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THE ZERO-INTEREST-RATE BOUND AND OPTIMAL MONETARY POLICY IN A SMALL OPEN ECONOMY*

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Abstract

To give a better understanding that, when the economy is caught up in a liquidity trap, it returns quickly to the normal state by using the exchange rate channel of monetary policy, we solved a central bank's intertemporal optimization problem in the framework of a small open economy. Given an adverse shock to aggregate demand, we computed the dynamic path of a short-term nominal interest rate in both discretion and commitment. We discovered that the timing to terminate a zero interest rate policy in the case of large openness would be earlier than that in the case of small openness. We find that the economy in case of large openness would be less overheated. Moreover, the difference between the optimal solution and discretionary solution becomes proportionately smaller to the degree of openness. Some simulation results reinforce these findings. Finally, this paper also suggests that the exchange rate peg enables the central bank to quickly end the zero interest rate policy.

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I. *Introduction*

The Bank of Japan (BOJ) adopted the so-called zero interest rate policy (ZIRP) to stimulate the Japanese economy from February 1999 to August 2000.[1] On February 12, 1999, the BOJ adopted this policy in order to avoid a possible intensification of deflationary pressure.

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[1] Thereafter, the decision of the BOJ policy board was made on March 19, 2001 as follows: (1) The main operating target for money market operations be changed from the current uncollateralized overnight call rate to the outstanding balance of the current accounts at the Bank of Japan. (2) The new procedures for money market operations continue to be in place until the consumer price index (excluding perishables, on nationwide statistics) registers stably a zero percent or an increase year on year. (3) For the time being, the balance outstanding at the Bank's current accounts be increased to around 5 trillion yen, or 1 trillion yen increase from the average 4 trillion yen outstanding in February 2001.

Moreover, the BOJ announced on April 13, 1999 that the monetary policy board would keep the overnight interest rate at zero until "deflationary concerns are dispelled." This announcement was intended to have the effect of lowering longer term interest rates by altering the expectations of market participants. This policy to escape from a liquidity trap by affecting market expectations has revived researchers' interest in the zero bound on nominal interest rates.

Krugman (1998) suggests that the BOJ should make a commitment to future monetary expansion and proposes an inflation target whereby the Japanese economy needs four percent per year for the next 15 years. When the nominal interest rate is bound at zero, an increase in inflation expectations brings about a reduction in the real interest rate and thereby stimulates the economy out of the liquidity trap. On the other hand, Woodford (1999) points out that, even when the current overnight interest rate is close to zero, the long-term nominal interest rate could be well above zero if future overnight rates are expected to be well above zero. Expectations theory of the term structure of interest rates implies that, in this situation, a central bank could lower the long-term nominal interest rate by committing itself to an expansionary monetary policy in the future, thereby stimulating aggregate demand.

Many suggestions are also offered for escaping from the liquidity trap in the framework of an open economy. Bernanke (2000) and Meltzer (2000) recommend that the BOJ lowers the yen to increase net exports and stop deflation by large-scale intervention. Their arguments rely on a portfolio-balance effect, whereby the relative supply of domestic and foreign currency denominated assets affects the exchange rate. In contrast to the above recommendations to manipulate the market rate by artificial intervention, Svensson (2000a) proposes to control market expectations about the future values of the currency. His argument does not rely on a portfolio-balance channel, but on a credible commitment to an expansionary policy. This method emphasizes an operation mechanism to change the current exchange rate by affecting market expectations of future exchange rates.

Since the ZIRP is adopted unavoidably to escape from an emergency situation, this policy seems to be terminated at some point in the future. Under this environment, an important point is the timing to end this policy. In other words, a central bank needs to specify and announce a contingency plan describing how long the ZIRP would be continued, i.e., when and under what circumstances the ZIRP should be terminated. The BOJ did this in its commitment of April 13, 1999, but its termination condition was very ambiguous. In practice, when the BOJ terminated a ZIRP on August 11, 2000, the government requested that the Policy Board postpone a vote on the proposed change of the guideline for money market operations until the next Monetary Policy Meeting.[2] At that time, the Director General of the Economic Planning Agency said that the BOJ had taken an optimistic view regarding the Japanese economy and the BOJ should have postponed terminating the ZIRP because deflationary concerns were not completely dispelled.

As stated above, we did not have a clear understanding of the definition of "deflationary concerns," which was a condition to end the ZIRP. More fundamentally, there was doubt as to whether the BOJ's termination condition was really appropriate. Many researchers and practitioners have said that the BOJ's policy might have been mistaken. Jung, et al. (2001)

[2] Pursuant to Article 19, Section 3 of the Bank of Japan Law, the Policy Board took a vote on this request and rejected it by majority vote.

investigated this problem in order to evaluate the BOJ's policy. Their main finding was that the optimal path is characterized by monetary policy inertia: the ZIRP should be continued for some time, even after the natural rate of interest returns to a positive level.

But Jung, et al. (2001) lacks the element of open economy, i.e., exchange rate mechanism.[3] The objective of this paper is to evaluate the ZIRP under the framework of a small open economy. The commitment of keeping the nominal interest rate at zero is expected to give rise to a weakening domestic currency and thereby lead to the improvement of output gap and deflation. As a result, it is supposed that the period of the ZIRP would be reduced.

The rest of the paper is organized as follows. Section 2 presents the setting of the demand and supply side under the framework of a small open economy, which is based on Gali and Monacelli (2002).[4] Section 3 and Section 4 characterize discretionary and commitment solutions to the problem. Section 5 gives a numerical example. Section 6 concludes the paper.

II. *The Model*

1. Central bank's loss function

The central bank chooses the path of the short-term nominal interest rates, starting from period 0, $\{i_0, i_1 \cdots\}$ to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t L_t, \tag{2.1}$$

where β is the discount factor and L is the loss function. Denoting domestic inflation by $\pi_{H,t}$, which is defined as the rate of change in the index of domestic goods prices, and output gap by x_t, the loss in a given period is given by

$$L_t = (\pi_{H,t} - \bar{\pi}_H)^2 + \lambda (x_t - \bar{x})^2, \tag{2.2}$$

where λ is a positive parameter representing the weight assigned to output stability, $\bar{\pi}_H$ denotes the domestic inflation target and $\bar{x}$ denotes the target level of the output gap. Denoting the real exchange rate by q_t and the share of domestic consumption allocated to imported goods, i.e., an index of openness by γ, domestic inflation and CPI inflation are linked as

$$\pi_t = \pi_{H,t} + \frac{\gamma}{1-\gamma} \Delta q_t, \tag{2.3}$$

which makes the gap between two measures of inflation dependent on the change in the real exchange rate and the coefficient given by the index of openness.

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[3] Of course, Japan is not a small open economy. But the essence of investigation made here applies to the case of a large open economy. Under existing environments where low and stable inflation has been achieved in many countries, in cases where the economy is hit by a large-scale negative shock, most countries have the possibility of being caught up in a liquidity trap like Japan.

[4] The characteristic of Gali and Monacelli (2002) is that it is laid out as a small open economy with Calvo-type staggered price-setting that is now common in the recent New Keynesian literatures. Another feature lies in the modeling of the rest of the world as a limiting case of an economy whose degrees of openness are negligible. This allows us to treat the rest of the world as if it were a closed economy.

2. Small open economy

Following Gali and Monacelli (2002), the small open economy outside the central bank is represented by two equations: an "IS curve" and an "AS curve."

$$x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{H,t+1} - r_t^n) - \frac{\omega - 1}{\sigma(1-\gamma)} E_t \Delta q_{t+1} \tag{2.4}$$

$$\pi_{H,t} = \beta E_t \pi_{H,t+1} + \kappa x_t \tag{2.5}$$

where i_t is the short-term nominal interest rate, r_t^n is the natural rate of interest, and σ, $\kappa \equiv \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}(\varphi + \frac{\sigma}{\omega})$ and $\omega \equiv 1 + \gamma(2-\gamma)(\sigma\eta - 1)$ are positive parameters. The natural rate of interest is an exogenous variable that could deviate from its steady-state level, thereby giving rise to fluctuations in the output gap and the inflation rate.[5] We also assume that η, α and φ are positive parameters.

Equation (2.4) states that output gap in period t is determined by the expected value of the output gap in period $t+1$, the deviation of the short-term real interest rate from the natural rate of interest in period t, and the expected change of the real exchange rate in period $t+1$. Equation (2.4) can be iterated forward to obtain

$$x_t = -\sigma^{-1} \sum_{j=0}^{\infty} E_t[(i_{t+j} - \pi_{H,t+j+1}) - r_{t+j}^n] + \frac{\omega - 1}{\sigma(1-\gamma)} q_t. \tag{2.6}$$

According to the expectations theory of the term structure of interest rates, the expression $\sum_{j=0}^{\infty} E_t[(i_{t+j} - \pi_{t+j+1}) - r_{t+j}^n]$ stands for the deviation of the long-term real interest rate from the corresponding natural rate of interest in period t, which implies that, given the path of the natural rate of interest, the output gap depends negatively on the long-term real interest rate. We find that the output gap would depend positively on the real exchange rate, assuming the sign of $\omega - 1$ is positive. This assumption is reasonable because depreciation of domestic currency increases net export, and accordingly the output gap improves.

Equation (2.5) is the New Keynesian Phillips Curve (NKPC) in a small open economy, which differs from a closed economy counterpart in that the coefficient of the output gap depends on the degree of openness and there is a domestic inflation rate instead of CPI inflation. Note that for $\gamma = 0$, the slope coefficient is absolutely identical to that of a closed economy NKPC. The degree of openness has an influence on inflation dynamics by only the size of the slope of the output gap.[6]

[5] More concretely, φ is the inverse of the elasticity of supplying goods, σ is the inverse of the intertemporal elasticity of substitution in consumption, η is the elasticity of substitution between domestic and foreign goods, and α denotes stickiness of prices following Calvo (1983), which assumes that each firm resets its price in any given period only with probability $1-\alpha$, independently of other firms and of the time that has elapsed since the last adjustment.

[6] The larger the degree of openness, the smaller the size of the slope of the output gap. This can be explained by the following logic. Let us assume that the output gap of the rest of the world is constant. Under this assumption, the increase in the domestic output gap means that the real exchange rate should depreciate in order for market clearing conditions to hold. The point to note is that if the degree of openness is large, the small size of depreciation of the real exchange rate is sufficient.

3. Uncovered interest parity

Under the assumption of complete international financial markets, the exchange rate fulfils the interest parity condition,

$$i_t - i_t^* = E_t e_{t+1} - e_t, \tag{2.7}$$

where i_t^* is the foreign short-term nominal interest rate and e_t is the nominal exchange rate (the price of foreign currency in terms of home currency). Combining this condition with equation (2.3) in order to incorporate the domestic inflation rate and the real exchange rate, we obtain the following equation,

$$\frac{1}{1-\gamma} q_t = \frac{1}{1-\gamma} E_t q_{t+1} - (i_t - E_t \pi_{H,t+1}) + (i_t^* - E_t \pi_{t+1}^*). \tag{2.8}$$

Equation (2.8) says that the present real exchange rate depends on the expected value of itself and the difference between the domestic real interest rate and a real interest rate of the rest of the world.

4. Zero bound constraint and adverse demand shock

We explicitly introduce the non-negativity constraint on short-term nominal interest rates,

$$i_t \geq 0, \tag{2.9}$$

like Jung, et al. (2001). In particular, this constraint is very important to an economy that is under unfavorable circumstances such as that of recent Japan. Woodford (1999) adopts the condition whereby the mean value of short-term nominal interest rates is no smaller than a prespecified positive level. This treatment makes the analysis much simpler, but at the price of losing reality. Therefore, this paper solves a dynamic optimization problem with explicit treatment of the non-negativity constraint in a forward-looking model notwithstanding its nonlinearity.[7]

Finally, it is assumed that a large negative demand shock to the natural interest rate in the initial period, denoted by ϵ_0^n, occurs, so that the natural rate of interest takes a large negative value in period 0. The natural rate of interest is assumed to converge to its steady state on and after period 1, but only gradually. That is,

$$r_t^n = \rho^t \epsilon_0^n + r_\infty^n \quad \text{for} \quad t = 0, \cdots \tag{2.10}$$

where r_∞^n is the steady-state value of the natural rate of interest, which is assumed to be non-negative, and ρ is a parameter satisfying $0 \leq \rho < 1$.[8]

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[7] Although a number of studies treat the non-negativity in an explicit way, or in an approximate way, they do not satisfactorily present enough to investigate how long a central bank should continue a ZIRP. Fuhrer and Madigan (1997) and Reifschneider and Williams (2000) do not solve an optimization problem, but just assume Taylor-type policy rules for setting the nominal interest rate with alternative inflation targets. Orphanides and Wieland (2000) solve a dynamic optimization problem with a non-negativity constraint in a backward-looking model instead of a forward-looking model.

[8] The assumption about the natural rate of interest like (2.10) is sufficient to fulfill the purpose of the present paper, which is to specify the optimal path of nominal interest rate in case of the occurrence of a one-time large negative demand shock.

III. *Optimization under Discretion*

1. First-order conditions and steady-state values

The central bank minimizes equation (2.1) subject to (2.4), (2.5), (2.8) and (2.9). For convenience, we assume that the short-term real interest rate of the rest of the world is equivalent to the steady-state value of the natural rate of interest (i.e. $r^n_\infty = i^*_t - E_t\pi^*_{t+1}$). The Lagrangean to the optimization problem is,

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ L_t + 2\phi_{1t}\Big[x_t - x_{t+1} + \Big(\frac{\omega}{\sigma}\Big)(i_t - \pi_{H,t+1}) - \sigma^{-1} r^n_t - \Big(\frac{\omega-1}{\sigma}\Big) r^n_\infty\Big]$$
$$+ 2\phi_{2t}[\pi_{H,t} - \kappa x_t - \beta\pi_{H,t+1}]\Big\}, \tag{3.1}$$

where ϕ_{1t} and ϕ_{2t} represent the Lagrange multipliers associated with the IS constraint and the AS constraint, respectively. The first-order conditions with respect to $\pi_{H,t}$, x_t and i_t are

$$\pi_{H,t} - \bar{\pi}_H + \phi_{2t} = 0 \tag{3.2}$$

$$\lambda(x_t - \bar{x}) + \phi_{1t} - \kappa\phi_{2t} = 0 \tag{3.3}$$

$$i_t\phi_{1t} = 0; \quad i_t \geq 0; \quad \phi_{1t} \geq 0 \tag{3.4}$$

Equation (3.4) is the Kuhn-Tucker condition regarding the non-negativity constraint on the nominal interest rate. If the non-negativity constraint is not binding, $\partial L/\partial i_t$ is equal to zero, so that ϕ_{1t} is zero also. On the other hand, if the constraint is binding, $\partial L/\partial i_t$ is non-negative, and so is ϕ_{1t}. The first-order conditions consist of equations (3.2)-(3.4), together with IS and AS equations.

The steady-state value of the various variables under discretionary policy, x_∞, $\pi_{H,\infty}$, i_∞, $\phi_{1\infty}$, and $\phi_{2\infty}$ are calculated by substituting constant values for each variable into the above first-order conditions. For convenience, we assume that $\bar{x} = (1-\beta)\kappa^{-1}\bar{\pi}_H$.[9] The interior solution of steady state values is given by[10]

$$x_\infty = (1-\beta)\kappa^{-1}\bar{\pi}_H; \quad \pi_{H,\infty} = \bar{\pi}_H; \quad i_\infty = r^n_\infty + \bar{\pi}_H; \quad \phi_{1\infty} = 0; \quad \phi_{2\infty} = 0. \tag{3.5}$$

2. Dynamic path

Given that the non-negativity constraint on nominal interest rates is not binding in the interior steady-state solution, and the assumption that the natural rate of interest converges monotonically to its steady-state value, it is straightforward to assume that the non-negativity

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[9] Without this assumption, the steady-state values take different values, which are more complicated by the dependence on $\bar{x}$. However, this assumption does not change the results of our analysis below.

[10] We can also calculate the corner solution of steady-state by substituting $i_\infty = 0$, and thereby x_∞ and π_∞ are given by $x_\infty = -(1-\beta)\kappa^{-1} r^n_\infty$; $\pi_{H,\infty}$. It is assumed in this paper that r^n_∞ and $\bar{\pi}_H$ are sufficiently large, so that the dynamic path of the endogenous variables converging to the interior solution is superior to that converging to the corner solution. Moreover, it is important to note that the interior solution is the first-best outcome, in the sense that the value of the central bank's loss function, defined by (2.2), is equal to zero. Thus, the focus of our interest is on the dynamic path that converges to the interior solution.

constraint is binding until some period, denoted by period T^d, but not binding afterwards.

In determining the responses of the path of the endogenous variables to shocks, it is convenient to work in terms of deviations from steady-state values. Thus we define $\hat{\pi}_{H,t}=\pi_{H,t}-\pi_{H,\infty}$ and $\hat{x}_t=x_t-x_\infty$ and rewrite IS and AS equations, together with first-order conditions. And then, we substitute ϕ_{1t} into (3.2) and (3.3) so as to characterize the path of the endogenous variables for the periods on and after T^d+1 and eliminate ϕ_{2t}. This procedure yields two first-order difference equations of the form,

$$\lambda\hat{x}_t+\kappa\hat{\pi}_{H,t}=0 \quad \text{for} \quad t=T^d+1,\cdots, \tag{3.6}$$

$$\hat{\pi}_{H,t+1}=\beta^{-1}(1+\lambda^{-1}\kappa^2)\,\hat{\pi}_{H,t} \quad \text{for} \quad t=T^d+1,\cdots, \tag{3.7}$$

combined with IS and AS equations. It is easy to see that equation (3.7) has a unique bounded solution, which is given by $\hat{\pi}_{H,t}=0$, from the fact that the coefficient of $\hat{\pi}_{H,t}$ on the right-hand side is greater than unity. Applying this to equation (3.6) and the IS equation, we obtain the following unique bounded solution for $t=T^d+1,\cdots$

$$z_t = 0 \qquad \text{for} \quad t=T^d+1,\cdots, \tag{3.8}$$

$$i_t = \frac{1}{\omega}r_t^n+\frac{\omega-1}{\omega}r_\infty^n+\bar{\pi}_H \qquad \text{for} \quad t=T^d+1,\cdots, \tag{3.9}$$

where $z_t\equiv[\hat{\pi}_{H,t}\ \ \hat{x}_H]'$. Equation (3.9) seems to be complicated and strange at first glance. However, noting that r_∞^n is the real interest rate in the rest of the world, we easily recognize that the right-hand side of (3.9) is the weighted average of the natural rate of interest and the world real interest rate. Substituting $\omega=1$ into (3.9), we obtain a familiar equation, which often appears in a closed economy.[11]

Next, we substitute $i_t=0$ into IS and AS equations for the periods during which a ZIRP is adopted. This substitution yields

$$z_{t+1}=Qz_t-\Xi\,[r_t^n \quad r_\infty^n \quad \bar{\pi}_H]' \qquad \text{for} \quad t=0,\ \cdots,\ T^d \tag{3.10}$$

where

$$Q\equiv\begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ -\sigma^{-1}\beta^{-1}\omega & 1+\sigma^{-1}\beta^{-1}\kappa\omega \end{bmatrix}, \qquad \Xi\equiv\begin{bmatrix} 0 & 0 & 0 \\ \sigma^{-1} & (\omega-1)\sigma^{-1} & \omega\sigma^{-1} \end{bmatrix}.$$

Combined with $z_{T^d+1}=0$ from (3.8), this difference equation has a unique bounded solution of the form

$$z_t=\sum_{k=t}^{T^d} Q^{-(k-t+1)}\,\Xi\,[r_t^n \quad r_\infty^n \quad \bar{\pi}_H]'. \tag{3.11}$$

The remaining work is to confirm that ϕ_{1t} is positive for $t=0,\ \cdots,\ T^d$. It is easy to recognize that $r_{T^d}^n\leq-(\omega-1)\,r_\infty^n-\omega\bar{\pi}_H$. Otherwise, both $\hat{\pi}_{H,T^d}$ and $\hat{\pi}_{T^d}$ will be positive from the monotonic structure of shocks and (3.11), so that ϕ_{1t} will be negative, contracting the Kuhn-Tucker condition. Therefore, the positiveness of ϕ_{1t} is well satisfied for the periods of ZIRP. It is also straightforward to find that $r_{T^d+1}^n\geq-(\omega-1)\,r_\infty^n-\omega\bar{\pi}_H$ by recalling (3.9). It is important to

[11] It is also noted that if the central bank sets a nominal interest rate equal to the right-hand side of (3.9) at all times, it can completely stabilize domestic inflation and the output gap, thereby achieving the minimized loss in terms of the central bank's preference.

perceive that the choice of T^d is different from that of a closed economy, in which the criterion for choosing T^d is zero. But in this open economy, the choice of T^d is taken in some negative value of the demand shock, which is proportionate to the size of openness and domestic inflation targeting

3. Implementation

Given the solution characterized in the previous subsection, the next issue we address is how to implement it. To deal with the problem of indeterminacy which has been pointed out by many economists, the central bank needs to adopt a feedback policy rule in which a policy instrument depends on endogenous variables. As an example of such feedback rules, consider

$$i_t = \max\{i_t^d + \theta_\pi(\hat{\pi}_{H,t} - \hat{\pi}_{H,t}^d) + \theta_x(\hat{x}_t - \hat{x}_t^d),\ 0\}, \tag{3.12}$$

where i_t^d, $\hat{\pi}_{H,t}^d$, and $\hat{\pi}_t^d$ are the solution, which is characterized in the previous subsection, and θ_π and θ_x are positive parameters representing the responsiveness of the short-term nominal interest rate to the deviations of $\hat{\pi}_{H,t}$ and $\hat{\pi}_t$ from the solution. When the central bank implements the feedback rule under the appropriate value of θ_π and θ_x, it observes $\{i_t^d, \hat{\pi}_{H,t}^d, \hat{x}_t^d, \hat{\pi}_{H,t}, \hat{x}_t\}$, simultaneously with the choice of i_t. It is important to note that equation (3.12) describes the "off-equilibrium path" of the short-term nominal interest rate, in that it specifies how the central bank behaves when the economy deviates from the solution.

Suppose there exists a path of i_t, $\hat{\pi}_{H,t}$, and $\hat{x}_t$ which differs from i_t^d, $\hat{\pi}_{H,t}^d$, and $\hat{x}_t^d$ but converges to the same interior steady-state value. Since i_t converges to $r_\infty^n + \bar{\pi}_H$, which is positive, $T \in [0, \infty]$ must exist such that $i_t \geq 0$ for $t \geq T+1$. Then, the system of equations consisting of (3.12), the IS equation, and the AS equation can be rewritten as

$$z_{t+1} = Rz_t + \Phi[i_t^d \quad \hat{\pi}_{H,t}^d \quad \hat{x}_t^d]' - \Xi[r_t^n \quad r_\infty^n \quad \bar{\pi}_H]', \qquad \text{for} \quad t \geq T+1 \tag{3.13}$$

where

$$R \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma^{-1}\omega(\theta_\pi - \beta^{-1}) & 1 + \sigma^{-1}\omega(\theta_x + \beta^{-1}\kappa) \end{bmatrix}, \quad \Phi \equiv \begin{bmatrix} 0 & 0 & 0 \\ \sigma^{-1}\omega & -\sigma^{-1}\omega\theta_\pi & -\sigma^{-1}\omega\theta_x \end{bmatrix}.$$

Since $\hat{\pi}_{H,t}$ and $\hat{x}_t$ are both non-predetermined variables, this difference equation has a unique bounded solution if the matrix R has two eigenvalues outside the unit circle, according to proposition 1 in Blanchard and Kahn (1980). The condition for determinacy in the present case is given by

$$\kappa(\theta_\pi - 1) + (1-\beta)\theta_x > 0. \tag{3.14}$$

Therefore, a unique bounded solution is obtained if $\theta_\pi > 1$ and $\theta_x > 0$. The satisfaction of these conditions rules out the possibility that there exists a path of i_t, $\hat{\pi}_{H,t}$, and $\hat{x}_t$ which differs from i_t^d, $\hat{\pi}_{H,t}^d$, and $\hat{x}_t^d$ but converges to the same interior steady-state value.[12]

[12] It is also noteworthy that the larger the degree of openness, the broader the region of parameters in which the condition of determinacy holds. Condition (3.14) for determinacy is a simple reflection of NKPC in a small open economy, which means that each unit permanent increase in the domestic inflation rate implies a permanent increase in output gap by $(1-\beta)/\kappa$ units, and that the compensation in terms of output gap for the increase in the domestic inflation rate decreases in accordance with the degree of openness.

IV. *Optimization under Commitment*

1. First-order conditions and steady-state values

An optimal plan must satisfy the first-order conditions

$$\hat{\pi}_{H,t} - (\beta\sigma)^{-1}\omega\phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0 \tag{4.1}$$

$$\lambda\hat{\pi}_t + \phi_{1t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0 \tag{4.2}$$

$$i_t\phi_{1t} = 0; \quad i_t \geq 0; \quad \phi_{1t} \geq 0 \tag{4.3}$$

obtained by differentiating the Lagrangean, given by equation (3.1), with respect to $\pi_{H,t}$, x_t and i_t respectively. Note that the lagged Lagrangean multipliers, ϕ_{1t-1} and ϕ_{2t-1}, appear in the first two equations out of the first-order conditions, which differ sharply from those obtained in the discretionary monetary policy.

In order to solve the optimization problem, we need to specify the values of the two multipliers in the initial period, i.e., add the stipulation that

$$\phi_{1-1} = 0; \quad \phi_{2-1} = 0. \tag{4.4}$$

These initial conditions stem from the assumption that the economy is in the interior steady-state before period 0. Therefore, the Lagrange multipliers in these periods should be equal to zero.[13]

Next, we need to specify the steady-state values of the endogenous variables. The same procedure as mentioned in the previous section allows us to obtain the interior solution given by (3.5). In this case, however, the corner solution does not satisfy the requirement of the steady-state values because ϕ_∞ is negative, which is inconsistent with the Kuhn-Tucker conditions.[14] Thus, the interior solution is a unique steady-state in the case of commitment.

2. Optimal dynamic path

We adopt the same method as in the previous section. It is assumed that the non-negativity constraint on the nominal interest rate is binding until some period, denoted by T^c, but not thereafter. To characterize the path of the endogenous variables for the periods on and after T^c+2, we substitute $\phi_{1T^c+1} = \phi_{1T^c+2} = \cdots = 0$ into the first-order conditions. Then we obtain a unique bounded solution that converges to the interior steady-state, which is given by[15]

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[13] It is also noticeable that time-inconsistency of the Barro-Gordon type that could arise in dynamic response to shocks, which is emphasized in Woodford (1999) and Clarida et al. (1999), does not occur here. The reason is that the assumption that $\bar{x} = (1-\beta)\kappa^{-1}\bar{\pi}_H$, which is made in the previous section, holds. This can be shown as follows. The optimal commitment plan is time-consistent only if $\phi_{1t}=0$ and $\phi_{2t}=0$ for all t. Substituting these values into the above first-order conditions, together with the IS and AS equations, we see that $\hat{\pi}_{H,t} = \hat{x} = 0$ for all t only if i_t is set as equation (3.9). Namely, the solution under commitment always coincides with the one under discretion.

[14] More concretely, $\phi_{1\infty} = -(\beta\sigma)\omega^{-1}(r^n_\infty + \bar{\pi}_H) < 0$.

[15] Details on the derivation are provided in the Appendix A

$$z_t = b\phi_{2t-1}$$
$$\phi_{2t} = \mu_1 \phi_{2t-1}$$
$$i_t = \frac{1}{\omega} r_t^n + \frac{\omega - 1}{\omega} r_\infty^n + \bar{\pi}_H + \delta\phi_{2t-1} \tag{4.7}$$

where μ_1 is a real eigenvalue of an associated matrix, satisfying $|\mu_1| < 1$,b is a column vector defined by $b \equiv [1-\mu_1 \quad \kappa\mu_1\lambda^{-1}]'$, and γ is a parameter defined by $\delta \equiv \mu_1(1-\mu_1)(1-\kappa\sigma\lambda^{-1}\omega\lambda^{-1})$. Note that we need the value of ϕ_{2T^c+1} as an initial condition in order to complete the solution for $t = T^c + 2, \cdots$.

Next, we characterize the path of the endogenous variables for $t = 0, \cdots, T^c$. Substituting $i_t = 0$ into the IS and AS equations yields

$$z_t = \sum_{k=t}^{T^c} Q^{-(k-t+1)} \Xi [r_t^n \quad r_\infty^n \quad \bar{\pi}_H]' + Q^{-(T^c - t+1)} z_{T^c+1} \tag{4.8}$$
$$\phi_t = C\phi_{t-1} - D\left[\sum_{k=t}^{T^c} Q^{-(k-t+1)} \Xi [r_t^n \quad r_\infty^n \quad \bar{\pi}_H]' + Q^{-(T^c-t+1)} z_{T^c+1}\right] \tag{4.9}$$

where $\phi_t \equiv [\phi_{1t} \; \phi_{2t}]'$, and C and D are 2×2 matrices given in Appendix A. Note that we need the value of z_{T^c+1} as a terminal condition, and the value of ϕ_{-1} as an initial condition.

Finally, for $t = T^c + 1$, we substitute $\phi_{1T^c+1} = 0$ into the first-order conditions to obtain

$$\begin{bmatrix} z_{T^c+1} \\ \phi_{2T^c+1} \end{bmatrix} = F^{-1} G z_{T^c+2} + F^{-1} H \phi_{T^c} \tag{4.10}$$

where F, G, and H are matrices given in the Appendix A. Note that we need the values of z_{T^c+2} and ϕ_{T^c} to complete the solution for $t = T^c + 1$.

Equations (4.5)-(4.10), the Kuhn-Tucker conditions (i.e., $i_t = 0$ for $t = 0, \cdots, T^c$, and $\phi_{1t} = 0$ for $t = T^c + 1, \cdots$), and the initial condition (4.5) are completely characterized by a unique optimal path of the endogenous variables. It should be noted that the jumping variables, π_t and x_t, depend on the current and past values of the Lagrange multipliers. This sharply contrasts with the case of discretion in which the path of the jumping variables is determined solely by the path of the natural rate of interest, which is exogenously given. An important implication of this difference is that the timing to terminate a ZIRP is endogenously determined in the commitment solution, while it is exogenously determined in the case of discretion.

The same argument as in Section 3.3 guarantees that the commitment solution characterized above can be implemented when the central bank follows a feedback policy rule of the form

$$i_t = max\{i_t^c + \theta_\pi(\hat{\pi}_{H,t} - \hat{\pi}_{H,t}^c) + \theta_x(\hat{x}_t - \hat{x}_t^c), 0\}, \tag{4.11}$$

where i_t^c, $\hat{\pi}_{H,t}^c$, and $\hat{x}_t^c$ represent the commitment solution, and θ_π and θ_x are parameters satisfying $\theta_\pi > 1$ and $\theta_x > 0$.

3. Timing to terminate the ZIRP and degree of openness

To investigate the relation between timing to terminate a ZIRP and degree of openness, we need to make a careful observation of equation (4.12). We eliminate ϕ_{2t} from equations

(4.1) and (4.2) to obtain a second-order difference equation with respect to ϕ_{1t}.

$$\phi_{1t}-[1+\beta^{-1}+\kappa\omega(\beta\sigma)^{-1}]\phi_{1t-1}+\beta^{-1}\phi_{1t-2}=-\kappa\hat{\pi}_{H,t}-\lambda\hat{x}_t+\lambda\hat{x}_{t-1} \quad \text{for} \quad t=0,\cdots,T^c+1, \tag{4.12}$$

where initial conditions are given by $\phi_{1-1}=\phi_{1-2}=0$. A unique solution to this difference equation is given by

$$\phi_{1t}=-\kappa A(L)\hat{\pi}_{H,t}-\lambda(1-L)A(L)\hat{\pi}_t, \tag{4.13}$$

where

$$A(L)\equiv\frac{1}{\eta_1-\eta_2}\left(\frac{\eta_1}{1-\eta_1 L}-\frac{\eta_2}{1-\eta_2 L}\right),$$

and L is a lag operator, and η_1 and η_2 are two real solutions to the characteristic equation, satisfying $\eta_1>1$ and $0<\eta_2<1$. Since η_1 is greater than unity, ϕ_{1t} tends to explode once it becomes positive, unless the right-hand side of equation (4.12) takes sufficiently large negative values. In other words, inflation and the output gap must overshoot the steady-state values (i.e., zero) after they take negative values in period 0 and subsequent periods. This condition is satisfied if the central bank continues the ZIRP for a sufficiently long period. By adopting such a policy, the output gap and inflation become sufficiently high, so that the expression on the right-hand side of (4.12) takes sufficiently large negative values to guarantee that ϕ_{1t} converges to zero within a finite period.

Noting that inflation and the output gap can be rewritten as a function of a sequence of exogenous shocks, and that a ZIRP continues in these periods, we solve this difference equation to obtain,[16]

$$A(L)^{-2}\phi_{1t}=(\beta\sigma)^{-1}B(L)\{-\omega i_{t-1}+r^n_{t-1}+\omega\bar{\pi}_H+(\omega-1)r^n_\infty\}, \tag{4.14}$$

where

$$A(L)=\left[1-\left(1+\frac{1}{\beta}+\frac{\kappa\omega}{\beta\sigma}\right)L+\frac{1}{\beta}L^2\right]^{-1},$$

$$B(L)=\left[\lambda\beta-(\kappa^2+\lambda+\lambda\beta)L+L^2\right].$$

Here, we find the following result,[17]

$$\frac{\partial\phi_{1t}}{\partial\gamma}<0, \tag{4.15}$$

which means that inflation and the output gap in the small openness case overshoot the steady-state values more than in the large openness case. Therefore, a ZIRP in the case of small openness continues longer than that in the case of large openness, i.e.,

$$0\leq T^c_l\leq T^c_s<\infty, \tag{4.16}$$

[16] The definition of $A(L)$ is exactly equivalent to that of (4.13)

[17] See Appendix B for more details on the derivation of (4.14) and (4.15)

TABLE 1. PARAMETER VALUES

$\lambda=0.048/4^2$
$\beta=0.990$
$\alpha=0.826$
$\sigma=0.157$
$\varphi=0.470$
$\eta=9.000$
$\bar{\pi}_H=0.000$
$\gamma^n_\infty=0.011$

where T^c_l and T^c_s indicate the timing to terminate a ZIRP in the case of large and small openness, respectively.

V. *Numerical Example*

In this section, we present some quantitative results, in particular, the optimal path of the short-term nominal interest rates, using the parameter values shown in Table 1. These parameters are borrowed from Woodford (1999), except the values of r^n_∞ and η. The steady-state value of the natural rate of interest (r^n_∞) is calculated under the assumption that the growth rate of potential output is three percent per year.[18] The value of η is set equal to 9 so as to satisfy the sign condition of $\omega-1$. As another candidate in the parameters, we set σ equal to 1, which corresponds to a log utility specification, and $\eta=3$ reasonably. The value of α, which is consistent with Buchi and Watanabe (2001) and Gali and Gertler (1999), is set equal to 0.826 in order to make the value of κ equal to that in Jung, et al. (2001). The values of parameters are adjusted so that the length of a period in our model is interpreted as a quarter.

Figure 1 shows the responses of endogenous variables to an adverse demand shock to the natural rate of interest under discretion. The solid line and dashed line represent the impulse responses in the case of $\gamma=0.7$ (large openness) and $\gamma=0.1$ (small openness), respectively. In the baseline case, shown in this figure, we assume that the initial shock to the natural rate of interest, ϵ^n_0 in equation (2.10), is equal to -0.15, which means a 60 percent decline in the annualized natural rate of interest. In addition, we assume that the persistence of the shock, which is represented by ρ in equation (2.10), is 0.5 per quarter. The path of the natural rate of interest is shown at the bottom of Figure 1.[19] As concretely seen by Figure 1, the short-term nominal interest rate in the case of large openness is set to zero for the first three periods until period 2, while, in the case of small openness, it is set to zero for the longer periods by a quarter. It is noteworthy that, before the natural rate of interest turns positive, the nominal interest rate is positive, which is in sharp contrast to that in a closed economy. By recalling equation (3.9), understanding is straightforward.

Figure 2 plots the corresponding impulse response functions of endogenous variables

[18] The definition of r^n_∞ is made as $r^n_t\equiv\sigma E_t[(y^p_{t+1}-y^p_t)-(g_{t+1}-g_t)]+(1-\beta)/\beta$, where y^p_t is the natural rate of output or potential output and g_t is a disturbance that fluctuates independently of changes in the real interest rate. See Jung et al. (2001) for more details.

[19] The path of it is denoted by a sequence of circles in Figure 1.

FIG. 1. OPTIMAL RESPONSES UNDER DISCRETION

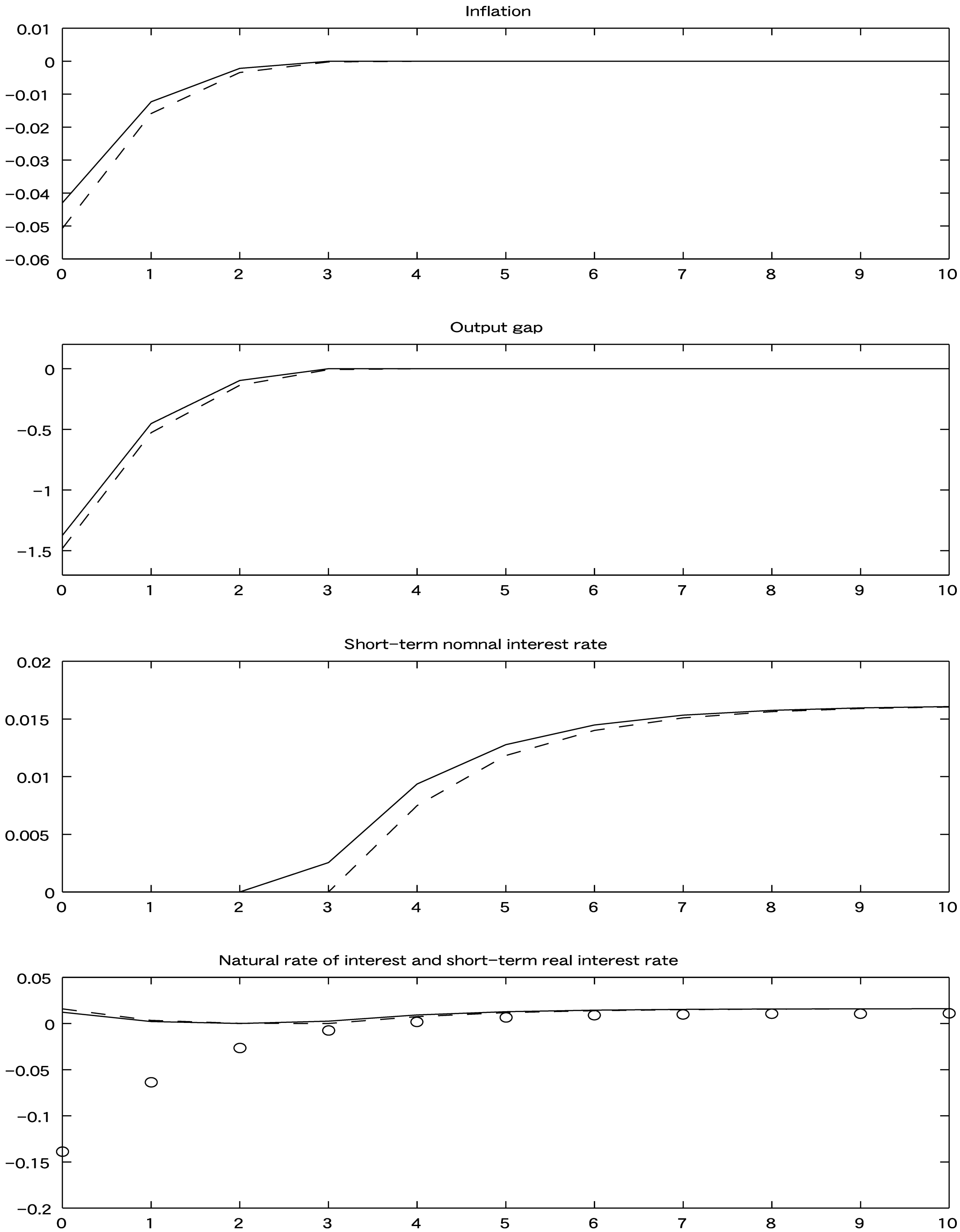

FIG. 2. OPTIMAL RESPONSES UNDER COMMITMENT

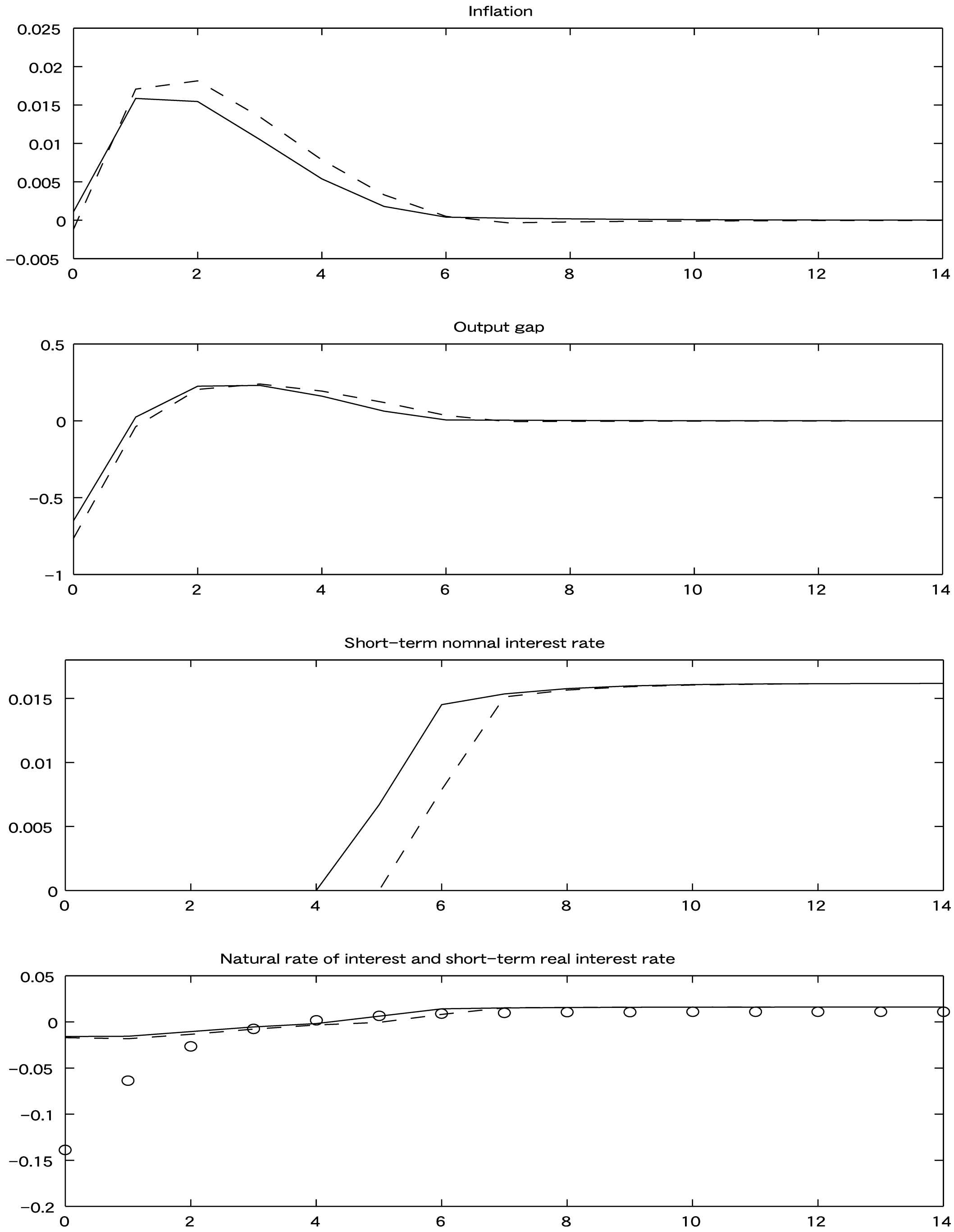

TABLE 2. VARIOUS SIZES OF INTITIAL SHOCK

T^d					
	$\epsilon_0^n=-0.05$	-0.10	-0.15	-0.20	-0.30
$\gamma=0.7$	1	2	2	3	3
0.5	1	2	2	3	3
0.3	1	2	2	3	3
0.1	1	2	3	3	4
0.0	1	2	3	3	4
T^c					
	$\epsilon_0^n=-0.05$	-0.10	-0.15	-0.20	-0.30
$\gamma=0.7$	1	3	4	5	6
0.5	2	3	4	5	6
0.3	2	4	5	5	6
0.1	2	4	5	6	7
0.0	2	4	5	6	7

Note: $\rho=0.5$, $\sigma=0.157$, $\eta=9$.

TABLE 3. VARIOUS PERSISTENCE OF SHOCK

T^d					
	$\rho=0.0$	0.1	0.3	0.5	0.7
$\gamma=0.7$	0	0	1	2	5
0.5	0	0	1	2	5
0.3	0	0	1	2	5
0.1	0	0	1	3	6
0.0	0	0	1	3	6
T^c					
	$\rho=0.0$	0.1	0.3	0.5	0.7
$\gamma=0.7$	1	2	3	4	7
0.5	2	2	3	4	7
0.3	2	2	3	5	8
0.1	2	2	3	5	8
0.0	2	2	4	5	9

Note: $\epsilon_0^n=-0.15$, $\sigma=0.157$, $\eta=9$.

under commitment.[20] The main features in the case of commitment are as follows. First, an important difference from the case of discretion is that a ZIRP is continued longer. This result comes from a historical dependent property of commitment, which means that a ZIRP is continued until the cumulative sum of deviation of the short-term real interest rate from the natural rate of interest is zero. This prolonged ZIRP lowers the long-term interest rate and heightens expected inflation, thereby stimulating aggregate demand.

[20] A key part in computing the commitment solution is how to find the timing to terminate a zero interest rate policy, T^c. We search T^c as follows: (1) We set T^c at a sufficiently high value, say 50, under which ϕ_{1T^c} is supposed to be negative, and compute the path of the variables; (2) If ϕ_{1T^c} is negative, we try $T^c=49$; and (3) We repeat this until ϕ_{1T^c} becomes non-negative.

TABLE 4. VARIOUS SIZES OF INITIAL SHOCK

T^d					
	$\epsilon_0^n=-0.05$	-0.10	-0.15	-0.20	-0.30
$\gamma=0.7$	0	0	1	1	2
0.5	0	0	1	1	2
0.3	0	1	1	2	2
0.1	0	1	2	2	3
0.0	0	2	2	3	3
T^c					
	$\epsilon_0^n=-0.05$	-0.10	-0.15	-0.20	-0.30
$\gamma=0.7$	0	1	2	3	4
0.5	0	1	2	3	4
0.3	0	2	3	4	5
0.1	1	3	4	5	7
0.0	2	4	5	6	8

Note: $\rho=0.5$, $\sigma=1$, $\eta=3$.

TABLE 5. VARIOUS PERSISTENCE OF SHOCK

T^d					
	$\rho=0.0$	0.1	0.3	0.5	0.7
$\gamma=0.7$	0	0	0	1	2
0.5	0	0	0	1	2
0.3	0	0	0	1	3
0.1	0	0	1	2	4
0.0	0	0	1	2	5
T^c					
	$\rho=0.0$	0.1	0.3	0.5	0.7
$\gamma=0.7$	0	0	1	2	4
0.5	0	1	1	3	4
0.3	1	1	2	3	5
0.1	1	2	2	4	8
0.0	2	2	3	5	10

Note: $\epsilon_0^n=-0.15$, $\sigma=1$, $\eta=3$.

Secondly, inflation and the output gap in the case of large openness are more improved for the first few periods and subsequently less overheated than those in the case of small openness. By making a closer observation of the NKPC and the IS equation, this result is easily understood. Note that the larger the degree of openness, the weaker the demand shock to the natural rate of interest. Then, at the initial period, inflation and the output gap to demand shock in the case of large openness decrease less than those in the other case. Since a ZIRP persists for a while and the trade-off between inflation and the output gap becomes favorable according to the degree of openness, the dynamic path of inflation and the output gap goes into reverse at some time or other.

Thirdly, a ZIRP in the case of small openness is continued longer than that in the case of large openness as under discretion. The short-term nominal interest rate in the case of large openness is set to zero for the first five periods until period 4, while, in the other case, it is set to zero for the longer periods by a quarter. It is important to note that the cumulative sum of the deviation of the short-term nominal interest rate from the weighted average of the natural rate of interest and the world real interest rate becomes proportionately smaller to the degree of openness through the foreign effect. This also justifies the fact that, in the case of large openness, the short-term nominal interest rate is positive in period 5 when the natural rate of interest turns positive, which is inconsistent with Jung et al. (2001).

Table 2 presents T^d and T^c for various combination of the degree of openness(γ) and initial shock (ϵ_0^n) in the case of $\rho=0.5$, $\sigma=0.157$ and $\eta=9$. Given the value of γ, both T^d and T^c become larger with the absolute value of ϵ_0^n. And both T^d and T^c become smaller with γ, given the value of ϵ_0^n. The response of T^d and T^c to γ is not as sensitive. In particular, when ϵ_0^n is -0.05, -0.10 or -0.20, T^d is constant, independent of the degree of openness. The reason for the lesser sensitivity to γ is that ω is small under the parameter configuration of σ and η.

Likewise, Table 3 presents T^d and T^c for various combination of the degree of openness (γ) and persistence of shock(ρ) in the case of $\epsilon_0^n=-0.15$, $\sigma=0.157$ and $\eta=9$. Given the value of γ, both T^d and T^c become larger with the value of ρ. And both T^d and T^c become smaller with γ, given the value of ρ. If a shock to the natural rate of interest is non-persistent, the responses of T^d and T^c to γ are insensitive. On the other hand, in the case of large persistence, the difference in the responses is clear to some extent. Table 4 and Table 5 present T^d and T ^{c} of $\rho=1$ and $\eta=3$, *ceteris paribus*. It is easy to see the same pattern of T^d and T^c as Table 2 and Table 3. However, in this case, since the value of ω is large, the responses of T^d and T^c to the degree of openness become somewhat distinct, except T^d in the case of $\rho=0.0$ and 0.1, shown in Table 5. In particular, if a shock to the natural rate of interest has great persistence or has such a large negative initial value, the change of T^c to γ becomes distinct.

Moreover, we find that the difference between T^d and T^c becomes proportionately smaller to the degree of openness from Table 2-5. While the difference between T^d and T^c is about 0-3 quarters in the case of $\gamma=0.7$, it ranges between 1 and 4 in the case of $\gamma=0.1$. Table 4, for example, shows that as γ becomes larger from 0 until 7 in the case of ϵ_0^n, the difference between T^d and T^c decreases gradually from 5 until 2.

VI. *Comparison with Svensson's Foolproof Way*

In this section we discuss the proposal of pegging the exchange rate by Svensson (2000) within the framework of the present model. Svensson (2000) recommends that a central bank in a liquidity trap should announce an upward-sloping price-level target path with a small positive long-run inflation target and, simultaneously, announce that the home currency will be devalued and that the exchange rate will be pegged to a crawling exchange-rate target until the price-target path has been reached. That is, the central bank makes a commitment to buy and sell unlimited amounts of foreign currency at the exchange rate target.

Assuming that uncovered interest parity holds exactly, we can rewrite the proposal of Svensson (2000) as the following equation,

$$\frac{1}{1-\gamma} q_t = \frac{1}{1-\gamma} E_t q_{t+1} + E_t \pi_{H,t+1} - \bar{\pi}_H. \tag{6.1}$$

Under the proposal of Svensson (2000), i.e., equation (6.1), we can construct a Lagrangean for solving an optimization problem and obtain the first-order conditions, which are the same as those in the case of commitment. By making the same simulation with earlier analysis, we find that the timing to terminate the ZIRP (T^c) is always zero, independent of the size of the demand shock. As required by the UIP condition, the nominal interest rate jumps to a positive level immediately upon the start of the crawling peg, which is the so-called foolproof way. This result can be easily proved without solving the optimization problem. Combining equation (6.1) with the UIP condition, we obtain the equation of the form, $i_t = \bar{\pi}_H + i_t^* - \pi^*$ (in the present paper, $i_t = \bar{\pi}_H + r_\infty^n$). This implies that, from the initial period, the zero bound on the nominal interest rate is not binding under the assumption of $\bar{\pi}_H \geq 0$ and $r_\infty^n \geq 0$.[21] Therefore, the ZIRP is quickly ended and the nominal interest rate remains well in the positive by taking the crawling peg, which needs its credibility to be absolutely essential.[22]

VII. *Conclusion*

To address the question of how long the ZIRP would be continued in the framework of a small open economy, we solved a central bank's intertemporal optimization problem. Given an adverse shock to aggregate demand, we have computed the dynamic path of the short-term nominal interest rate in both discretion and commitment. We found that the timing to terminate a ZIRP in the case of large openness would be earlier than that in the case of small openness. We also found that the economy in the case of large openness would be less overheated.

Notwithstanding the use of the exchange rate channel of monetary policy, the solution is that the ZIRP should be continued until the cumulative sum of deviation of the short-term real interest rate from the natural rate of interest is zero. This shows that the BOJ's announcement still lacks the element of history dependence because it focuses only on a forward looking stance. However, the difference between the optimal solution and the BOJ's policy in termination timing becomes proportionately smaller to the degree of openness. Nevertheless, noting that Japan's exports and imports were about 11 and 10 percent of the GDP in 2001 respectively, the BOJ's termination timing might be premature, even though it had taken the exchange rate channel into account.

This paper also suggests that the exchange rate peg enables the central bank to quickly end the ZIRP and a positive nominal interest rate to be an equilibrium, under the condition that the credibility of the peg is absolutely established. This implies that, if the termination of the ZIRP is accompanied by the exchange rate peg, its effect is intensified. However, the BOJ was skeptical about the exchange rate peg in spite of its perception that the transmission mechanism of the interest rate to the exchange rate would be limited.

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[21] Combining $i_t = \bar{\pi}_H + i_t^* - \pi^*$ with the IS equation and establishing a new Lagrangean, we easily find that Kuhn-Tucker conditions are excluded from the first-order conditions.

[22] Svensson (2003) says that the exchange peg can induce private-sector expectations of a higher future price level because the present exchange rate is directly related to the expected future exchange rate, which is also related to private-sector expectations of the future price level. Another important thing to note is that the peg may have a negative effect on the trading partners, which must be discussed in more detail.

A Optimal path under commitment

This section characterizes the optimal path under commitment. We start by characterizing the path of the endogenous variables for the periods on and after T^c+2. Substituting $\phi_{1T^c+1}=\phi_{1T^c+2}=\cdots=0$ into (4.1) and (4.2) yields a system of a difference equation of the form

$$\hat{\pi}_{H,t}+\phi_{2t}-\phi_{2t-1}=0 \tag{A.1}$$

$$\lambda\hat{x}_t-\kappa\phi_{2t}=0. \tag{A.2}$$

Eliminating $\hat{x}_t$ using the AS equation, we have a difference equation with respect to $\hat{\pi}_{H,t}$ and ϕ_{2t} of the form

$$\begin{bmatrix}\hat{\pi}_{H,t+1}\\ \phi_{2t}\end{bmatrix}=\begin{bmatrix}\beta^{-1}(1+\kappa^2/\lambda) & -\beta^{-1}\kappa^2/\lambda\\ -1 & 1\end{bmatrix}\begin{bmatrix}\hat{\pi}_{H,t}\\ \phi_{2t-1}\end{bmatrix}\quad \text{for}\quad t=T^c+2,\cdots, \tag{A.3}$$

where ϕ_{2T^c+1} is given as an initial condition. This difference equation system has one predetermined variable, ϕ_2, and one non-predetermined variable, $\hat{\pi}_H$, and the two-by-two matrix on the right-hand side of the equation has two real eigenvalues, which are denoted by $|\mu_1|<1$ and $|\mu_2|>1$. Since the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, a unique bounded solution exists that converges to the interior steady-state, whose dynamic path is given by

$$\hat{\pi}_{H,t}=(1-\mu_1)\phi_{2t-1}, \tag{A.4}$$

$$\phi_{2t}=\mu_1\phi_{2t-1}. \tag{A.5}$$

We obtain the optimal path of $\hat{\pi}_t$ in the corresponding periods by substituting this solution into (4.5), and the optimal path of i_t in the corresponding periods by substituting it into the IS equation. Equations (4.5)-(4.7) follow from this.

Next, we characterize the path of the endogenous variables for $t=0,\cdots,T^c+1$. As for $t=0,\cdots,T^c$, we substitute $i_t=0$ into (2.4) to obtain

$$\hat{x}_t-\hat{x}_{t+1}-\frac{\omega}{\sigma}(\hat{\pi}_{H,t+1}+\bar{\pi}_H)-\frac{1}{\sigma}r_t^n-\frac{\omega-1}{\sigma}r_\infty^n=0. \tag{A.6}$$

This equation, and equations (4.1), (4.2), and (2.5) characterize the optimal path for $t=0,\cdots,T^c$, which is given by

$$z_t=\sum_{k=t}^{T^c}Q^{-(k-t+1)}\,\Xi\,[r_t^n\quad r_\infty^n\quad \bar{\pi}_H]'+Q^{-(T^c-t+1)}z_{T^c+1}, \tag{A.7}$$

$$\phi_t=C\phi_{t-1}-D\left[\sum_{k=t}^{T^c}Q^{-(k-t+1)}\,\Xi\,[r_t^n\quad r_\infty^n\quad \bar{\pi}_H]'+Q^{-(T^c-t+1)}z_{T^c+1}\right], \tag{A.8}$$

where

$$C\equiv\begin{bmatrix}\beta^{-1}+\kappa(\beta\sigma)^{-1}\omega & \kappa\\ (\beta\sigma)^{-1}\omega & 1\end{bmatrix},\quad D\equiv\begin{bmatrix}\kappa & \lambda\\ 1 & 0\end{bmatrix}.$$

As for $t=T^c+1$, we substitute $\phi_{1T^c+1}=0$ into (4.1) and (4.2) to obtain

$$\hat{\pi}_{H,T^c+1}-(\beta\sigma)^{-1}\omega\phi_{1T^c}+\phi_{2T^c+1}-\phi_{2T^c}=0, \tag{A.9}$$

$$\lambda\hat{x}_{T^c+1}-\beta^{-1}\phi_{1T^c}-\kappa\phi_{2T^c+1}=0. \tag{A.10}$$

Rearranging (A.9), (A.10), and (2.5) yields

$$\begin{bmatrix} z_{T^c+1} \\ \phi_{2T^c+1} \end{bmatrix}=F^{-1}Gz_{T^c+2}+F^{-1}H\phi_{T^c}, \tag{A.11}$$

where

$$F\equiv\begin{bmatrix} 1 & -\kappa & 0 \\ 1 & 0 & 1 \\ 0 & \lambda & -\kappa \end{bmatrix},\quad G\equiv\begin{bmatrix} \beta & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad H\equiv\begin{bmatrix} 0 & 0 \\ (\beta\sigma)^{-1}\omega & 1 \\ \beta^{-1} & 0 \end{bmatrix}.$$

B Derivation of (4.14) and (4.15)

The AS equation can be expressed as

$$\hat{\pi}_{H,t}=(1+\beta L^{-1}+\beta^2 L^{-2}+\cdots)\kappa\hat{x}_t \tag{B.1}$$

where L is the lag operator, $\hat{\pi}_{H,t}=\pi_{H,t}-\pi_{H,\infty}$ and $\hat{x}_t=x_t-x_\infty$. Similarly, we can rewrite the IS equation as

$$\begin{aligned}\hat{x}_t-\hat{x}_{t-1}&=\left(\frac{\omega}{\sigma}\right)\hat{\pi}_{H,t+1}+\Lambda_t\\ &=\left(\frac{\omega}{\sigma}\right)(1+\beta L^{-1}+\beta^2 L^{-2}+\cdots)\kappa\hat{x}_{t+1}+\Lambda_t,\end{aligned} \tag{B.2}$$

where

$$\Lambda_t=\frac{1}{\sigma}\{-\omega i_t+r_t^n+\omega\pi_H+(\omega-1)r_\infty^n\}.$$

Rearranging (A.2) yields

$$C(L)\hat{x}_t=\Lambda_t, \tag{B.3}$$

where

$$C(L)=\left[1-\left(1+\frac{\omega\kappa}{\sigma}\right)L^{-1}-\left(\frac{\omega\kappa\beta}{\sigma}\right)L^{-2}-\left(\frac{\omega\kappa\beta}{\sigma}\right)^2 L^{-3}-\cdots\right].$$

We substitute (A.3) into (A.1) to obtain

$$C(L)(1-\beta L^{-1})\hat{\pi}_{H,t}=\kappa\Lambda_t. \tag{B.4}$$

Multiplying both sides of (4.12) by $C(L)(1-\beta L^{-1})L^2\hat{\pi}_{H,t}$ yields

$$\begin{aligned}C(L)(1-\beta L^{-1})L^2A(L)^{-1}\phi_t&=C(L)(1-\beta L^{-1})L^2\{-\kappa\hat{\pi}_{H,t}-\lambda(1-L)\hat{x}_t\}\\ &=\{-\kappa^2 L-\lambda(1-\beta L^{-1})L(1-L)\}\Lambda_{t-1}\\ &=B(L)\Lambda_{t-1}.\end{aligned} \tag{B.5}$$

Substituting $C(L)(1-\beta L^{-1})L^2=\beta A(L)^{-1}$ into (A.5) yields equation (4.14).

We differentiate (4.14) with respect to ω to obtain

$$-\frac{2\kappa}{\beta\sigma}A(L)^{-1}\phi_{1t-1}+A(L)^{-2}\frac{\partial\phi_{1t}}{\partial\omega}=(\beta\sigma)^{-1}B(L)(-i_{t-1}+\bar{\pi}_H+r^n_\infty). \tag{B.6}$$

Rearranging (A.6) yields

$$\frac{\partial\phi_{1t}}{\partial\omega}=\frac{2\kappa}{\beta\sigma}A(L)\phi_{1t-1}-\frac{1}{\beta\sigma}A(L)^2B(L)i_{t-1}+\frac{1}{\beta\sigma}A(L)^2B(L)(\bar{\pi}_H+r^n_\infty). \tag{B.7}$$

The first term of RHS in (A.7) is

$$\begin{aligned}\frac{2\kappa}{\beta\sigma}A(L)\phi_{1t-1}&=\frac{2\kappa}{\beta\sigma}\frac{1}{1-\eta_1 L}(1+\eta_2 L+\eta_2 L^2+\cdots)\phi_{1t-1}\\&=-\frac{2\kappa}{\beta\sigma}\eta_1^{-1}\left(\sum_{i=0}^{\infty}\eta_1^{-j}L^{-j}\right)\left(\sum_{i=0}^{\infty}\eta_2^{j}L^{j}\right)\phi_{1t}<0.\end{aligned} \tag{B.8}$$

Similarly, the third term of RHS in (A.7) is

$$\frac{1}{\beta\sigma}A(L)^2B(L)(\bar{\pi}_H+r^n_\infty)=-\frac{\beta\sigma}{\omega^2}(\bar{\pi}_H+r^n_\infty)<0. \tag{B.9}$$

Noting both (A.8) and (A.9) have postive values, together with the fact that the second term of RHS in (A.7) is zero by the ZIRP, we obtain the following result,

$$\frac{\partial\phi_{1t}}{\partial\omega}<0. \tag{B.10}$$

Noting that $\partial\omega/\partial\gamma>0$, we obtain

$$\frac{\partial\phi_{1t}}{\partial\gamma}=\frac{\partial\phi_{1t}}{\partial\omega}\frac{\partial\omega}{\partial\gamma}<0, \tag{B.11}$$

which proves (4.15).

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