

## THE LONG RUN EQUILIBRIUM IN A GAME OF “BATTLE OF THE SEXES” \*

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### *Abstract*

Kandori, Mailath, and Rob (1993) considered a single, homogeneous population of agents playing a 2-player 2-action, coordination game and found that the risk-dominant equilibrium is the stochastically stable outcome. However, this robustness does not extend to the case where there are distinct populations of player 1's and player 2's. Here, with some restrictions on the adjustment dynamics other than the 'Darwinian property' we show that the risk-dominant equilibrium would be the stochastically stable outcome in the “battle of the sexes” game.

*Key words:* Evolutionary games; Heterogeneous population; Risk dominance

*JEL Classification:* C72, D83

There is some literature on the stochastically stable equilibria in games with best response learning when the players experiment randomly. Kandori, Mailath and Rob (1993) and Young (1993) are two well-known studies on this topic.

In particular, Kandori, Mailath, and Rob (1993) showed that the risk-dominant equilibrium is the long run equilibrium in a 2-player, 2-action coordination game with a single homogeneous population, provided that a very weak assumption, named the 'Darwinian property', is satisfied. The coordination game they considered is an important one with many implications in the real world. However, it is by no means the only game to which their concept of the long run equilibrium can be applied. The unfortunate fact is that beyond the 2-player, 2-action coordination game with a single homogeneous population, the 'Darwinian property' is not strong enough to find out which state will be the long run equilibrium in general.

In this paper we will look at coordination games similar to those of Kandori, Mailath, and Rob (1993), except for the fact that there are two populations of players instead of one homogeneous population. We will assume that the players are playing games only with players from the other population. We can observe this kind of situation in the real world, such as the case of sellers and buyers in the retail market or the case of males and females when they interact with each other.

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These games with two populations of players were mentioned briefly in the last section of Kandori, Mailath, and Rob (1993), and they gave one example to show the difficulties in analyzing the two-population case compared with the one homogeneous population case. Here, we will look more systematically at the differences between single- and two-population cases.

In this paper we will mainly concentrate on ‘battle of the sexes’ games, as shown below. In other words, games such that while coordination is always desired by both populations of players, each prefers a different coordination equilibrium because their payoffs are asymmetric.

		player 2		$(a > 1, b > 1)$
		A	B	
player 1	A	1, $a$	0, 0	
	B	0, 0	$b$ , 1	

The main result of this paper shows us what conditions are needed for the adjustment dynamics of the ‘risk-dominant equilibrium’ to be the long run equilibrium in ‘battle of the sexes’ games. Harsanyi and Selten (1988) defined ‘risk-dominant equilibrium’, and in the game shown above the equilibrium (A, A) is the risk-dominant equilibrium if  $a > b$ , and the equilibrium (B, B) is the risk-dominant equilibrium if  $a < b$ .<sup>1</sup> Intuitively, (A, A) is the risk-dominant equilibrium if player 2 is more strongly attracted to the equilibrium (A, A) than player 1 is to (B, B).

In the last section, we will examine the case of local interaction, which was introduced in Ellison (1993). We can see that in local interaction games with two populations, not only is the convergence to the long run equilibrium much faster, but that only the ‘Darwinian property’ on the deterministic dynamics is enough to guarantee that the risk-dominant equilibrium is the long run equilibrium.

## I. Modeling

We will use basically the same model as Kandori, Mailath, and Rob (1993), except for that there are two populations, row players (player 1) and column players (player 2). Each population has  $N$  players. The game will be played in discrete times,  $t = 1, 2, 3, \dots$ . In each period every player will choose the strategy she will use and be paired with another randomly chosen player from the other population, and the two players will get the payoffs decided by the strategies they already chose.

We will look at  $2 \times 2$  games and the basic payoff table of the games is as we described in the introduction.

The scenario here is that while both populations of players want to coordinate, player 2 prefers equilibrium  $E_A$  in which both players play strategy A, and player 1 prefers equilibrium  $E_B$  in which both players play strategy B. These kinds of games are known by the name ‘battle of the sexes’.

Let  $z_1$  denote the number of player 1’s who are playing strategy A, and let  $z_2$  denote the

<sup>1</sup> Actually, what Harsanyi and Selten defined was that equilibrium (A, A) ‘risk dominates’ equilibrium (B, B) if  $a > b$ . Here, we will just call the equilibrium (A, A) as the ‘risk dominant equilibrium’ in this case.

number of player 2's who are playing A. Then, the expected payoff of a player  $i$  from playing strategy  $s$  when  $z_j$  of player  $j$ 's are playing A, which we will denote as  $\pi_s^i(z_j)$ , is as follows,

$$\begin{aligned}\pi_A^1(z_2) &= \frac{z_2}{N}, \text{ and } \pi_B^1(z_2) = \frac{(N-z_2)}{N} b. \\ \pi_A^2(z_1) &= \frac{z_1}{N} a, \text{ and } \pi_B^2(z_1) = \frac{(N-z_1)}{N}.\end{aligned}$$

As in Kandori, Mailath, and Rob (1993), we will assume that the players are myopic and choose the best response against the distribution of the other player's strategy in the previous period, i.e. in period  $t$  by comparing  $\pi_A^i(z_j^{t-1})$  and  $\pi_B^i(z_j^{t-1})$ ,  $i \neq j$ .

If we denote the deterministic dynamics by

$$(z_1^{t+1}, z_2^{t+1}) = b(z_1^t, z_2^t) = (b_1(z_1^t, z_2^t), b_2(z_1^t, z_2^t)),$$

then, the ‘Darwinian property’, (D), means,

$$(D) \text{ sign}(b_i(z_1^t, z_2^t) - z_i^t) = \text{sign}(\pi_A^i(z_j^t) - \pi_B^i(z_j^t)), i \neq j, \text{ for } i = 1, 2.$$

We will assume that the ‘Darwinian’ property is satisfied by the deterministic dynamics throughout the rest of this paper.

Another thing to note here is that the Darwinian property (D) allows inertia in adjustment. This means that all the players may not immediately change to the best response, even though the population is moving in that direction by the assumption of the Darwinian property (D). Like the Kandori, Mailath, and Rob (1993) model, this inertia helps to justify the myopic actions of players.

As in Kandori, Mailath, and Rob (1993), we assume that players do not always choose the best response, but they experiment or mutate with a small probability  $2\epsilon$ , and randomly choose A or B with the probability,  $\epsilon > 0$  respectively. This probability to experiment,  $2\epsilon$ , is same for all the players and independent across the players and the populations.

Let's reintroduce Kandori, Mailath, and Rob (1993)'s formal definition of the long run equilibria.

First, define the transition probability as

$p_{ijkl} = \text{Prob}(z^{t+1} = (k, l) | z^t = (i, j))$ , where  $z^t = (z_1^t, z_2^t)$ ,  $0 \leq i, j, k, l \leq N$ , and  $i, j, k, l$  are integers.

$P = [p_{ijkl}]$  is a Markov matrix and due to the possibility of mutations all elements in the matrix  $P$  are strictly positive.

The stationary distribution  $\mu \in \Delta$  is  $\mu P = \mu$ , where  $\Delta \equiv \{s \in R^{2N+2} | s_{ij} \geq 0 \text{ for } i, j = 0, 1, \dots, N \text{ and } \sum_{ij} s_{ij} = 1\}$ .

This  $\mu$  exists uniquely satisfying stability and ergodicity.<sup>2</sup>

**Definition:** The long run equilibria are the set of states  $\{z = (i, j) | \mu_{ij}^* > 0, \mu^* \in \Delta\}$  where  $\mu^*$  is the limit distribution defined by  $\mu^* = \lim_{\epsilon \rightarrow 0} \mu(\epsilon)$ .

Kandori, Mailath, and Rob (1993) proved the existence of the long run equilibrium in the single homogeneous population case. Even though there are two populations in our model, it is basically the same because there are still only a finite number of possible states as the

<sup>2</sup> Look at Fudenberg and Levine (1998) Chapter 5 for more about this.

numbers of players and strategies are finite. Therefore, we omit the proof on the existence of the long run equilibrium.

**Theorem** (Kandori, Mailath, and Rob (1993)): The limit distribution  $\mu^*$  exists and is unique.

Once we know the existence of the long run equilibrium, the next question would be how to find it. For this purpose we will use the radius and coradius concepts of Ellison (2000) in the next section.

## II. Long Run Equilibria in the Battle of Sexes Game

Before we apply the radius and coradius concepts and find the long run equilibrium of the game, we need to introduce some notations. First, we need to define  $z_i^*$ , ( $i = 1, 2$ ) as the smallest integer that is not smaller than  $\bar{z}_i^*$  when  $\pi_A^j(\bar{z}_i^*) = \pi_B^j(\bar{z}_i^*)$ , where  $i \neq j$ . In the game we are considering now, they are  $\bar{z}_1^* = \left(\frac{N}{1+a}\right)$ ,  $\bar{z}_2^* = \left(\frac{bN}{1+b}\right)$ .

Obviously, there can be three restpoints of the underlying deterministic paths,  $(z_1, z_2) = (0, 0)$ ,  $(z_1, z_2) = (N, N)$ , and  $(z_1, z_2) = (\bar{z}_1^*, \bar{z}_2^*)$ . These are the only possible states that can be the long run equilibrium.

Here, we will eliminate the possibility that  $(\bar{z}_1^*, \bar{z}_2^*)$  can be a long run equilibrium by assuming that it cannot be achieved by fractions of players playing each pure strategy. Then, we have two candidates for the long run equilibrium,  $(z_1, z_2) = (0, 0)$  and  $(z_1, z_2) = (N, N)$ .

Now let's call the set of states from which the deterministic paths go to  $E_B$  (where  $(z_1, z_2) = (0, 0)$ ) as 'the basin of attraction of  $E_B$ ' and denote it by  $B_B$ . A similar definition applies to the basin of attraction of  $E_A$  (where  $(z_1, z_2) = (N, N)$ ) and notation  $B_A$ .

As we mentioned at the end of the previous section, Ellison (2000) proved that if the radius of  $E_s$  is greater than the coradius of  $E_s$ , then the long run equilibrium of the game belongs to  $E_s$ .

We will not repeat the formal definition of radius and coradius here,<sup>3</sup> but roughly speaking, the "radius of  $B_s$ " is the least number of mutations necessary to leave  $B_s$  when play begins in  $E_s$ , and the "coradius of  $B_s$ " is the least number of mutations necessary to reach  $B_s$  when play begins in a state from which the largest number of mutations are needed to reach  $B_s$ .

Therefore, in practice we can find the long run equilibrium by comparing the numbers of mutations to leave and enter each basin of attraction.

In our case of "battle of the sexes", this is as simple as figuring out how many mutations will be needed for the state to change from  $E_A$  to  $B_B$  and from  $E_B$  to  $B_A$ .

Clearly, the reason why it is so simple is that all states belong to either  $B_A$  or  $B_B$ . Therefore, on the radius side, once the play leaves  $B_A$ , it is automatically in  $B_B$ . On the coradius side, once the play begins in  $B_B$ , one option that is always available is to reach  $E_B$  following the deterministic path, then enter  $B_A$  from  $E_B$ . As a result, the coradius of  $B_A$  cannot be bigger than the number of mutations needed from  $E_B$  to  $B_A$ , while clearly it cannot be smaller.

<sup>3</sup> For the formal definitions of radius and coradius, see page 23 and 24 of Ellison (2000).

Now, let's look at this game using figure 1 (a). The states can be denoted in a 2-dimensional diagram and the directions of the deterministic paths will be like the arrows in figure 1 (a). When  $z_2 < z_2^*$ , B is the best response for player 1 and  $z_1$  decreases. On the other hand, when  $z_2 \geq z_2^*$ ,  $z_1$  increases. In the same way, we can see that the direction of the movement of  $z_2$  depends on the comparison of  $z_1$  and  $z_1^*$ .

FIG. 1 (a)

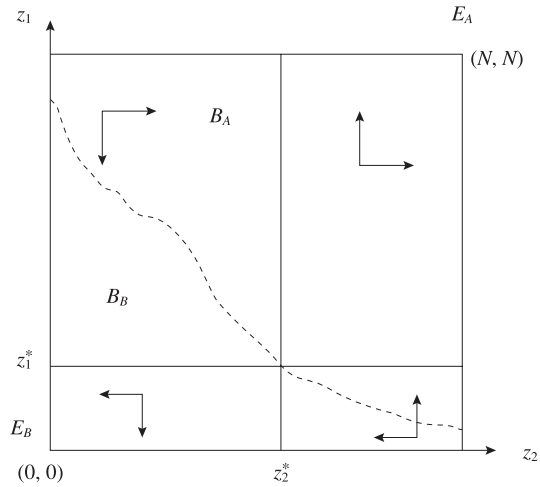
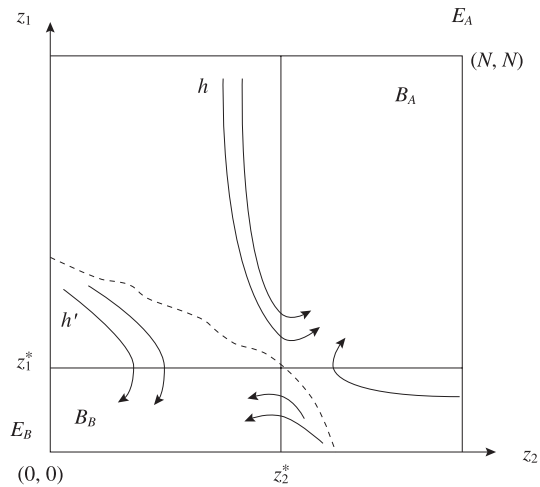


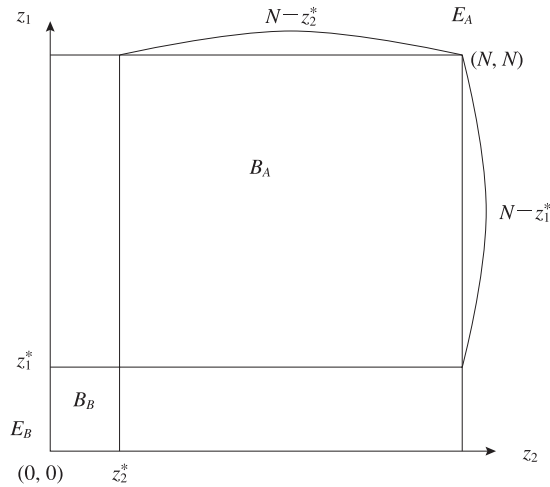
FIG. 1 (b)



Before we look at the battle of the sexes game, let's see the case where the Darwinian property is strong enough to predict the long run equilibrium.

**Proposition 1:** In a game with  $(a-1)(b-1) < 0$ , for any dynamics with the Darwinian property  $E_A$  is the long run equilibrium if  $z_1^* + z_2^* \leq \min\{N - z_1^*, N - z_2^*\}$ , and  $E_B$  is the long run equilibrium if  $(N - z_1^* + 1) + (N - z_2^* + 1) < \min\{z_1^*, z_2^*\}$ .

FIG. 2



The proof is trivial once we realize that  $z_1^* + z_2^*$  is the most number of mutations needed for the state to change from  $E_B$  to one in  $B_A$ , while  $\min\{N - z_1^*, N - z_2^*\}$  is the least number of mutations needed for the state to change from  $E_A$  to one in  $B_B$ . We can see this in figure 2.

Now let's go back to battle of the sexes games, where  $(a-1)(b-1) > 0$ . In general, it is impossible to figure out the long run equilibrium in battle of the sexes games only with the assumption of Darwinian property (D) and the reasons are as follows.

In figure 1 (a), (b) we can see why different dynamics would lead to different long run equilibria in games with two populations of players, unlike games with a single population as in Kandori, Mailath, and Rob (1993) where any dynamics with the Darwinian property (D) always lead to the risk-dominant equilibrium in the long run.

In the single population case, all states can be denoted on a line and the sizes of the basins of attraction can be simply measured and compared by their lengths. Therefore, we can easily see that the restpoint with the bigger basin of attraction would be the long run equilibrium. In contrast, here the size of the basin of attraction depends on the adjustment speed of each population, and in this two dimensional diagram we have no clear standard in comparing the sizes of the basins of attraction. For example, in figure 1 (a) we cannot see an obvious way to say whether  $B_B$  is bigger than  $B_A$  or the other way around.

Furthermore, if one looks at figure 1 (b), one will notice that a simple comparison of the sizes of  $B_A$  and  $B_B$  is not enough to find the long run equilibria. Even though  $B_A$  seems bigger than  $B_B$ , the deterministic path can make it very easy to get out of  $B_A$  (i.e. without many mutations), while it may need more mutations to get out of the seemingly smaller basin of attraction,  $B_B$ . This is because of the fact that some deterministic paths in  $B_A$  approach  $B_B$  as

they proceed. Sometimes consecutive small numbers of mutations over several periods may cost less to exit a basin of attraction than one big number of mutations in just one period. This possibility makes it more difficult to analyze the game, because just a simple comparison of the sizes of the basins of attraction will not be enough in figuring out the long run equilibria.

First, we will see conditions that make one cluster of mutations always need less numbers of mutations than consecutive clusters of small numbers of mutations. We refer to the dynamics that satisfy the conditions as dynamics (DA).

Before we define Dynamics (DA), we want to define  $x_1 \equiv z_1 - z_1^*$  and  $x_2 \equiv z_2 - z_2^*$ , and  $x_i^t$  as  $x_i$  at time  $t$  so that  $x_i^{t+1} = b_i(z_1^t, z_2^t) - z_i^*$ . With a little abuse of notation, the deterministic path can be alternatively denoted by  $x_i^{t+1} = b_i(x_1^t, x_2^t)$ .

**Dynamics (DA):**

$$\begin{cases} x_i^{t+1} = x_i^t + \min\{f_i(x_i^t, x_{-i}^t), N - z_i^* - x_i^t\}, & \text{if } x_{-i}^t \geq 0 \\ \quad = x_i^t + \max\{f_i(x_i^t, x_{-i}^t), -z_i^* - x_i^t\}, & \text{if } x_{-i}^t < 0 \end{cases} \quad (i=1, 2)$$

where  $f_i$  is a function

$f_i: N^* \times N^* \rightarrow N^*$  ( $N^* = \{-N, -N+1, \dots, -1, 0, 1, \dots, N-1, N\}$ ) such that

- (i)  $f_i(x_i, x_{-i})x_{-i} \geq 0$ ,
- (ii)  $(f_i(x_i, x_{-i}) - f_i(x_i, \tilde{x}_{-i}))(x_{-i} - \tilde{x}_{-i}) \geq 0$ ,
- (iii)  $(f_i(x_i, x_{-i}) - f_i(\tilde{x}_i, x_{-i}))(x_i - \tilde{x}_i) \geq 0$ .

First, condition (i) means that the Dynamics (DA) satisfy the Darwinian property.

Second, at (ii) ' $f_i(x_i, x_{-i}) \geq f_i(x_i, \tilde{x}_{-i})$  if  $x_{-i} \geq \tilde{x}_{-i}$ ' means roughly that  $x_i$  moves faster when  $x_{-i}$  is farther away from zero. It can be interpreted that players adjust faster to the best response when the other population is more concentrated on one strategy, that is, when a bigger gain in payoff is expected from strategy changes. This is a kind of 'monotonicity' assumption because the adjustment speeds are higher when the expected payoff gains from the adjustment are bigger.

Third, condition (iii) means that the change to the best response happens at a higher speed when more players have already changed to the best response, provided that the expected payoff gains from this change are the same. The implication of this condition would be that players might be slower to change to a new strategy when not many from their population have changed to the new strategy yet.<sup>4</sup>

Here,  $\min\{f_i(x_i^t, x_{-i}^t), N - z_i^* - x_i^t\}$  and  $\max\{f_i(x_i^t, x_{-i}^t), -z_i^* - x_i^t\}$  are needed to prevent the cases where  $z_i^{t+1} < 0$  or  $z_i^{t+1} > N$ .

The practical reason we need (DA) is that it has the property which guarantees that a sequence of mutations over several periods will not reduce the number of mutations needed to exit any basin of attraction compared with simultaneous mutations in just one period. The next lemma shows this.

**Lemma 1:** In the case that the deterministic dynamics belong to (DA), for any state  $(x_1^0, x_2^0) \in B_B$  if  $(b_1(x_1^0, x_2^0) + k_1, b_2(x_1^0, x_2^0) + k_2) \in B_A$ ,  $k_1, k_2 \geq 0$ , then  $(x_1^0 + k_1, x_2^0 + k_2) \in B_A$ .

*Proof:* Consider two initial states  $(x_1^\tau, x_2^\tau)$  and  $(\tilde{x}_1^\tau, \tilde{x}_2^\tau) = (x_1^\tau + k_1, x_2^\tau + k_2)$ ,  $k_1, k_2 \geq 0$ , at  $t = \tau$ . Then, at  $t = \tau + 1$  we have  $(x_1^{\tau+1}, x_2^{\tau+1}) = (x_1^\tau + f_1(x_1^\tau, x_2^\tau), x_2^\tau + f_2(x_1^\tau, x_2^\tau))$  and  $(\tilde{x}_1^{\tau+1}, \tilde{x}_2^{\tau+1}) =$

<sup>4</sup> We need this condition mainly for the technical purpose of proving lemma 2. However, we can do without this condition, if the deterministic dynamics are functions of only the expected payoffs, that is,  $f_i(x_i, x_{-i}) = f_i(x_{-i})$ .

$(x_1^{\tau} + k_1 + f_1(x_1^{\tau} + k_1, x_2^{\tau} + k_2), x_2^{\tau} + k_2 + f_2(x_1^{\tau} + k_1, x_2^{\tau} + k_2)) \geq (x_1^{\tau} + k_1 + f_1(x_1^{\tau} + k_1, x_2^{\tau}), x_2^{\tau} + k_2 + f_2(x_1^{\tau}, x_2^{\tau} + k_2)) \geq (x_1^{\tau} + f_1(x_1^{\tau}, x_2^{\tau}), x_2^{\tau} + f_2(x_1^{\tau}, x_2^{\tau}))$ . The last two inequalities came from conditions (ii) and (iii) of the (DA). This means that  $x_1^{\tau+1} \leq \tilde{x}_1^{\tau+1}$  and  $x_2^{\tau+1} \leq \tilde{x}_2^{\tau+1}$ . Therefore, we know that for all  $t \geq \tau + 1$ ,  $x_i^t \leq \tilde{x}_i^t$ ,  $i = 1, 2$ .

If  $(x_1^{\tau}, x_2^{\tau}) \in B_A$ , then the deterministic path will ultimately lead  $(x_1^t, x_2^t)$  to  $(N, N)$ . Because  $x_i^t \leq \tilde{x}_i^t$ , for all  $t > \tau$ ,  $i = 1, 2$ , the deterministic path will also lead  $(\tilde{x}_1^t, \tilde{x}_2^t)$  to  $(N, N)$ , and as a result,  $(\tilde{x}_1^{\tau}, \tilde{x}_2^{\tau}) \in B_A$ .

Now, we can prove the lemma using this result.

Compare  $b(x_1^0 + k_1, x_2^0 + k_2) = (x_1^0 + k_1 + f_1(x_1^0 + k_1, x_2^0 + k_2), x_2^0 + k_2 + f_2(x_1^0 + k_1, x_2^0 + k_2))$  with  $(x_1^1 + k_1, x_2^1 + k_2)$ . We can see that  $x_1^0 + k_1 + f_1(x_1^0 + k_1, x_2^0 + k_2) \geq x_1^0 + k_1 + f_1(x_1^0, x_2^0) = x_1^1 + k_1$  and  $x_2^0 + k_2 + f_2(x_1^0 + k_1, x_2^0 + k_2) \geq x_2^0 + k_2 + f_2(x_1^0, x_2^0) = x_2^1 + k_2$ . From what we showed above, if  $(x_1^1 + k_1, x_2^1 + k_2) \in B_A$ , then  $(x_1^0 + k_1, x_2^0 + k_2) \in B_A$  because  $b(x_1^0 + k_1, x_2^0 + k_2) \in B_A$ . Q.E.D.

We can say the same thing for the case where the state changes from  $B_A$  to one in  $B_B$ .

What we can see from lemma 1 is that if  $k_1 + k_2$  mutations are needed for the state to change from  $B_B$  to  $B_A$  after the state changes following the deterministic path from  $(x_1^0, x_2^0)$  to  $(x_1^1, x_2^1)$ , the state can be changed with no more than  $k_1 + k_2$  mutations if it changes from  $(x_1^0, x_2^0)$  into  $B_A$  without following the deterministic path any further. Therefore, the least number of mutations will be needed for the state to change from  $E_B$  into  $B_A$ , if this change happens in one period rather than over many periods.

Now, we are able to decide the long run equilibria by comparing the adjustment speeds of two populations. What we are saying in the next proposition is that if the payoffs of the game we are looking at satisfy  $b < a$  (i.e.  $E_A$  is the risk-dominant equilibrium according to Harsanyi and Selten's definition), then unless there exists a big difference in adjustment speeds between the two populations,  $E_A$  will be the long run equilibrium.

Let  $\bar{f}_i \equiv \max_{\{x_i \geq 0, x_{-i} < 0\}} \{f_i(x_i, x_{-i})\}$ , that is, the maximum distance that the state may change in one period when  $x_i \geq 0, x_{-i} < 0$ . Also, let's define  $\lambda_1 \equiv \frac{N - z_1^* - \bar{f}_1}{z_1^* + z_2^* - (N - z_2^*)}$ ,  $\lambda_2 \equiv \frac{z_1^* + z_2^* - (N - z_1^*)}{N - z_2^* - \bar{f}_2}$ , and see the following proposition.

**Proposition 2:** Consider dynamics (DA) satisfying  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right| < \lambda_1$ , for all  $(x_1, x_2)$  such

that  $x_1 \geq 0, x_2 < 0, x_1 \geq -\lambda_1 x_2 + \bar{f}_1$ , and  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right| > \lambda_2$ , for all for all  $(x_1, x_2)$  such that  $x_1 < 0, x_2 \geq 0, x_1 \geq -\lambda_2(x_2 - \bar{f}_2)$ . Then, there exists a unique long run equilibrium in games where  $b < a$ , and the unique long run equilibrium is the risk-dominant equilibrium,  $E_A$ .

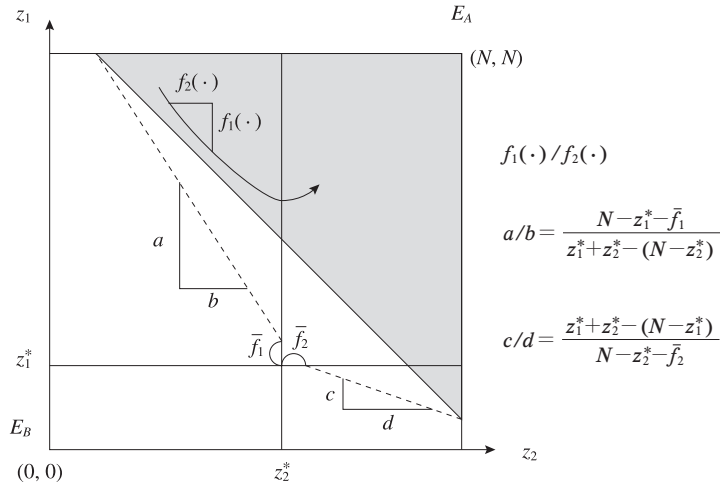
*Proof:* From lemma 1 it is obvious that comparing the minimum mutations needed for the state to change from  $E_B$  to a state in  $B_A$  and those from  $E_A$  to a state in  $B_B$  would be enough to find the long run equilibrium. The state will certainly change from  $E_B$  to one in  $B_A$ , if there occur  $z_1^*$  mutations in population 1 and  $z_2^*$  mutations in population 2. The shaded area in figure 3 denotes the set of states that can be reached from  $E_A$  with no more than  $z_1^* + z_2^*$  mutations. The slope of the dotted line in the area  $x_1 \geq 0, x_2 < 0$  is  $\lambda_1$ . Let's limit our consideration to the right side of the dotted line, in other words,  $(x_1, x_2)$ 's such that  $x_1 \geq 0, x_2 < 0, x_1 \geq -\lambda_1 x_2 + \bar{f}_1$ . Here,



the slope of the deterministic path from these points would be  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right|$  which is the ratio of two adjustment speeds of the players. If it is  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right| < \lambda_1$ , as you can see from figure 3, the deterministic path from any state in the shaded area would remain on the right side of the dotted-line and will ultimately end up in the area where  $x_1, x_2 > 0$  and go back to  $E_A$ . The same logic can be applied to the deterministic paths from points where  $x_1 < 0, x_2 \geq 0, x_1 \geq -\lambda_2(x_2 - \bar{f}_2)$ , and as a result, all the states in the shaded area where  $x_1 < 0, x_2 \geq 0$ , also belong to  $B_A$ . Therefore, it takes more than  $z_1^* + z_2^*$  mutations for the state to change from  $E_A$  to a state in  $B_B$ , while at most  $z_1^* + z_2^*$  mutations would be enough for the state to change from  $E_B$  to a state in  $B_A$ . As a result, the risk-dominant equilibrium  $E_A$  is the long run equilibrium. Q.E.D.

The reason that  $\bar{f}_i (i = 1, 2)$  was introduced is that the deterministic path may lead the state to change from the area  $x_1 \geq 0, x_2 < 0$  to  $x_1 < 0, x_2 \geq 0$ , and vice versa. Then, it would be difficult to predict where the state will finally end up, in  $B_A$  or  $B_B$ .  $\bar{f}_i (i = 1, 2)$  were introduced so that we can exclude this possibility. This would not weaken proposition 2 very much if there is considerable inertia and only very small portions of the populations change their strategies in each period, that is, if  $\bar{f}_i (i = 1, 2)$  is small.

FIG. 3



One thing to notice here is that proposition 2 is a sufficient condition for  $E_A$  to be the long run equilibrium.

To get some sense of this proposition let's look at a numerical example. Let's assume that the population is very large and the inertia is big so that  $\bar{f}_i (i = 1, 2)$  is trivially small.

A simple calculation shows that, when  $b < a$ ,

$$\frac{N - z_1^*}{z_1^* + z_2^* - (N - z_2^*)} = \frac{a + ab}{ab + 2b - a} > 1 \text{ and } \frac{z_1^* + z_2^* - (N - z_1^*)}{N - z_2^*} = \frac{1 + 2b - a}{a + b} < 1.$$

If we substitute  $a, b$  with numerical values such as  $a=4, b=2$ , then the sufficient condition for  $E_A$  to be the long run equilibrium is that  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right| < \frac{3}{2}$ , when  $x_1 \geq 0, x_2 < 0$ ,  $x_1 \geq -\frac{3}{2}x_2$ , and  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right| > \frac{1}{6}$ , when  $x_1 < 0, x_2 \geq 0, x_1 \geq -\frac{1}{6}x_2$ .

Because  $\left| \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \right|$  is the rate of the two populations' adjustment speeds, the result means that for  $E_B$  to be the long run equilibrium there has to be quite a difference in the adjustment speeds between the two populations.

Remark: One thing that should be mentioned here is that Young (1993) also pointed out that the risk-dominant equilibrium will be the long run equilibrium in two-population games. However, Young employed an assumption that the players will remember only randomly chosen  $K$  periods out of the most recent  $N$  periods. ( $K < N$ ) This assumption that the players may remember only the remote past without remembering the most recent past trivializes the analysis in the following sense.

Let's assume that in figure 3 the game was at  $E_A$  at first, then  $N - z_2^* + 1$  number of player 2's mutate and choose B for one period. Of course, given that all player 1's are playing A, player 2's will go back to play A in the following periods. However, if all player 1's only remember the period when many of player 2's played B, without noticing the fact that they went back to play A, all player 1's will change to play B regardless of what player 2's are actually doing at the moment. If more than  $N - z_1^* + 1$  player 1's have changed to play B and player 2 notices this change, then player 2 will begin to play B until everybody will play (B,B). Therefore,  $N - z_2^* + 1$  mutations are enough to make the state change from  $E_A$  to  $E_B$ , while for the same reason  $z_1^*$  mutations are enough for the change from  $E_B$  to  $E_A$ . As a result, the risk-dominant equilibrium  $E_A$  will be the long run equilibrium.

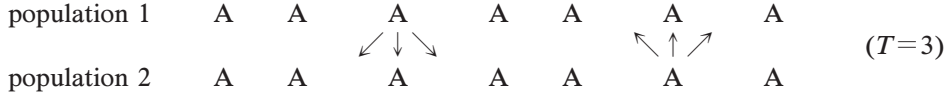
Even though it is a very clear cut result, one can see that this selective memory assumption totally trivializes the analysis of the areas where  $x_1 \geq 0, x_2 < 0$  or  $x_1 < 0, x_2 \geq 0$ .

Therefore, even though Young got a similar result as proposition 2 with different assumptions, analyzing the game without his specific memory system, as we did in this paper, is meaningful, especially in the sense that we can get some insight into how the characteristics of the deterministic path affect the long run equilibrium, not to mention the fact that one may feel it is more realistic.

### III. Local Interaction

Here we will again consider the 'battle of the sexes' games with  $N$  players in each population, but this time we will assume that the players will be matched only locally.

Let's assume that players of each population are arrayed in a circle, as in the model in Ellison (1993), and that the two circles are facing each other and each player is playing only with her  $T$  nearest counterparts with equal probability,  $\frac{1}{T}$ , for each of them.



Let  $w_1$  be the minimum number of A's played among  $T$  nearest players in population 1 in order to make A the best response for player 2. Also define  $w_2$  as the minimum number of B's played among  $T$  nearest players in population 2 in order to make B the best response for player 1. Clearly,  $w_i$ 's are dependent on the payoffs  $a, b$ , and when  $a, b > 1$ , obviously  $w_1, w_2 \leq \frac{1}{2} T$  if  $T$  is even, and  $w_1, w_2 \leq \frac{1}{2} (T+1)$  if  $T$  is odd. In the case when  $a=4, b=2$ , and  $T=4$ , it would be  $w_1=1$  and  $w_2=2$ .

Now, we can prove the following proposition.

**Proposition 3:** In a local interaction game where  $N$  is big enough relative to  $T$ ,  $E_A$  will be the long run equilibrium if  $w_1 < w_2$ , and  $E_B$  will be the long run equilibrium if  $w_1 > w_2$ .<sup>5</sup>

*Proof:* (1) We want to show that if  $w_1 < w_2$ , for any state there is always a way to reach  $(N, N)(=E_A)$  with at most  $w_1 + (T - w_2)$  mutations needed. Consider the state where all the players are playing B. If there occurs a situation where a cluster of neighboring players choose strategy A such as  $w_1$  of player 1's and  $T - w_2$  of player 2's, the deterministic path will go to  $E_A$ . This can be shown because  $w_1$  number of player 1's playing A will make  $T - w_1 + 1$  adjacent player 2's choose A, and because from the assumption  $w_1 < w_2$  it is  $T - w_1 + 1 > T - w_2$ , we can see that the number of player 2's choosing A will increase. In the same way,  $(T - w_2)$  number of player 2's who are playing A will make  $w_2 + 1$  player 1's choose A, and as  $w_1 < w_2 + 1$ , the number of player 1's playing A will increase. Then, the increased number of players who play A will make more players choose A, and this will keep going on until  $E_A$  is reached.

(2) The rest of the proof is the same as Theorem 1 (a) of Ellison (1993). Q.E.D.

One can see that it can be  $w_1 < w_2$ , only when  $b < a$ , and  $w_1 > w_2$ , only when  $b > a$ . Since  $E_A$  is the risk-dominant equilibrium when  $b < a$ , and  $E_B$  is the risk-dominant equilibrium when  $b > a$ , proposition 3 in essence predicts that the risk-dominant equilibrium will be the long run equilibrium.

Clearly, one property of the local interaction model is that the convergence time is much shorter than in the basic model.

However, another good thing about the proposition is that the Darwinian property (D) is the only restriction needed on the deterministic dynamics.

The fact that we need much fewer restrictions on the dynamics here compared with the case in section 2 can be understood as follows. In the local interaction model of Ellison (1993) with a single homogeneous population, the Darwinian property was enough to make the risk-dominant equilibrium the long run equilibrium, and there were two reasons for this. First, inside each small neighborhood the risk-dominant equilibrium will be chosen for the same reason as Kandori, Mailath, and Rob (1993). Second, a cluster of players who choose the risk-dominant strategy keeps making the nearby players choose the same strategy. Even

<sup>5</sup> In the case where  $w_1 = w_2$ , there are many equilibria other than  $E_B, E_A$ , and it is difficult to see what will happen.

though both of them work to make the risk-dominant equilibrium the long run equilibrium, actually the second one was enough to make the risk-dominant equilibrium the long run equilibrium.

In this two population case the first reason does not hold any more, because, as we saw in the previous section, the Darwinian property is not strong enough to select the risk-dominant equilibrium as the long run equilibrium. However, the second reason still holds, because the local interaction model here is basically the same as that in Ellison, even though in this model we have two populations of players arrayed in circles rather than one as in Ellison. Therefore, the Darwinian property should be enough to single out the risk-dominant equilibrium as the long run equilibrium in this case of local interaction.

We can also easily see that for the same reason that the number of players who play A keeps expanding, the number of players who play B keeps shrinking when  $w_1 < w_2$ .

Moreover, we can see that there would be many equilibrium states if  $w_1 = w_2$ . In this case there are many states where the clusters of players who play A and those of players who play B neither expand nor shrink, but coexist while maintaining a kind of stalemate.

However, if the payoffs are asymmetric and  $a \neq b$ , then we can always find  $T$  large enough to make  $w_1 \neq w_2$ , and the risk-dominant equilibrium will be the long run equilibrium by proposition 3.

Following is an example, when  $T=3$ ,  $w_1=2$ , and  $w_2=1$ .

Each of the A's and B's denotes a player who plays that strategy, and a bold character denotes the player who will change strategy in the next period.

t=1							
population 1	A	A	<b>A</b>	B	<b>B</b>	A	A
population 2	A	A	A	B	<b>A</b>	A	A
t=2							
population 1	A	A	B	B	B	<b>A</b>	A
population 2	A	A	<b>A</b>	B	B	A	A

It can be easily seen that once B is used by three neighboring players, it will be spread to the entire population via the domino-effect.

#### IV. Conclusion

In this paper we have shown that the nice result of Kandori, Mailath, and Rob (1993) can be extended to two population cases with some restrictions, including the condition that we called the dynamics DA. The long run equilibrium would be the risk-dominant equilibrium.

In addition, we have shown that in the case of local interaction, the long run equilibrium will be again the risk dominant equilibrium as long as only the Darwinian property is satisfied.

Also, we could see that the speed of adjustment for each population is important in deciding the long run equilibrium in these two population cases, unlike the one population cases.

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