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OPTIMAL CONSUMPTION/INVESTMENT DECISIONS IN MARKOVIAN DYNAMIC SYSTEMS

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Abstract

We investigate optimal consumption/investment decision problems in a continuous financial market where the price fluctuations of assets are assumed to follow Markov diffusions. Sufficient conditions for the verification of Hamilton-Jacobi-Bellman equation will be given.

I. Introduction

Consider a financial market on which $n+1$ assets are traded in continuous time. The $(n+1)$-th asset is called bond which is assumed to pay a constant return over time. The price process of the bond is given by $p_{n+1,t} = p_{n+1,0}e^{rt}$, where r is a known positive constant. The other n assets have unpredictable price fluctuations which are modeled as follows. Let $\boldsymbol{p}_t = (p_{1,t}, \ldots, p_{n,t})', t \in [0,T]$ be an $\mathbb{R}_+^n$ valued stochastic process whose i-th component represents price of the i-th asset. We assume $\{\boldsymbol{p}_t\}_0^T$ is described by a system of stochastic differential equations as

$$\mathbf{diag}\{p_{1,t}^{-1}, \ldots, p_{n,t}^{-1}\}d\boldsymbol{p}_t \equiv d\boldsymbol{Z}_t = \boldsymbol{\mu}(t, \boldsymbol{Z}_t)dt + \boldsymbol{\Sigma}^{1/2}d\boldsymbol{W}_t, \tag{1.1}$$

where $\{\boldsymbol{W}_t\}_0^T$ is an n-dimensional standard Brownian motion on $(\Omega, \sigma(\boldsymbol{p}_s, 0 \leq s \leq T), P)$, $\boldsymbol{\mu} : [0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function and $\boldsymbol{\Sigma}$ is a $n \times n$ fixed positive definite matrix. Additional conditions on $\boldsymbol{\mu}$ will be imposed later.

Given initial wealth $X_0 = x_0$ at time 0, an investor continuously invests his wealth in this Markovian financial market until time $T < \infty$. At the same time, he uses some portion of his wealth for consumption. Assume that his investment is so small compared to the market's volume that the prices of assets are not affected by his consumption/investment strategy. The wealth process $\{X_s\}_0^T$ is expressed as

$$\begin{aligned} X_s &= x_0 + \int_0^s X_\theta \sum_{i=1}^n \frac{\pi_{i,\theta} dp_{i,\theta}}{p_{i,\theta}} + \frac{dp_{n+1,\theta}}{p_{n+1,\theta}} d\theta - \int_0^s c_\theta X_\theta d\theta \\ &= x_0 + \int_0^s X_\theta \boldsymbol{\pi}'_\theta d\boldsymbol{Z}_\theta + \int_0^s X_\theta \left\{(1 - \mathbf{1}'\boldsymbol{\pi}_s)r - c_\theta\right\} d\theta \end{aligned} \tag{1.2}$$

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where $\boldsymbol{\pi}_\theta = (\pi_{1,\theta}, \ldots, \pi_{n,\theta})'$ is the proportion of wealth invested in uncertain assets at time θ and c_θ is the proportion consumed. We assume that $\int_0^T \|\boldsymbol{\pi}_s\|^2 X_s^2 ds < \infty$ and $\int_0^T c_s X_s ds < \infty$. Hereafter, we only consider $\{\boldsymbol{Z}_s\}_0^T$ instead of $\{\boldsymbol{p}_s\}_0^T$ since they contain the same information. We abbreviate consumption/investment strategy $(\boldsymbol{\pi}_s, c_s), s \in [t, T]$ as $(\boldsymbol{\pi}, c)$ for simplicity. Let $\mathfrak{F}_s = \sigma(\boldsymbol{p}_\theta, 0 \leq \theta \leq s) = \sigma(\boldsymbol{Z}_\theta, 0 \leq \theta \leq s)$ and $\mathfrak{A} = \{(\boldsymbol{\pi}, c) : (\boldsymbol{\pi}_s, c_s)$ is $\mathfrak{F}_s$-progressively measurable such that $X_s \geq 0$ a.s.$\}$. The problem is to find an optimal strategy $(\boldsymbol{\pi}^*, c^*) \in \mathfrak{A}$ which maximizes the expected utility from cumulative consumption and terminal wealth:

$$(1.3) \qquad E^P \left[\int_0^T e^{-\delta s} U_1(c_s X_s) ds + e^{-\delta T} U_2(X_T) \right],$$

where $U_i(\alpha), i = 1, 2$ are strictly increasing and strictly concave utility functions and $\delta \in \mathbb{R}$ is a known positive constant discount rate.

Since the price dynamics (1.1) has a Markovian structure, it is convenient to set the initial point other than zero. We consider the following maximization problem instead of (1.3):

$$(1.4) \qquad \sup_{(\boldsymbol{\pi}, c) \in \mathfrak{A}_M} E^P \left[\int_t^T e^{-\delta s} U_1(c_s X_s) ds + e^{-\delta T} U_2(X_T) \middle| \mathfrak{F}_t \right],$$

where $\mathfrak{A}_M \subset \mathfrak{A}$ is the set of Markov controls. In this case, the problem (1.4) is equivalent to

$$(1.5) \qquad u(t, x, \boldsymbol{z}) = \sup_{(\boldsymbol{\pi}, c) \in \mathfrak{A}_M} E^P \left[\int_t^T e^{-\delta s} U_1(c_s X_s^{t,x,\boldsymbol{z}}) ds + e^{-\delta T} U_2(X_T^{t,x,\boldsymbol{z}}) \right],$$

with the dynamics

$$(1.6) \qquad X_s^{t,x,\boldsymbol{z}} = x + \int_t^s X_\theta^{t,x,\boldsymbol{z}} \{(1 - \boldsymbol{1}'\boldsymbol{\pi}_s) r - c_\theta\} d\theta + \int_t^s X_\theta^{t,x,\boldsymbol{z}} \boldsymbol{\pi}_\theta' d\boldsymbol{Z}_\theta^{t,\boldsymbol{z}}$$

$$(1.7) \qquad \boldsymbol{Z}_s^{t,\boldsymbol{z}} = \boldsymbol{z} + \int_t^s \boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta^{t,\boldsymbol{z}}) d\theta + \boldsymbol{\Sigma}^{1/2}(\boldsymbol{W}_s - \boldsymbol{W}_t).$$

The problem (1.3) can be analyzed via the martingale approach developed by Karatzas, Lehoczky and Shreve (1987), and, Cox and Huang (1989), and shown to have an optimal strategy. In general, it is not easy to obtain optimal strategy in an explicit form. However, if the price dynamics has a Markovian representation as in (1.7), we can derive Hamilton-Jacobi-Bellman (HJB) equations for the maximization problem (1.5). The HJB equations are inevitably involved with degeneracy. Thus the standard argument on the validity of HJB equations based on the non-degeneracy assumption does not apply to the problem. We amend this drawback by applying Krylov's (1980) result on stochastic solutions and its extension by Kuwana (1995b) to Cauchy problems associated with the martingale approach.

The optimal consumption/investment decision problems in continuous time originated from the work by Merton (1971). Various ramifications and generalizations have been made and there is a substantial literature on the subject. A list of important related researches can be found in Duffie (1996, Chapter 9).

II. *The Martingale Approach*

In this section, we review the martingale approach applied to the investment/consumption decision problem. The discussion here entirely relies on Karatzas, Lehoczky and Shreve (1987).

It is convenient to work with the investment and consumption level processes

$$\begin{aligned} \boldsymbol{\Pi}_s &= \boldsymbol{\pi}_s X_s \\ C_s &= c_s X_s, \quad s \in [0,T], \end{aligned}$$

rather than the rate processes $\{\boldsymbol{\pi}_s\}_0^T$ and $\{c_s\}_0^T$. Since $X_s e^{-rs} = x_0 - \int_0^s rX_\theta e^{-r\theta}d\theta + \int_0^s e^{-r\theta}dX_\theta$ and by (1.1) and (1.2), we have a strong solution for X_s in terms of $\{\boldsymbol{\Pi}_\theta\}_0^T$ and $\{C_\theta\}_0^T$ as follows:

$$X_s = e^{rs}\left\{x_0 + \int_0^s e^{-r\theta}\{\boldsymbol{\Pi}'_\theta(\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta) - r\mathbf{1}) - C_\theta\}d\theta + \int_0^s e^{-r\theta}\boldsymbol{\Pi}'_\theta \boldsymbol{\Sigma}^{\frac{1}{2}} d\boldsymbol{W}_\theta\right\}. \tag{2.1}$$

In order to eliminate $\boldsymbol{\Pi}$ term of the drift from the r.h.s. of (2.1), we define a probability measure $\tilde{P}$ on $(\Omega, \mathfrak{F}_T)$ as

$$\tilde{P}(A) = E^P[1_A M_T^0] \quad \text{for} \quad A \in \mathfrak{F}_T,$$

where

$$\begin{aligned} M_s^t = \exp\Bigg[&- \int_t^s (\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta) - r\mathbf{1})'\boldsymbol{\Sigma}^{-\frac{1}{2}} d\boldsymbol{W}_\theta \\ &- \frac{1}{2}\int_t^s (\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta) - r\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta) - r\mathbf{1})d\theta\Bigg], \quad 0 \le t \le s \le T. \end{aligned}$$

We assume the following condition:

Condition 2.1. $\{M_s^0\}_0^T$ *is a P-martingale.*

Remark. A sufficient condition for Condition 2.1 is of course the Novikov condition:

$$E\exp\left[\frac{1}{2}\int_0^T ||\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(s, \boldsymbol{Z}_s) - r\mathbf{1})||^2 ds\right] < \infty.$$

Now, by the Girsanov theorem, $\tilde{\boldsymbol{W}}_s = \int_0^s \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta) - r\mathbf{1})d\theta + \boldsymbol{W}_s$ is a standard Brownian motion on $(\Omega, \mathfrak{F}_T, \tilde{P})$ and thus (2.1) can be rewritten as

$$X_s e^{-rs} - x_0 + \int_0^s e^{-r\theta} C_\theta d\theta = \int_0^s e^{-r\theta} \boldsymbol{\Pi}_\theta' \boldsymbol{\Sigma}^{\frac{1}{2}} d\tilde{\boldsymbol{W}}_\theta, s \in [0,T] \quad \text{a.s. } (\tilde{P}). \tag{2.2}$$

The r.h.s. is a $\tilde{P}$ local martingale. By the condition $\int_0^T \|\boldsymbol{\Pi}_s\|^2 ds < \infty$ and the constancy of $\boldsymbol{\Sigma}$, this local martingale is indeed a martingale (e.g. Protter (1991) p.66). Thus we have

$$E^{\tilde{P}}\left[X_s e^{-rs} + \int_0^s e^{-r\theta} C_\theta d\theta\right] = x_0 \tag{2.3}$$

for all $0 \le s \le T$.

Conversely, suppose a consumption level process $\{C_s\}_0^T$ and a terminal wealth X_T satisfy (2.3). We show that there exists a corresponding investment level process. Define a process $\{Y_s\}_0^T$ as

$$Y_s = e^{rs}\left\{E^{\tilde{P}}\left[X_T e^{-rT} + \int_0^T C_\theta e^{-r\theta} d\theta \middle| \mathfrak{F}_s\right] - \int_0^s C_\theta e^{-r\theta} d\theta\right\}, \quad s \in [t, T]. \tag{2.4}$$

We apply a martingale representation theorem to the $\tilde{P}$-conditional expectation in the definition of Y_s which is a $\tilde{P}$-martingale with respect to $\mathfrak{F}_s$. We cannot, however, directly apply a 'classical' version of representation theorems because of the measurability. Most theorems require the martingale to be adapted to $\sigma(\tilde{\boldsymbol{W}}_\theta, \theta \le s)$. Karatzas, Lehoczky and Shreve (1987) avoid the measurability consideration by converting the $\tilde{P}$- martingale to a P-martingale. We employ Karatzas and Xue (1991) Theorem 5.1 here who use Jacod's (1977) extension to the representation theorem. Then there exists an $\mathfrak{F}_s$-progressively measurable process $\{\boldsymbol{H}_s\}_0^T$ such that $\int_0^T \|\boldsymbol{H}_\theta\|^2 d\theta < \infty$ a.s. $(\tilde{P})$ and

$$E^{\tilde{P}}\left[X_T e^{-rT} + \int_0^T C_\theta e^{-r\theta} d\theta \middle| \mathfrak{F}_s\right] = x_0 + \int_0^s \boldsymbol{H}_\theta' d\tilde{\boldsymbol{W}}_\theta, \quad s \in [0,T] \quad \text{a.s. } (\tilde{P}) \tag{2.5}$$

Now by (2.5) and letting $\boldsymbol{\Pi}_s = e^{rs}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}_s$, the process $\{Y_s\}_0^T$ defined in (2.4) is equivalent to $\{X_s\}_0^T$ in (2.2). Also, it is clear that $\int_0^T \|\boldsymbol{\Pi}_\theta\|^2 dt < \infty$ a.s. $(\tilde{P})$ from Cauchy-Schwarz inequality. Hence we have the following proposition:

Proposition 2.2. *Suppose Assumption 2.1.1 is satisfied. Then given a consumption level process $\{C_s\}_0^T$ and a terminal wealth X_T,*
(a) there exists a corresponding investment level process $\{\boldsymbol{\Pi}_s\}_0^T$ if and only if

$$E^{\tilde{P}}\left[X_T e^{-rT} + \int_0^T e^{-r\theta} C_\theta d\theta\right] = x_0. \tag{2.6}$$

(b) if (2.6) is satisfied, then the corresponding wealth process is given by

$$(2.7)\qquad X_s = e^{rs}\left\{E^{\tilde{P}}\left[X_T e^{-rT} + \int_0^T C_\theta e^{-r\theta}d\theta\middle|\mathfrak{F}_s\right] - \int_0^s C_\theta e^{-r\theta}d\theta\right\}, s \in [0,T].$$

We consider the utility maximization problem in three stages: (1) maximization of the expected utility from consumption, (2) maximization of the expected utility from terminal wealth and (3) combine the results of (1) and (2).

Hereafter, we assume that $U_i(x).i = 1,2$ is a strictly increasing, strictly concave C^1 function with $\lim_{a\downarrow 0} dU_i(a)/da = \infty$ and $\lim_{a\to\infty} dU_i(a)/da = 0$. Additional assumptions will be imposed whenever they are necessary. We define the inverse function $I_i(y), i = 1,2$ of $dU_i(x)/dx$ as $I_i(y) = x$ if $y = dU_i(x)/dx$ and $I_i(y) = 0$ otherwise. From concavity and the definition of I_i, we have an inequality: $U_i(I_i(y)) - U_i(c) \geq y(I_i(y) - c), c \geq 0, y > 0$. This inequality will be used in the maximizations.

First we consider the maximization from consumption only. Let

$$u_1(x_0) = \sup_{\boldsymbol{\Pi},C} u_1^{\boldsymbol{\Pi},C}(x_0) = \sup_{\boldsymbol{\Pi},C} E\left[\int_0^T e^{-\delta\theta}U_1(C_\theta)d\theta\right],$$

and, $(\boldsymbol{\Pi}^{1*}, C^{1*})$ and $\{X_s^{1*}\}_0^T$ be an optimal decision and corresponding wealth process respectively. It is clear that we must have $X_T^{1*} = 0$ a.s. $(\tilde{P})$, since otherwise the utility could be increased by allocating the wealth to consumption.

For $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$, we define a function

$$(2.8)\qquad \begin{aligned}\mathcal{H}_1(t,y,z) &= E\left[\int_t^T M_\theta^t e^{-r(\theta-t)} I_1(e^{y+(\delta-r)(\theta-t)}M_\theta^t)d\theta\right]\\ &= e^{-y}E\left[\int_t^T e^{-\delta(\theta-t)}\exp[K_\theta^{t,y,z}]I_1(\exp[K_\theta^{t,y,z}])d\theta\right],\end{aligned}$$

where

$$(2.9)\qquad \begin{aligned}K_s^{t,y,z} &= y + (\delta - r)(s-t) + \log M_s^t\\ &= y + \int_t^s \{\delta - r - \frac{1}{2}(\boldsymbol{\mu}(\theta, \boldsymbol{Z}_\theta^{t,z}) - r\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}(\theta,\boldsymbol{Z}_\theta^{t,z}) - r\mathbf{1})\}d\theta\\ &\quad - \int_t^s (\boldsymbol{\mu}(\theta,\boldsymbol{Z}_\theta^{t,z}) - r\mathbf{1})'\boldsymbol{\Sigma}^{-\frac{1}{2}}d\boldsymbol{W}_\theta, \quad s \in [t,T]\end{aligned}$$

and $\boldsymbol{Z}_s^{t,z}$ is defined by (1.7). In general, it is difficult to compute $\mathcal{H}_1(t,y,z)$ and $\mathcal{G}_1(t,y,z)$ defined below directly. However, when $U_1(x) = b_1\log(x+m_1), b_1, m_1 > 0$, we can explicitly evaluate $\mathcal{G}_1(t,y,z)$ and $\mathcal{H}_1(t,y,z)$. See Kuwana (1995a) for a detailed discussion.

We assume some conditions on $\mathcal{H}_1$:

Condition 2.3. *We assume that* $\mathcal{H}_1(t,y,z)<\infty$ *for all* $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^n$. $\mathcal{H}_1$ *is continuous and for each* $(t,z)\in[0,T]\times\mathbb{R}^n$ *strictly decreasing with* $\mathcal{H}_1(t,-\infty,z)=\infty$ *and* $\mathcal{H}_1(t,\infty,z)=0$.

Under Condition 2.3, there exists a well-defined inverse function with respect to y, i.e. for $x\in(0,\infty)$

$$\mathcal{K}_1(t,x,z)=y,\quad \text{if } \mathcal{H}_1(t,y,z)=x.$$

Let

$$C_s^{1*}=I_1(\exp[K_s^{0,\mathcal{K}_1(0,x_0,z),z}]). \tag{2.10}$$

Then $\{C_s^{1*}\}_0^T$ is optimal. To see this, let $\{C_s\}_0^T$ be any consumption level process. By Proposition 2.2 and the concavity inequality mentioned before, we have

$$\begin{aligned}
&E\left[\int_0^T e^{-\delta\theta}U_1(C_\theta^{1*})d\theta\right]-E\left[\int_0^T e^{-\delta\theta}U_1(C_\theta)d\theta\right]\\
\geq& E\left[\int_0^T e^{-\delta\theta}\exp[K_\theta^{0,\mathcal{K}_1(0,x_0,z),z}]I_1(\exp[K_\theta^{0,\mathcal{K}_1(0,x_0,z),z}])d\theta\right]\\
&\quad -E\left[\int_0^T e^{-\delta\theta}\exp[K_s^{0,\mathcal{K}_1(0,x_0,z),z}]C_\theta d\theta\right]\\
=&\exp[\mathcal{K}_1(0,x_0,z)]\left\{E\left[\int_0^T M_\theta^0 e^{-r\theta}I_1(\exp[K_\theta^{0,\mathcal{K}_1(0,x_0,z),z}])d\theta\right]\right.\\
&\quad\left.-E\left[\int_0^T M_\theta^0 e^{-r\theta}C_\theta d\theta\right]\right\}\\
=&\exp[\mathcal{K}_1(0,x_0,z)]\left\{\mathcal{H}_1(0,\mathcal{K}_1(0,x_0,z),z)-E^{\tilde{P}}\left[\int_0^T e^{-r\theta}C_\theta d\theta\right]\right\}\\
&\quad(\text{since } E[1_A M_\theta^0]=E[1_A M_T^0], A\in\mathfrak{F}_\theta, \theta\in[0,T] \text{ and by Fubini's theorem})\\
\geq&\exp[\mathcal{K}_1(0,x_0)]\,(x_0-x_0)\\
=&\,0.
\end{aligned}$$

Thus $\{C_s^{1*}\}_0^T$ maximizes $u_1(x_0)$. Hence by Proposition 2.2, we have the following proposition.

Proposition 2.4. *Suppose Conditions 2.1 and 2.3 are satisfied. Then there exist an investment level process* $\{\boldsymbol{\Pi}_s^{1*}\}_0^T$ *corresponding to* C^{1*} *defined by* (2.10) *and a terminal wealth* $X_T^{1*}=0$. $(\boldsymbol{\Pi}^{1*},C^{1*})$ *maximizes* $u_1^{\Pi,C}(x_0)$. *The optimal wealth process is given by*

$$\begin{aligned}
X_s^{1*}&=e^{rs}\left\{E^{\tilde{P}}\left[\int_0^T C_\theta^{1*}e^{-r\theta}d\theta\middle|\mathfrak{F}_s\right]-\int_0^s C_\theta^{1*}e^{-r\theta}d\theta\right\}\\
&=E\left[\int_s^T e^{-r(\theta-s)}M_\theta^s I_1(\exp[K_\theta^{0,\mathcal{K}_1(0,x_0,z),z}])d\theta\middle|\mathfrak{F}_s\right],
\end{aligned}\tag{2.11}$$

and the maximized expected utility is expressed as

$$u_1(x_0) = \mathcal{G}_1(0, \mathcal{K}_1(0, x_0, z), z) \tag{2.12}$$

where

$$\mathcal{G}_1(t, y, z) = E\left[\int_t^T e^{-\delta(\theta-t)} U_1(I_1(\exp[K_\theta^{t,y,z}]))d\theta\right]. \tag{2.13}$$

Next we consider the maximization from terminal wealth only. The analysis goes similarly to the above argument. Let

$$u_2(x_0) = \sup_{\Pi, C} u_2^{\Pi,\dot{C}}(x_0) = \sup_{\Pi, C} E\left[e^{-\delta T} U_2(X_T)\right],$$

and, (Π_{2*}, C_{2*}) and $\{X_s^{2*}\}_0^T$ be an optimal strategy and corresponding wealth process respectively. It is obvious that we must have $C_s^{2*} = 0$ a.s. $(\tilde{P})$. Define

$$\begin{aligned} \mathcal{H}_2(t, y, z) &= E\left[e^{-r(T-t)} M_T^t I_2(\exp[K_T^{t,y,z}])\right] \\ &= e^{-y} E\left[e^{-\delta(T-t)} \exp[K_T^{t,y,z}] I_2(\exp[K_T^{t,y,z}])\right], \end{aligned} \tag{2.14}$$

where $\{K_\theta^{t,y,z}\}_t^T$ is defined by (2.9). Similarly to what we have assumed on $\mathcal{H}_1$, we make an assumption on $\mathcal{H}_2$ as follows:

Condition 2.5. *We assume that $\mathcal{H}_2(t, y, z) < \infty$ for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$. $\mathcal{H}_2$ is continuous and for each $(t, z) \in [0, T] \times \mathbb{R}^n$ strictly decreasing with $\mathcal{H}_2(t, -\infty, z) = \infty$ and $\mathcal{H}_2(t, \infty, z) = 0$.*

Under Condition 2.5, there exists a well-defined inverse function with respect to y, i.e. for $x \in (0, \infty)$

$$\mathcal{K}_2(t, x, z) = y, \quad \text{if } \mathcal{H}_2(t, y, z) = x.$$

Let

$$X_T^{2*} = I_2(\exp[K_T^{0,\mathcal{K}_2(0,x_0,z),z}]), \tag{2.15}$$

then

$$E^{\tilde{P}}\left[e^{-rT} X_T^{2*}\right] = \mathcal{H}_2(0, \mathcal{K}_2(0, x_0, z), z) = x_0.$$

Therefore by Proposition 2.2, there exists an investment level process $\{\Pi_s^{2*}\}_0^T$ corresponding to $\{C_s^{2*}\}_0^T$ and X_T^{2*}. Also we have an expression of the wealth process:

$$\begin{aligned} X_s^{2*} &= e^{-r(T-s)} E^{\tilde{P}}[X_T^{2*} | \mathfrak{F}_s] \\ &= E\left[e^{-r(T-s)} M_T^s I_2(\exp[K_T^{0,\mathcal{K}_2(0,x_0,z),z}]) \middle| \mathfrak{F}_s\right]. \end{aligned} \tag{2.16}$$

The optimality of $(\boldsymbol{\Pi}^{2*}, C^{2*})$ can be shown in the same manner. Let X_T be any terminal wealth such that $E^{\bar{P}}[e^{-rT}X_T] \leq x_0$. Then we have

$$\begin{aligned}
&E[e^{-\delta T}U_2(X_T^{2*})] - E[e^{-\delta T}U_2(X_T)] \\
&\geq \exp[K_T^{0,\mathcal{K}_2(0,x_0,z),z}]\left\{E\left[I_2(\exp[K_T^{0,\mathcal{K}_2(0,x_0,z),z}])M_T^0\right] - E[X_T M_T^0]\right\} \\
&\geq \exp[K_T^{0,\mathcal{K}_2(0,x_0,z),z}]\{\mathcal{H}_2(0,\mathcal{K}_2(0,x_0,z),z) - x_0\} \\
&=0.
\end{aligned}$$

Thus $\{X_s^{2*}\}_0^T$ is an optimal wealth process. Hence we have:

Proposition 2.6. *Suppose Conditions 2.1 and 2.5 are satisfied. Then there exists an investment level process $\{\boldsymbol{\Pi}_s^{2*}\}_0^T$ and a consumption level process $\{C_s^{2*}\}_0^T$ (which is zero a.s.) such that corresponding wealth $\{X_s^{2*}\}_0^T$ (which is given by (2.16)) maximizes $u_2^{\boldsymbol{\Pi},C}(x_0)$. and the maximized expected utility is expressed as*

$$u_2(x_0) = \mathcal{G}_2(0, \mathcal{K}_2(0,x_0,z), z) \tag{2.17}$$

where

$$\mathcal{G}_2(t,y,z) = E\left[e^{-\delta(T-t)}U_2(I_2(\exp[K_T^{t,y,z}]))\right]. \tag{2.18}$$

Now we combine the results so far. Recall the original maximization problem:

$$u(x_0) = \sup_{\boldsymbol{\Pi},C} u^{\boldsymbol{\Pi},C}(x_0) = \sup_{\boldsymbol{\Pi},C} E\left[\int_0^T e^{-\delta\theta}U_1(C_\theta)d\theta + e^{-\delta T}U_2(X_T)\right].$$

For convenience, we write $X_s = X_s(x;\boldsymbol{\Pi},C)$ to indicate dependency on the initial wealth $X_0 = x$ and the strategy $(\boldsymbol{\Pi},C)$. Also we write the optimal processes, which are considered above, as $\boldsymbol{\Pi}_s^{i*} = \boldsymbol{\Pi}_s^{i*}(x), C_s^{i*} = C_s^{i*}(x), i = 1,2$. The next lemma shows the reduction of the original maximization problem to partial maximization problems.

Lemma 2.7. *Let*

$$x^* = \arg\sup_{\xi\in[0,x]}\{u_1(\xi) + u_2(x_0 - \xi)\}.$$

Then the value function satisfies

$$u(x) = u_1(x^*) + u_2(x_0 - x^*) \tag{2.19}$$

and the corresponding optimal processes are given by

$$X_s^* = X_s^{1*}(x^*;\boldsymbol{\Pi}^{1*}(x^*),C^{1*}(x^*)) + X_s^{2*}(x-x^*;\boldsymbol{\Pi}^{2*}(x_0-x^*),0) \tag{2.20}$$

$$\boldsymbol{\Pi}_s^* = \boldsymbol{\Pi}_s^{1*}(x^*) + \boldsymbol{\Pi}_s^{2*}(x-x^*) \tag{2.21}$$

$$C_s^* = C_s^{1*}(x^*) \tag{2.22}$$

Proof. Existence of the maximizer x^* follows from the continuity of $u_i(\cdot)$ on a compact set $[0,x]$. It is unique since strict concavity of u_i is inherited from U_i. Recall the stochastic equation for X_s:

$$X_s(x;\boldsymbol{\Pi},C)=x+\int_0^s e^{-r\theta}\{\boldsymbol{\Pi}_\theta'(\boldsymbol{\mu}(\theta,\boldsymbol{Z}_\theta)-r\mathbf{1})-C_\theta\}d\theta+\int_0^s e^{-r\theta}\boldsymbol{\Pi}_\theta'\boldsymbol{\Sigma}^{\frac{1}{2}}d\boldsymbol{W}_\theta.$$

Linearity of the equation in $x,\boldsymbol{\Pi}$ and C gives an identity

$$X_s(x_1+x_2;\boldsymbol{\Pi}^1+\boldsymbol{\Pi}^2,C^1+C^2)=X_s(x_1;\boldsymbol{\Pi}^1,C^1)+X_s(x_2;\boldsymbol{\Pi}^2,C^2) \tag{2.23}$$

for any $\{X_s(x_i;\boldsymbol{\Pi}^i,C^i)\}_0^T, i=1,2$ satisfying $E^{\tilde{P}}\left[X_T(x_i;\boldsymbol{\Pi}^i,C^i)e^{-rT}+\int_0^T e^{-r\theta}C_\theta^i d\theta\right]=x_i, i=1,2$. By (2.23), for any $\xi\in[0,x_0]$, $(\boldsymbol{\Pi}^{1*}(\xi)+\boldsymbol{\Pi}^{2*}(x_0-\xi),C^{1*}(\xi))$ is a well-defined strategy. Thus we have

$$\begin{aligned}u(x_0)&\geq u^{\boldsymbol{\Pi}^{1*}(\xi)+\boldsymbol{\Pi}^{2*}(x_0-\xi),C^{1*}(\xi)}(x_0)\\&=u_1^{\boldsymbol{\Pi}^{1*}(\xi),C^{1*}(\xi)}(x_0)+u_2^{\boldsymbol{\Pi}^{2*}(x_0-\xi),0}(x_0)\\&=u_1(\xi)+u_2(x_0-\xi)\quad,\quad\forall\xi\in[0,x].\end{aligned}$$

Conversely, given $(\boldsymbol{\Pi},C)$ whose terminal wealth satisfies (2.6), let

$$\alpha=E^{\tilde{P}}\left[\int_0^T e^{-r\theta}C_\theta^i d\theta\right]\in[0,x_0].$$

Then by Proposition 2.2, there exists an investment level process $\boldsymbol{\Pi}^1$ corresponding to consumption level process C and the terminal wealth $X_T(\alpha;\boldsymbol{\Pi}^1,C)=0$. Further, let $\boldsymbol{\Pi}_s^2=\boldsymbol{\Pi}_s-\boldsymbol{\Pi}_s^1$. Then the terminal wealth $X_T(x_0-\alpha;\boldsymbol{\Pi}^2,0)$ satisfies the condition (2.6) with x_0 replaced by $x_0-\alpha$. By (2.23), the wealth process $X_s(x_0;\boldsymbol{\Pi},C)$ can be decomposed as $X_s(x_0;\boldsymbol{\Pi}^1+\boldsymbol{\Pi}^2,C^1+C^2)=X_s(\alpha;\boldsymbol{\Pi}^1,C)+X_s(x_0-\alpha;\boldsymbol{\Pi}^2,0)$. Hence we have a converse inequality:

$$u^{\boldsymbol{\Pi},C}(x_0)=u_1^{\boldsymbol{\Pi}^1,C}(\alpha)+u_2^{\boldsymbol{\Pi}^2,0}(x_0-\alpha)\leq u_1(\alpha)+u_2(x_0-\alpha).$$

This completes the proof. □

III. *Smoothness of the Value Functions*

When we derived Propositions 2.4 and 2.6, we left the Conditions 2.3 and 2.5 undiscussed. We fill this gap by using the results from Kuwana (1995b). At the same time, we further deduce more convenient expression for the maximizer in Lemma 2.7. The next lemma shows finiteness and smoothness of the functions $\mathcal{G}_i$, $\mathcal{H}_i$ and $\mathcal{K}_i, i=1,2$.

Lemma 3.1. *Suppose* $\|\boldsymbol{\mu}(t,z)-r\mathbf{1}\|$ *and* $\|\boldsymbol{\mu}(t,z)-r\mathbf{1}\|^2$ *satisfy the Lipschitz condition in* z *uniformly in* $t\in[0,T]$. *Further, assume that* $\boldsymbol{\mu}(t,z)$ *is continuously differentiable in* z. *Let*

$U_i(x), i=1,2$ be strictly increasing, strictly concave and twice continuously differentiable with $\lim_{a\downarrow 0} dU_i(a)/da=\infty$ and $\lim_{a\to\infty} dU_i(a)/da=0$. Suppose either condition:
(a) there exist $K, m>0$ such that

$$\sum_{i=1}^{2}\left\{|U_i(I_i(e^y))|+|e^y I_i(e^y)|+\left|e^y\frac{dI_i(e^y)}{dy}\right|\right\}\le K(1+|y|^m), \tag{3.1}$$

or,
(b) $\mu(t,z)$ is bounded and there exist $K, a>0$ and $0<\gamma<2$ such that

$$\sum_{i=1}^{2}\left\{|U_i(I_i(e^y))|+|e^y I_i(e^y)|+\left|e^y\frac{dI_i(e^y)}{dy}\right|\right\}\le Ke^{a|y|^\gamma}, \tag{3.2}$$

is satisfied. Then we have
(1) $\mathcal{G}_i(t,y,z)$, $\mathcal{H}_i(t,y,z), i=1,2$ are finite and continuous for all $t\in[0,T]$ and $y\in\mathbb{R}$,
(2) $\mathcal{H}_i(t,y,z), i=1,2$ are strictly decreasing with $\lim_{y\to-\infty}\mathcal{H}_i(t,y,z)=-\infty$ and $\lim_{y\to\infty}\mathcal{H}_i(t,y,z)=0$,
(3) $\mathcal{G}_i(t,y,z)$, $\mathcal{H}_i(t,y,z), i=1,2$ are continuously differentiable in y and

$$\frac{\partial}{\partial y}\mathcal{G}_i(t,y,z)=e^y\frac{\partial}{\partial y}\mathcal{H}_i(t,y,z), i=1,2. \tag{3.3}$$

Proof. The finiteness of $\mathcal{G}_i(t,y,z)$ and $\mathcal{H}_i(t,y,z)$ easily follows from Theorem 1.4 of Kuwana (1995b). By the condition $\lim_{x\downarrow 0}\frac{dU_i(x)}{dx}=\infty$, I_i is continuously differentiable. Thus $U_i(I_i(e^y))$, $e^y I_i(e^y), i=1,2$ are all continuously differentiable. Hence the continuity and continuous differentiability of $\mathcal{G}_i(t,y,z)$ and $\mathcal{H}_i(t,y,z)$ follow from Corollary 3.4 and a similar argument in the proof of Theorem 3.5 of Kuwana (1995b). It is not hard to see that $I_i(x), i=1,2$ is strictly decreasing. Thus $\mathcal{H}_i(t,y,z)$ is strictly decreasing in y. (3.3) is a simple consequence of the $\mathcal{L}$-differentiation rule described in Krylov (1980). To see this, note that $\mathcal{L}\text{-}\frac{\partial}{\partial y}K_s^{t,y,z}=1$. Then

$$\begin{aligned}\frac{\partial}{\partial y}\mathcal{H}_1(t,y,z)&=e^{-y}E\left[\int_t^T e^{-\delta(\theta-t)}\mathcal{L}\text{-}\frac{\partial}{\partial y}\left(\exp[K_\theta^{t,y,z}]I_1(\exp[K_\theta^{t,y,z}])\right)d\theta\right]\\&\quad-e^{-y}E\left[\int_t^T e^{-\delta(\theta-t)}\exp[K_\theta^{t,y,z}]I_1(\exp[K_\theta^{t,y,z}])d\theta\right]\\&=e^{-y}E\left[\int_t^T e^{-\delta(\theta-t)}\exp[2K_\theta^{t,y,z}]I_1'(\exp[K_\theta^{t,y,z}])d\theta\right].\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{\partial}{\partial y}\mathcal{G}_1(t,y,z) &= E\left[\int_t^T e^{-\delta(\theta-t)}\mathcal{L}\text{-}\frac{\partial}{\partial y}U_1(I_1(\exp[K_\theta^{t,y,z}]))d\theta\right]\\ &= E\left[\int_t^T e^{-\delta(\theta-t)}\exp[2K_\theta^{t,y,z}]I_1'(\exp[K_\theta^{t,y,z}])\mathcal{L}\text{-}\frac{\partial}{\partial y}K_\theta^{t,y,z}d\theta\right]\\ &= e^y\frac{\partial}{\partial y}\mathcal{H}_1(t,y,z).\end{aligned}$$

The identity for $\mathcal{G}_2(t,y,z)$ can be proved similarly. □

By using above results, the martingale solution to the original maximization problem given in Lemma 2.7 can be expressed in a more convenient form.

Proposition 3.2. *Suppose Condition 2.1 and conditions of Lemma 3.1 are satisfied. Then the maximized expected utility is given by*

$$u(x_0) = \mathcal{G}(0,\mathcal{K}(0,x_0,z),z).$$

Here,

$$\begin{aligned}\mathcal{G}(t,y,z) &= \mathcal{G}_1(t,y,z)+\mathcal{G}_2(t,y,z)\\ &= E\left[\int_t^T e^{-\delta(\theta-t)}U_1(I_1(\exp[K_\theta^{t,y,z}]))d\theta\right.\\ &\qquad\left. + e^{-\delta(T-t)}U_2(I_2(\exp[K_T^{t,y,z}]))\right],\end{aligned}\tag{3.4}$$

and for each $(t,z)\in[0,T]\times\mathbb{R}^n$, $\mathcal{K}(t,x,z)$ is the inverse function of

$$\begin{aligned}\mathcal{H}(t,y,z) &= \mathcal{H}_1(t,y,z)+\mathcal{H}_2(t,y,z)\\ &= e^{-y}E\left[\int_t^T e^{-\delta(\theta-t)}\exp[K_\theta^{t,y,z}]I_1(\exp[K_\theta^{t,y,z}])d\theta\right.\\ &\qquad\left. + e^{-\delta(T-t)}\exp[K_T^{t,y,z}]I_2(\exp[K_T^{t,y,z}])\right].\end{aligned}\tag{3.5}$$

The optimal consumption and wealth processes are expressed as

$$C_s^* = I_1(\exp[K_\theta^{0,\mathcal{K}(0,x_0,z),z}])$$

and

$$X_s^* = E^{\tilde{P}}\left[\int_s^T e^{-r(\theta-s)}I_1(\exp[K_\theta^{0,\mathcal{K}(0,x_0,z),z}])d\theta + e^{-r(T-s)}I_2(\exp[K_T^{0,\mathcal{K}(0,x_0,z),z}])\middle|\mathfrak{F}_s\right].$$

Proof. By the smoothness property proved in Lemma 3.1, the maximizer x^* in Lemma 2.7 satisfies

$$\mathcal{G}_{1,y}(0,\mathcal{K}_1(0,x^*,z),z)\mathcal{K}_{1,x}(0,x^*,z)=\mathcal{G}_{2,y}(0,\mathcal{K}_2(0,x_0-x^*,z),z)\mathcal{K}_{2,x}(0,x_0-x^*,z).$$

From (3.3) and the fact $\mathcal{K}_{i,x}(t,x,z)\mathcal{H}_{i,y}(t,\mathcal{K}_i(t,x,z),z)=1, i=1,2$, the above relation reduces to

$$\mathcal{K}_1(0,x^*,z)=\mathcal{K}_2(0,x_0-x^*,z)=y^*,$$

which is equivalent to

$$\mathcal{H}(0,y^*,z)=\mathcal{H}_1(0,y^*,z)+\mathcal{H}_2(0,y^*,z)=x_0$$

Thus $\mathcal{K}(0,x_0,z)=y^*=\mathcal{K}_1(0,x^*,z)=\mathcal{K}_2(0,x_0-x^*,z)$ and

$$u(x_0)=\mathcal{G}_1(0,\mathcal{K}_1(0,x^*,z),z)+\mathcal{G}_2(0,\mathcal{K}_2(0,x_0-x^*,z),z)=\mathcal{G}(0,\mathcal{K}(0,x_0,z),z).$$

The optimal processes easily follow from (2.10), (2.11), (2.16), (2.20) and (2.22). □

IV. *Cauchy Problems Associated with the Martingale Solution*

We state technical conditions which are sufficient for the smoothness of $\mathcal{G}(t,y,z)$ and $\mathcal{H}(t,y,z)$:

Condition 4.1. $\|\boldsymbol{\mu}(t,z)-r\mathbf{1}\|$ *and* $\|\boldsymbol{\mu}(t,z)-r\mathbf{1}\|^2$ *are Lipschitz continuous in* z. $\boldsymbol{\mu}(t,z)\in C^{1,2}[0,T]\times\mathbb{R}^n$ *satisfies*

$$\sum_i\left\|\frac{\partial\boldsymbol{\mu}(t,z)}{\partial z_i}\right\|+\sum_{i,j}\left\|\frac{\partial^2\boldsymbol{\mu}(t,z)}{\partial z_i\partial z_j}\right\|\le K(1+\|z\|^a)$$

for some $K,a>0$.

Condition 4.2. *(a)* $\boldsymbol{\mu}(t,z)$ *satisfies* $E\exp\left[\frac{1}{2}\int_0^T\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(s,Z_s)-r\mathbf{1})\|^2ds\right]<\infty$. *and Condition 4.1.*
(b) $U_i(x), i=1,2$ *is strictly increasing, strictly concave, three times continuously differentiable on* $(0,\infty)$, $\lim_{a\downarrow 0}dU_i(a)/da=\infty$ *and* $\lim_{a\to\infty}dU_i(a)/da=0$. *Further there exist constants* $K,a>0$ *such that*

$$\sum_{i=1}^2\left\{|U_i(I_i(e^y))|+|e^yI_i(e^y)|+\left|e^y\frac{dI_i(e^y)}{dy}\right|+\left|e^y\frac{d^2I_i(e^y)}{dy^2}\right|\right\}\le K(1+|y|^a).$$

Condition 4.3. *(a)* $\boldsymbol{\mu}(t,z)$ *is bounded and satisfies Condition 4.1.*
(b) $U_i(x), i=1,2$ *is strictly increasing, strictly concave, three times continuously differentiable on* $(0,\infty)$, $\lim_{a\downarrow 0}dU_i(a)/da=\infty$ *and* $\lim_{a\to\infty}dU_i(a)/da=0$. *Further* $U_i(x), i=1,2$ *satisfies the exponential growth condition:*

$$\sum_{i=1}^2\left\{|U_i(I_i(e^y))|+|e^yI_i(e^y)|+\left|e^y\frac{dI_i(e^y)}{dy}\right|+\left|e^y\frac{d^2I_i(e^y)}{dy^2}\right|\right\}\le Ke^{a|y|^\gamma}$$

for some $K, a > 0, 0 < \gamma < 2$.

It is noted that there is a trade-off between growth of $\mu(t,z)$ and $U(x)$. The logarithmic utility $U_i(x) = \log(x + m_i), m_i > 0$ and the HARA utility $U_i(x) = x_i^{\alpha}, 0 < \alpha_i < 1$ do not satisfy 4.2 (b).

Now we apply the stochastic solution technique (e.g. Theorem 3.6 of Kuwana (1995b)) to $\mathcal{G}(t,y,z)$ and $e^y\mathcal{H}(t,y,z)$.

Proposition 4.4. *Suppose either Condition 4.2 or 4.3 is satisfied.* $\mathcal{G}(t,y,z)$ *and* $\mathcal{H}(t,y,z)$ *are unique smooth solutions to the following Cauchy problems respectively:*

$$(4.1)\qquad \begin{cases} 0 = \dfrac{\partial \mathcal{G}}{\partial t} + \dfrac{1}{2}(\mu(t,z) - r\mathbf{1})'\Sigma^{-1}(\mu(t,z) - r\mathbf{1})\dfrac{\partial^2 \mathcal{G}}{\partial y^2} - (\mu(t,z) - r\mathbf{1})'\dfrac{\partial^2 \mathcal{G}}{\partial y \partial z} \\ \quad + \dfrac{1}{2}\mathbf{tr}\Sigma\dfrac{\partial^2 \mathcal{G}}{\partial z \partial z'} + \left(\delta - r - \dfrac{1}{2}(\mu(t,z) - r\mathbf{1})'\Sigma^{-1}(\mu(t,z) - r\mathbf{1})\right)\dfrac{\partial \mathcal{G}}{\partial y} \\ \quad + \mu(t,z)'\dfrac{\partial \mathcal{G}}{\partial z} - \delta\mathcal{G} + U_1(I_1(e^y)) \\ \mathcal{G}(T,y,z) = U_2(I_2(e^y)), \end{cases}$$

$$(4.2)\qquad \begin{cases} 0 = \dfrac{\partial \mathcal{H}}{\partial t} + \dfrac{1}{2}(\mu(t,z) - r\mathbf{1})'\Sigma^{-1}(\mu(t,z) - r\mathbf{1})\dfrac{\partial^2 \mathcal{H}}{\partial y^2} - (\mu(t,z) - r\mathbf{1})'\dfrac{\partial^2 \mathcal{H}}{\partial y \partial z} \\ \quad + \dfrac{1}{2}\mathbf{tr}\Sigma\dfrac{\partial^2 \mathcal{H}}{\partial z \partial z'} + \left(\delta - r + \dfrac{1}{2}(\mu(t,z) - r\mathbf{1})'\Sigma^{-1}(\mu(t,z) - r\mathbf{1})\right)\dfrac{\partial \mathcal{H}}{\partial y} \\ \quad + r\mathbf{1}'\dfrac{\partial \mathcal{H}}{\partial z} - r\mathcal{H} + I_1(e^y) \\ \mathcal{H}(T,y,z) = I_2(e^y). \end{cases}$$

Remark. The pair of Cauchy problems (4.1) and (4.2) contain one derived by Karatzas, Lehoczky and Shreve (1987) as a special case if we set $\mu(t,z) = \mu$, $\mathcal{G}(t,y,z) = G(t,e^y)$ and $\mathcal{H}(t,y,z) = \mathcal{X}(t,e^y)$.

V. *Verification of the HJB Equation*

As far as the optimal investment choice is concerned, the martingale approach only guarantees its existence and does not provide any explicit solution. In the case of Markovian dynamics, we can make use of the HJB equation to give an expression of the optimal investment choice in terms of the value function. In this section, we present a verification result for the HJB equation by applying Proposition 4.4.

In Proposition 3.2, we derived a convenient expression for the martingale solution when $t = 0$. By a similar argument, we see the maximized expected utility, optimal consumption and wealth processes for the Markovian problem can be represented as follows:

$$(5.1)\qquad u(t,x,z) = e^{-\delta t}\mathcal{G}(t,\mathcal{K}(t,x,z),z),$$

$$C_s^{t,x,z,*} = I_1(K_s^{t,\mathcal{K}(t,x,z)}), \tag{5.2}$$

and

$$\begin{aligned}
X_s^{t,x,z,*} &= E^{\tilde{P}}\left[\int_s^T e^{-r(\theta-s)} I_1(\exp[K_\theta^{t,\mathcal{K}(t,x,z)}])d\theta \right.\\
&\qquad \left. + e^{-r(T-s)} I_2(\exp[K_T^{t,\mathcal{K}(t,x,z)}])\middle|\mathfrak{F}_s\right]\\
&= E\left[\int_s^T M_\theta^s e^{-r(\theta-s)} I_1(\exp[K_\theta^{t,\mathcal{K}(t,x,z)}])d\theta \right.\\
&\qquad \left. + M_T^s e^{-r(T-s)} I_2(\exp[K_T^{t,\mathcal{K}(t,x,z)}])\middle|\mathfrak{F}_s\right] \qquad (5.3)\\
&= \exp[-K_s^{t,\mathcal{K}(t,x,z)}]E\left[\int_s^T e^{-\delta(\theta-s)} \exp[K_\theta^{s,K_s^{t,\mathcal{K}(t,x,z)}}] I_1(\exp[K_\theta^{s,K_s^{t,\mathcal{K}(t,x,z)}}])d\theta \right.\\
&\qquad \left. + e^{-\delta(T-s)} \exp[K_T^{t,K_s^{s,\mathcal{K}(t,x,z)}}] I_2(\exp[K_T^{s,K_s^{t,\mathcal{K}(t,x,z)}}])\right]\\
&\qquad \text{(by the Markov property)}\\
&= \mathcal{H}(s, K_s^{t,\mathcal{K}(t,x,z)}, \boldsymbol{Z}_s^{t,z}).
\end{aligned}$$

We show that $u(t,x,z)$ satisfies a degenerate HJB equation:

$$\sup_{\pi\ c}\left\{\frac{\partial u}{\partial t} + e^{-\delta t}U_1(cx) + (r-c)x\frac{\partial u}{\partial x} + \boldsymbol{\pi}'(\boldsymbol{\mu}(t,z) - r\mathbf{1})xu_x \right. \tag{5.4}$$
$$\left. + \boldsymbol{\mu}(t.z)'\frac{\partial u}{\partial z} + \frac{1}{2}\boldsymbol{\pi}'\boldsymbol{\Sigma}\boldsymbol{\pi}x^2\frac{\partial^2 u}{\partial x^2} + x\boldsymbol{\pi}'\boldsymbol{\Sigma}\frac{\partial^2 u}{\partial x \partial z} + \frac{1}{2}\mathbf{tr}\boldsymbol{\Sigma}\frac{\partial^2 u}{\partial z \partial z'}\right\} = 0$$

$$u(T,x,z) = e^{-\delta T}U_2(x). \tag{5.5}$$

First, it is noted that each derivative of $\mathcal{K}(t,x,z)$ is well-defined and smooth. By differentiating $\mathcal{H}(t,\mathcal{K}(t,x,z),z) = x$, we obtain

$$\frac{\partial \mathcal{K}(t,x,z)}{\partial t} = -\left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-1}\cdot\left(\frac{\partial \mathcal{H}}{\partial t}\right)\Bigg|_{y=\mathcal{K}(t,x,z)}, \quad \frac{\partial \mathcal{K}(t,x,z)}{\partial x} = \left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-1}\Bigg|_{y=\mathcal{K}(t,x,z)},$$

$$\frac{\partial \mathcal{K}(t,x,z)}{\partial z} = -\left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-1}\cdot\frac{\partial \mathcal{H}}{\partial z}\Bigg|_{y=\mathcal{K}(t,x,z)}, \quad \frac{\partial^2 \mathcal{K}(t,x,z)}{\partial x^2} = -\left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-3}\cdot\frac{\partial^2 \mathcal{H}}{\partial y^2}\Bigg|_{y=\mathcal{K}(t,x,z)},$$

$$\frac{\partial^2 \mathcal{K}(t,x,z)}{\partial x \partial z} = \left\{-\left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-2}\cdot\frac{\partial^2 \mathcal{H}}{\partial y \partial z} + \left(\frac{\partial \mathcal{H}}{\partial y}\right)^{-3}\cdot\frac{\partial^2 \mathcal{H}}{\partial y^2}\cdot\frac{\partial \mathcal{H}}{\partial z}\right\}\Bigg|_{y=\mathcal{K}(t,x,z)},$$

$$\frac{\partial^2\mathcal{K}(t,x,z)}{\partial z\partial z'} = \left\{ -\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-1}\cdot\frac{\partial^2\mathcal{H}}{\partial z\partial z'} + 2\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-2}\cdot\frac{\partial^2\mathcal{H}}{\partial y\partial z}\cdot\frac{\partial\mathcal{H}}{\partial z'} - \left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-3}\cdot\frac{\partial^2\mathcal{H}}{\partial y^2}\cdot\frac{\partial\mathcal{H}}{\partial z}\cdot\frac{\partial\mathcal{H}}{\partial z'}\right\}\Bigg|_{y=\mathcal{K}(t,x,z)}.$$

By differentiating the right hand side of (5.1) and by using (3.3) together with the derivatives of $\mathcal{K}(t,x,z)$, each derivative of $u(t,x,z)$ can be computed as

$$\frac{\partial u(t,x,z)}{\partial t} = \left\{e^{-\delta t}\left(\frac{\partial\mathcal{G}}{\partial t} - \delta\mathcal{G} - e^y\frac{\partial\mathcal{H}}{\partial t}\right)\right\}\Bigg|_{y=\mathcal{K}(t,x,z)}, \quad \frac{\partial u(t,x,z)}{\partial x} = e^{\mathcal{K}(t,x,z)-\delta t}$$

$$\frac{\partial u(t,x,z)}{\partial z} = \left\{e^{-\delta t}\left(\frac{\partial\mathcal{G}}{\partial z} - e^y\frac{\partial\mathcal{H}}{\partial z}\right)\right\}\Bigg|_{y=\mathcal{K}(t,x,z)}, \quad \frac{\partial^2 u(t,x,z)}{\partial x^2} = e^{y-\delta t}\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-1}\Bigg|_{y=\mathcal{K}(t,x,z)},$$

$$\frac{\partial^2 u(t,x,z)}{\partial x\partial z} = \left\{-e^{y-\delta t}\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-1}\cdot\frac{\partial\mathcal{H}}{\partial z}\right\}\Bigg|_{y=\mathcal{K}(t,x,z)},$$

$$\frac{\partial^2 u(t,x,z)}{\partial z\partial z'} = e^{-\delta t}\left\{\frac{\partial^2\mathcal{G}}{\partial z\partial z'} - e^y\frac{\partial^2\mathcal{H}}{\partial z\partial z'} + e^y\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-1}\cdot\frac{\partial\mathcal{H}}{\partial z}\cdot\frac{\partial\mathcal{H}}{\partial z'}\right\}\Bigg|_{y=\mathcal{K}(t,x,z)}.$$

By using above computations, we prove the next proposition.

Proposition 5.1. *Suppose either Condition 4.2 or 4.3 holds. Then $u(t,x,z)$ satisfies the degenerate HJB equation (5.4)–(5.5). Moreover, the solution to the equation can be obtained by solving degenerate Cauchy problems (4.1) and (4.2).*

Proof. The boundary condition (5.5) is obvious. The maximizer $(\boldsymbol{\pi}^*, c^*)$ for the l.h.s. of (5.4) is given by

$$\text{(5.6)} \quad \boldsymbol{\Pi}^* = \boldsymbol{\pi}^* x = -\left\{\Sigma^{-1}(\boldsymbol{\mu}(t,z) - r\mathbf{1})\frac{\partial u(t,x,z)}{\partial x} + \frac{\partial^2 u(t,x,z)}{\partial x\partial z}\right\}\cdot\left(\frac{\partial^2 u(t,x,z)}{\partial x^2}\right)^{-1},$$

$$\text{(5.7)} \quad C^* = c^* x = I_1\left(e^{\delta t}\frac{\partial u(t,x,z)}{\partial x}\right).$$

Then, from (3.3), (4.1), (4.2) and the derivatives computed above, we have that

$$
\begin{aligned}
&\text{l.h.s. of (5.4)}\\
=&\frac{\partial u}{\partial t}+e^{-\delta t}U_1\left(I_1\left(e^{\delta t}\frac{\partial u}{\partial x}\right)\right)+\left\{rx-I_1\left(e^{\delta t}\frac{\partial u}{\partial x}\right)\right\}\frac{\partial u}{\partial x}+\boldsymbol{\mu}(t,z)'\frac{\partial u}{\partial z}\\
&+\frac{1}{2}\mathbf{tr}\boldsymbol{\Sigma}\frac{\partial^2 u}{\partial z\partial z'}-\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^{-1}\left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\left\{(\boldsymbol{\mu}(t,z)-r\mathbf{1})\frac{\partial u}{\partial x}+\boldsymbol{\Sigma}\frac{\partial^2 u}{\partial x\partial z}\right\}\right\|^2\\
=&e^{-\delta t}\left\{U_1(I_1(e^y))-\delta\mathcal{G}+\frac{\partial\mathcal{G}}{\partial t}+\boldsymbol{\mu}(t,z)\frac{\partial\mathcal{G}}{\partial z}+\frac{1}{2}\mathbf{tr}\boldsymbol{\Sigma}\frac{\partial^2\mathcal{G}}{\partial z\partial z'}-e^y\left(I_1(e^y)-r\mathcal{H}+\frac{\partial\mathcal{H}}{\partial t}\right.\right.\\
&\left.\left.\left.+\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2\frac{\partial\mathcal{H}}{\partial y}+r\mathbf{1}'\frac{\partial\mathcal{H}}{\partial z}+\frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial z\partial z'}\right)\right\}\right|_{y=\mathcal{K}(t,x,z)}\\
=&e^{-\delta t}\left\{\left(\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2-(\delta-r)\right)\frac{\partial\mathcal{G}}{\partial y}\right.\\
&+(\boldsymbol{\mu}(t,z)-r\mathbf{1})'\frac{\partial^2\mathcal{G}}{\partial y\partial z}-\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2\frac{\partial^2\mathcal{G}}{\partial y^2}-e^y\left(I_1(e^y)-r\mathcal{H}+\frac{\partial\mathcal{H}}{\partial t}\right.\\
&\left.\left.\left.+\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2\frac{\partial\mathcal{H}}{\partial y}+r\mathbf{1}'\frac{\partial\mathcal{H}}{\partial z}+\frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial z\partial z'}\right)\right\}\right|_{y=\mathcal{K}(t,x,z)}\\
=&e^{y-\delta t}\left\{\frac{\partial\mathcal{H}}{\partial t}+\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2\frac{\partial^2\mathcal{H}}{\partial y^2}-(\boldsymbol{\mu}(t,z)-r\mathbf{1})'\frac{\partial^2\mathcal{H}}{\partial y\partial z}\right.\\
&+\frac{1}{2}\mathbf{tr}\boldsymbol{\Sigma}\frac{\partial^2\mathcal{H}}{\partial z\partial z'}+\left(\delta-r+\frac{1}{2}\|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\|^2\right)\frac{\partial\mathcal{H}}{\partial y}\\
&\left.\left.+r\mathbf{1}'\frac{\partial\mathcal{H}}{\partial z}-r\mathcal{H}+I_1(e^y)\right\}\right|_{y=\mathcal{K}(t,x,z)}\\
=&0.
\end{aligned}
$$

Hence $u(t,x,z)$ satisfies (5.4). □

By using the expression of derivatives of u, the maximizer $(\boldsymbol{\pi}^*,c^*)$ given by (5.6) and (5.7) can be written as

$$
\begin{aligned}
\boldsymbol{\pi}^*x&=-e^{-y+\delta t}\frac{\partial\mathcal{H}}{\partial y}\left.\left\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}(t,z)-r\mathbf{1})e^{y-\delta t}-e^{y-\delta t}\left(\frac{\partial\mathcal{H}}{\partial y}\right)^{-1}\frac{\partial\mathcal{H}}{\partial z}\right\}\right|_{y=\mathcal{K}(t,x,z)}\\
&=-\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}(t,z)-r\mathbf{1})\left(\frac{\partial\mathcal{K}}{\partial x}\right)^{-1}-\left(\frac{\partial\mathcal{K}}{\partial x}\right)^{-1}\frac{\partial\mathcal{K}}{\partial z}
\end{aligned}
\tag{5.8}
$$

$$
c^*x=I_1\left(e^{\mathcal{K}(t,x,z)}\right).\tag{5.9}
$$

Now we have a solution to our problem in a feedback form.

Proposition 5.2. *The strategy* $(\pi^{t,x,z,*}, c^{t,x,z,*})$ *given by*

$$\begin{aligned}\pi_s^{t,x,z,*}X_s^{t,x,z,*} = &-\Sigma^{-1}(\mu(s,Z_s^{t,z})-r\mathbf{1})\left(\frac{\partial\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z})}{\partial x}\right)^{-1}\\ &-\left(\frac{\partial\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z})}{\partial x}\right)^{-1}\frac{\partial\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z})}{\partial z}\end{aligned} \tag{5.10}$$

$$c_s^{t,x,z,*}X_s^{t,x,z,*} = I_1\left(\exp[\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z})]\right) \tag{5.11}$$

is optimal. Here $X_s^{t,x,z,*}$ *is the optimal wealth process given by (5.3).*

Proof. By (5.3),

$$\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z}) = \mathcal{K}(s,\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z}),Z_s^{t,z}) = K_s^{t,\mathcal{K}(t,x,z)}.$$

Thus (5.11) coincides with (5.2). By applying Ito's lemma to (5.3), we have

$$\begin{aligned}
&dX_s^{t,x,z,*}\\
=&d\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})\\
=&\frac{\partial\mathcal{H}}{\partial t}ds+\frac{\partial\mathcal{H}}{\partial y}dK_s^{t,\mathcal{K}(t,x,z)}+\frac{\partial\mathcal{H}}{\partial z}dZ_s^{t,z}+\frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial y^2}\|\Sigma^{-\frac{1}{2}}(\mu(s,Z_s^{t,z})-r\mathbf{1})\|^2ds\\
&-\frac{\partial^2\mathcal{H}}{\partial y\partial z'}(\mu(s,Z_s^{t,z})-r\mathbf{1})ds+\frac{1}{2}\mathbf{tr}\frac{\partial^2\mathcal{H}}{\partial z\partial z'}\Sigma ds\\
=&\Bigg\{-\|\Sigma^{-\frac{1}{2}}(\mu(s,Z_s^{t,z})-r\mathbf{1})\|^2\frac{\partial\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})}{\partial y}\\
&+(\mu(s,Z_s^{t,z})-r\mathbf{1})'\frac{\partial\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})}{\partial z}+r\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})\\
&-I_1\left(\exp[\mathcal{K}(s,X_s^{t,x,z,*},Z_s^{t,z})]\right)\Bigg\}ds\\
&+\Bigg\{-\Sigma^{-\frac{1}{2}}(\mu(s,Z_s^{t,z})-r\mathbf{1})\frac{\partial\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})}{\partial y}\\
&+\Sigma^{\frac{1}{2}}\frac{\partial\mathcal{H}(s,K_s^{t,\mathcal{K}(t,x,z)},Z_s^{t,z})}{\partial z}\Bigg\}'dW_s\\
=&\Big\{(\pi_s^{t,x,z,*})'(\mu(s,Z_s^{t,z})-r\mathbf{1})+r-c_s^{t,x,z,*}\Big\}X_s^{t,x,z,*}ds+X_s^{t,x,z,*}(\pi_s^{t,x,z,*})'dW_s.
\end{aligned}$$

This shows the optimality of $\pi_s^{t,x,z,*}$. $\square$

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