omega^i$.

By Alaoglu's Theorem, $\{p \in A'_\omega : |<x, p>| \leq 1$, for all $x \in A_\omega\}$ is $\sigma(H, \Lambda(H))^{A'_\omega}$-compact, where $\sigma(H, \Lambda(H))^{A'_\omega}$ is the restriction of $\sigma(H, \Lambda(H))$ to A'_ω. So we can assume that $\bar{p}^F$ converges in the topology $\sigma(H, \Lambda(H))^{A'_\omega}$ to some $\bar{p} \in A'_\omega$,

We shall show that $y \in P_i(\bar{x}) \cap A_\omega$ implies $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i$. Suppose for $y \in P_i(\bar{x}) \cap A_\omega$, $\bar{p} \cdot y < \bar{p} \cdot$

ω. Since P_i has an open graph in $X \times X_i$, there exist y^F for large enough F such that $y^F \in P_i(x^F)$ where y^F is in F^+ and τ_∞-converges to y, and x^F is in F^+. Since all y^F and x^F are in F, we may write $y^F \in P_i^F(x^F)$ where $P_i^F(x^F) = P_i(x^F) \cap F$. Note that y^F converges to y in the topology τ_∞, it also converges to y in the topology $\tau^{A'_\omega}$, because the former is stronger than the latter. Since $\bar{p}^F$ converges to $\bar{p}$ in the $\sigma(H, \Lambda(H))^{A_\omega}$ for large enough F, we obtain $\bar{p}^F \cdot y^F < \bar{p}^F \cdot \omega$ (see Lemma A in Yannelis-Zame (1986)). This contradicts the fact that $(x^F, \bar{p}^F)$ is a quasi-equilibrium for the restricted economy. We establish the desired result: $y \in P_i(\bar{x}) \cap A_\omega$ implies $\bar{p} \cdot y \geq \bar{p} \cdot \omega$.

The next step is to prove $\bar{p} \neq 0$. By Proposition 1, order interval $[-\omega, \omega]$ is $\sigma(\Lambda(H), H)$-compact so that it is $\sigma(\Lambda(H), H)$-bounded and thus also τ-bounded. Since $x \in [0, \omega]^m$, by A.3, we can take $y \in P_i(\bar{x}) \cap A_\omega^+$ with y being a τ_∞-interior point of A_ω^+. Consider $F \in \mathcal{F}$ containing F_1, y and $[-\omega, \omega]$. Since P_i has an open graph, and x^{iF} converges to $\bar{x}^i$ in the $\sigma(\Lambda(H), H)$-topology, there exist an $\varepsilon > 0$ such that $\{y\} + \varepsilon[-\omega, \omega] \subset P_i(x^F)$. We can pick $\omega \in [-\omega, \omega]$ so that $\bar{p}^F \cdot \omega = 1$. Therefore $y - \varepsilon\omega \in P_i(x^F)$. Note that $(x^F, \bar{p}^F)$ is a quasi-equilibrium for this F dimensional restricted economy so that we have $\bar{p}^F \cdot y \geq \bar{p}^F \omega^i + \varepsilon$. We know that $\bar{p}^F$converges to $\bar{p}$ in the $\sigma(H, \Lambda(H))$-topology. If $\bar{p} = 0$, then the last inequality becomes absurd. It follows $\bar{p} \neq 0$. Since ω is a τ_∞-interior point of A_ω^+ and $\bar{p} \neq 0$, we can say $\bar{p} \cdot \omega > 0$.

Finally, we shall show that $\inf\{\bar{p} \cdot z : z \in A_\omega^+\} < \bar{p} \cdot \omega^i$. Suppose the opposite is true, i.e., for every $i \in I$, $\inf\{\bar{p} \cdot z : z \in A_\omega^+\} \geq \bar{p} \cdot \omega^i$. Since $\dfrac{1}{m}\omega$ is an interior point of A_ω^+, there exists $\varepsilon > 0$ such that $\bar{p} \cdot (\dfrac{\omega}{m} + \varepsilon[-\omega, \omega]) \geq \bar{p} \cdot \omega^i$ holds for every $i \in I$. By summing over i, we obtain $\bar{p} \cdot m\varepsilon [-\omega, \omega] \geq 0$ which leads to $\bar{p} = 0$. This contracts to $\bar{p} \neq 0$.

Proof of Theorem 1. By Proposition 3, there exist $(\bar{x}, \bar{p}) \in ((A_\omega)^m \times A'_\omega)$ such that $y \in P_i(\bar{x}) \cap A_\omega$ implies $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i\ \ \forall i \in I$. By properness of preferences, there are $v_i \in \Lambda(H)^+$ and $\alpha > 0$ such that $\bar{x}^i + \alpha v_i \in P_i(\bar{x}) \cap A_\omega$. Thus $\bar{p} \cdot (\bar{x}^i + \alpha v_i) \geq \bar{p} \cdot \omega^i$. By continuity of $\bar{p}$, we have $\bar{p} \cdot \bar{x}^i \geq \bar{p} \cdot \omega^i$, $\forall i \in I$. By summing over i, $\bar{p} \cdot (\sum_{i=1}^m \bar{x}^i) \geq \bar{p} \cdot (\sum_{i=1}^m \omega^i)$. Then from the feasibility of $\bar{x}$, we have $\bar{p} \cdot (\sum_{i=1}^m \bar{x}^i) = \bar{p} \cdot (\sum_{i=1}^m \omega^i)$ and therefore, $\bar{p} \cdot \bar{x}^i = \bar{p} \cdot \omega^i$, $\forall i \in I$. Thus we conclude $y \in P_i(\bar{x}) \cap A_\omega$ implies $\bar{p} \cdot y \geq \bar{p} \cdot \omega^i = \bar{p} \cdot x^i$, $\forall i \in I$. Now $(\bar{x}, \bar{p})$ satisfies all the requirements of a quasi-equilibrium except for the continuity of $\bar{p}$ in the τ-topology on $\Lambda(H)$. We shall show in the below that $\bar{p}$ can be extended to be an element of H using Lemma 1.

We now consider the first case where A_ω is τ-dense in $\Lambda(H)$ and P_i is F-proper at $\bar{x}^i$. By F-properness at $\bar{x}$, there exists a τ-open and convex cone $\Gamma_i = \{\lambda(v_i - u) : u \in U_i, \lambda \in \mathbb{R}_{++}\}$ for each i where v_i and U_i are the properness vector and the τ-neighborhood of zero, respectively. Then it follows that $\{\bar{x}^i\} + \Gamma_i$ is also τ-open and convex. Since A_ω is τ-dense in $\Lambda(H)$, A_ω^+ is τ-dense in $\Lambda(H)^+$ (by Lemma 3 in Podczeck (1996)). Thus we have

$$(\{\bar{x}^i\} + \Gamma_i) \cap A_\omega^+ \neq \emptyset. \tag{1}$$

Since $\{\bar{x}^i\} + \Gamma_i$ is τ-open and convex, and Γ_i is a cone, $\bar{x}^i$ belongs to the τ-closure of $\{\bar{x}^i\} + \Gamma_i$. Again by F-properness, there is a $\gamma \in \Gamma_i$ such that $\bar{x}^i + \gamma \in \Lambda(H)^+$ implies

$$\bar{x}^i + \alpha\gamma \in P_i(\bar{x}) \tag{2}$$

for a sufficiently small positive real number α. Since $\bar{x}^i \in A_\omega^+$, $\bar{x}^i + \gamma \in A_\omega^+$ implies $\bar{x}^i + \alpha\gamma \in A_\omega^+$ if $0 \leq \alpha \leq 1$. Then it is clear

$$\bar{x}^i+\alpha\gamma\in(\{\bar{x}^i\}+\Gamma_i)\cap A_\omega^+. \tag{3}$$

Obviously $\bar{p}\cdot\bar{x}^i\leq\bar{p}\cdot(\bar{x}^i+\alpha\gamma)$ and we can conclude $\bar{p}\cdot\bar{x}^i\leq\bar{p}\cdot y$ for all $y\in(\{\bar{x}^i\}+\Gamma_i)\cap A_\omega^+$. Thus we showed that the assumptions of Lemma 1 hold. Let $Z=\{\bar{x}^i\}+\Gamma_i$, $M_+=A_\omega^+$, $x=\bar{x}^i$, and $p=\bar{p}$. Then by Lemma 1, there exists a $\pi_i\in H$ such that $\pi_i\cdot\bar{x}^i\leq\pi_i\cdot y$, $\forall y\in\{\bar{x}^i\}+\Gamma_i$ for each $i\in I$. Let $\bar{\pi}=\vee_{i=1,\cdots,m}\pi_i$. We shall show that $\bar{\pi}$ is a quasi-equilibrium price functional. Since H is a vector lattice space, $\bar{\pi}$ is a τ-continuous linear functional on $\Lambda(H)$, i.e., $\bar{\pi}\in H$. From $\sum_i\bar{x}^i=\omega$ we have $\bar{p}\cdot\omega=\sum_i\bar{p}\cdot\bar{x}^i=\sum_i\pi_i\cdot\bar{x}^i\leq\bar{\pi}\cdot\omega$. But on A_ω, $\bar{\pi}\leq\bar{p}$ so that $\bar{p}\cdot\omega\geq\bar{\pi}\cdot\omega$. Therefore, $\bar{p}\cdot\omega=\bar{\pi}\cdot\omega$. But we know that on A_ω $(\bar{p}-\bar{\pi})\cdot\omega=0$ on A_ω. Since ω is positive in A_ω, we get $\bar{p}=\bar{\pi}$ on A_ω. This implies that $\bar{\pi}$ is an extension of $\bar{p}$. Since for all $i\in I$ $\pi_i\cdot\bar{x}^i\leq\pi_i\cdot y$, $\forall y\in\{\bar{x}^i\}+\Gamma_i$, we have $\bar{\pi}\cdot x^i\leq\bar{\pi}\cdot y$ $\forall y\in\{\bar{x}^i\}+\Gamma_i$ or $\forall y\in P_i(\bar{x})$. Hence $(\bar{x},\bar{\pi})$ is a quasi-equilibrium of $\mathcal{E}$.

Next, we turn to the second case where P_i is E-proper at $\bar{x}^i$ relative to A_ω. From E-properness, there exists a $v_i\in A_\omega^+$ such that $\bar{x}^i+v_i\in A_\omega^+$ and $\bar{x}^i+\alpha v_i\in P_i(\bar{x})$, where α is a small enough positive real number. We can also construct a τ-open and convex cone $\Gamma_i=\{\alpha v_i\colon\alpha>0\}$.

We shall show that $P_i(\bar{x})+\Gamma_i$ can be a set Z in Lemma 1. Since $\bar{x}^i+v_i$ can be rewritten as $\bar{x}^i+\alpha v_i+(1-\alpha)v_i$, we have $\bar{x}^i+v_i\in(P_i(\bar{x})+\Gamma_i)$. It follows immediately that

$$(P_i(\bar{x})+\Gamma_i)\cap A_\omega^+\neq\emptyset. \tag{4}$$

From the fact that P_i is open and convex and that Γ_i is a open convex cone, $P_i(\bar{x})+\Gamma_i$ is τ-open and convex. The next step is to show $\bar{x}^i\in\mathrm{cl}(P_i(\bar{x})+\Gamma_i)$. Since $\bar{x}^i+\alpha v_i\in P_i(\bar{x})$ for all $\alpha>0$, we have $\bar{x}^i\in\mathrm{cl}P_i(\bar{x})$ and, thus, $\bar{x}^i\in\mathrm{cl}(P_i(\bar{x})+\Gamma_i)$, where 'cl' denotes τ-closure of a relevant set.

To apply Lemma 1, we need to verify one more condition:

$$\bar{p}\cdot\bar{x}^i\leq\bar{p}\cdot z,\ \forall z\in(P_i(\bar{x})+\Gamma_i)\cap A_\omega^+.$$

We choose $z\in(P_i(\bar{x})+\Gamma_i)\cap A_\omega^+$. Note that $\bar{x}^i\in A_\omega^+$ and that by E-properness there is A_i which is radial at $\bar{x}^i$. Thus there exists $\lambda\,(0<\lambda\leq1)$ such that $z_\lambda=(1-\lambda)\bar{x}^i+\lambda z\in A_i\cap A_\omega^+$. Since z is also in $P_i(\bar{x})+\Gamma_i$, it can be decomposed into $z_1+\gamma$ where $z_1\in P_i(\bar{x})$ and $\gamma\in\Gamma_i$. Thus $z_\lambda=(1-\lambda)\bar{x}^i+\lambda(z_1+\gamma)$. From the convexity of $P_i(\bar{x})$ along with $\bar{x}^i\in\mathrm{cl}P_i(\bar{x})$, we have $(1-\lambda)\bar{x}^i+\lambda z_1=z_\lambda-\lambda\gamma\in\mathrm{cl}P_i(\bar{x})$. But $z_\lambda-\lambda\gamma$ also belongs to the set $\{z_\lambda\}-\Gamma_i$. This implies that

$$(\{z_\lambda\}-\Gamma_i)\cap\mathrm{cl}P_i(\bar{x})\neq\emptyset. \tag{5}$$

From E-properness, we know that $y\in A_i\cap\Lambda(H)^+$ but $y\notin P_i(\bar{x})$ implies $(\{y\}-\Gamma_i)\cap P_i(\bar{x})=\emptyset$. Since Γ_i is open, this condition can be written as

$$y\in A_i\cap(H)^+,\text{ but }y\notin P_i(\bar{x})\text{ implies }(\{y\}-\Gamma_i)\cap\mathrm{cl}P_i(\bar{x})=\emptyset. \tag{6}$$

Considering (6), we can say that (5) implies that $z_\lambda\in P_i(\bar{x})$ and therefore $z_\lambda\in P_i(\bar{x})\cap A_\omega^+$. By the property of $(\bar{x},\bar{p})$ seen in Proposition 3, we have $\bar{p}\cdot\bar{x}^i\leq\bar{p}\cdot z_\lambda$, which implies that when $\lambda=1$, $\bar{p}\cdot\bar{x}^i\leq\bar{p}\cdot z$. This shows the desired result.

We are now ready to apply Lemma 1. Let $Z=P_i(\bar{x})+\Gamma_i$, $M_+=A_\omega^+$, $x=\bar{x}^i$, $p=\bar{p}$. Then by the same argument as above, there exists an $\bar{\pi}\in H$ which extends $\bar{p}$ and $(\bar{x},\bar{\pi})$ is a quasi-equilibrium.

Proof of Proposition 4. Let $I_1=\{i\in I\colon\text{for }z\in\Lambda(H)^+,\ \inf\bar{\pi}\cdot z<\bar{\pi}\cdot\omega^i\}$ and $I_2=\{i\in I\colon\text{for }z\in\Lambda(H)^+,\ \inf\bar{\pi}\cdot z\geq\bar{\pi}\cdot\omega^i\}$. Then $I_1\cup I_2=I$, and $I_1\cap I_2=\emptyset$. According to Theorem 1, $\mathcal{E}$ has a non-trivial quasi-equilibrium $(\bar{x}, \bar{\pi}) \in (\Lambda\ (H))^m \times H$, where $\bar{\pi} \neq 0$ and $\inf\{\bar{\pi}\cdot z\colon z \in A_\omega^+\} < \bar{\pi}\cdot\omega^i$ for some $i \in I$. Thus I_1 is not empty.

We shall first show that the equilibrium conditions are satisfied for each agent in I_1. Fix $i \in I_1$ and let $y \in P_i(\bar{x})$. Then $\bar{\pi}\cdot y \geq \bar{\pi}\cdot\omega^i$. We need to show $\bar{\pi}\cdot y > \bar{\pi}\cdot\omega^i$. Suppose $\bar{\pi}\cdot y = \bar{\pi}\cdot\omega^i$. Let $k = \inf \bar{\pi}\cdot z$, where $z \in \Lambda(H)^+$. We know that $\bar{\pi}\cdot\omega^i > k$. We can take $\lambda < 1$ to be sufficiently close to 1 and then, by continuity of P_i, we have $\lambda y \in P_i(\bar{x})$ with $\bar{\pi}\cdot(\lambda y) = \lambda(\bar{\pi}\cdot y) < \bar{\pi}\cdot y = \bar{\pi}\cdot\omega^i$. This contradicts the quasi-equilibrium conditions. We conclude that $\bar{\pi}\cdot y > \bar{\pi}\cdot\omega^i$ for all i in I_1.

Now suppose I_2 is non-empty. Then for $i \in I_2$, we have

$$\bar{\pi}\cdot\ \bar{x}^i \leq \bar{\pi}\cdot\omega^i \leq \inf \bar{\pi}\cdot X_i$$

which implies for any $z^i \in X_i$,

$$\bar{\pi}\cdot\ (\omega^i - z^i) \leq 0$$

and, in turn,

$$\bar{\pi}\ \cdot\ \sum_{I_2}(\omega^i - z^i) \leq 0.$$

Since $\mathcal{E}$ is irreducible, we can have an allocation $\tilde{x}$ such that $\tilde{x}^i \in P_i(\bar{x})\ \ \forall i \in I_1$ with

$$\sum_{I_1}\tilde{x}^i = \sum_{I_1}\bar{x}^i + \sum_{I_2}(\omega^i - \tilde{x}^i).$$

Thus we have

$$\bar{\pi}\ \cdot\ \sum_{I_1}(\tilde{x}^i - \bar{x}^i) = \bar{\pi}\ \cdot\ \sum_{I_2}(\omega^i - \tilde{x}^i) \leq 0.$$

Then for all $i \in I_1$, we have

$$\bar{\pi}\cdot\ \tilde{x}^i \leq \bar{\pi}\cdot\omega^i$$

which is a contradiction to $\tilde{x}^i \in P_i(\bar{x})\ \ \forall i \in I_1$. Hence I_2 is empty and for all $i \in I$, $y \in P_i(\bar{x})$ implies $\bar{\pi}\cdot y > \bar{\pi}\cdot\omega^i$. We conclude that $(\bar{x}, \bar{\pi})$ is an equilibrium.

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