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SYNERGETIC INTERACTIONS WITHIN THE PAIR-WISE HIERARCHY OF ECONOMIC LINKAGES SUB-SYSTEMS

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Abstract

This paper clarifies and extends Miyazawa's suggestion to classify the types of synergetic interactions within the preset pair-wise hierarchy of economic linkages sub-systems. Such a classification is based on a partitioned input-output system and exploits techniques that produce *left* and *right* matrix multipliers for the Leontief Inverse.

I. *Introduction*

Consider an input-output system represented by the following block matrix, A, of direct inputs:

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1}$$

where A_{11} and A_{22} are the quadrat matrices of direct inputs within the first and second regions, and A_{12} and A_{21} are the rectangular matrices showing the direct inputs purchased by the second region and vice versa. It is possible to interpret the matrix A as a two-region system in which the second region represents the rest of the economy minus the first region.

In a recent paper, Sonis and Hewings (1993) explored the possibilities of connecting the idea of a hierarchical structure of economic linkages sub-systems in the multiregional input-output system with additive block-matrix decompositions of direct inputs matrix and multiplicative block-matrix decompositions of the corresponding Leontief Inverse. In a private communication, Miyazawa (1993) suggested a further extension through the identification of four types of synergetic interactions namely;

(1) "push or pull" linkages type:

$$A=A_1+A_2=\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}. \tag{2}$$

(2) "push or pull" order-replaced type:

$$A=A_1+A_2=\begin{bmatrix}0 & A_{12}\\ 0 & A_{22}\end{bmatrix}+\begin{bmatrix}A_{11} & 0\\ A_{21} & 0\end{bmatrix}. \tag{3}$$

(3) "intra- & inter-" linkages type:

$$A=A''_1+A''_2=\begin{bmatrix}A_{11} & 0\\ 0 & A_{22}\end{bmatrix}+\begin{bmatrix}0 & A_{12}\\ A_{21} & 0\end{bmatrix}. \tag{4}$$

(4) "intra- & inter-" three sub-systems type:

$$A=\begin{bmatrix}A_{11} & 0\\ 0 & A_{22}\end{bmatrix}+\begin{bmatrix}0 & 0\\ A_{21} & 0\end{bmatrix}+\begin{bmatrix}0 & A_{12}\\ 0 & 0\end{bmatrix}. \tag{5}$$

or

$$A=\begin{bmatrix}A_{11} & 0\\ 0 & A_{22}\end{bmatrix}+\begin{bmatrix}0 & A_{12}\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ A_{21} & 0\end{bmatrix}. \tag{6}$$

In this paper, we expand and elaborate further these considerations. The elaborations are based on the distinction between the inner (regional) matrix multipliers and the outer (hierarchical) block-matrix multipliers: the regional matrix multipliers are the components of block-matrix hierarchical multipliers. It is important to underline that the inner matrix multipliers of the Miyazawa type are well known in the partitioned input-output analysis (see Miyazawa, 1976), while the hierarchical block-matrix multipliers provide a relatively new perspective (Sonis and Hewings, 1993, 1994, 1995). Surprisingly, the proposed procedure of the multiplicative decompositions of the Leontief inverse provides the possibility to include in one scheme the formal results of inversion of block-matrices from their start in the Frobenius School to more recent studies (*cf.* Henderson and Searle, 1981). Moreover, well known methods of matrix inversion, such as Gauss-Fourier-Jourdan direct and dual elimination method, and the bordering method correspond to some decompositions of the matrix into the ordered sum of sub-matrices; thus, this paper will explore the possible correspondence between the hierarchies of economic sub-systems and the multiplicity of direct (non-iterative) methods of matrix inversion.

II. *Basic Pair-wise Hierarchies of Economic Linkages Sub-systems of the Block-matrix Input-output System.*

The building blocks of the pair-wise hierarchies of sub-systems of intra/interregional linkages of the block-matrix Input-Output system are the four matrices A_{11}, A_{12}, A_{21} and A_{22}, corresponding to four basic block-matrices:

$$A_{11}=\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix};\quad A_{12}=\begin{bmatrix}0 & A_{12}\\ 0 & 0\end{bmatrix};\quad A_{21}=\begin{bmatrix}0 & 0\\ A_{21} & 0\end{bmatrix};\quad A_{22}=\begin{bmatrix}0 & 0\\ 0 & A_{22}\end{bmatrix} \tag{7}$$

In this paper, we will usually consider the decomposition of the block-matrix (1) into the sum of two block-matrices, such that each of them is the sum of the block-matrices (7) A_{11}, A_{12}, A_{21} and A_{22}. Only 14 types of such pair-wise hierarchies of economic sub-systems can be identified by the following decompositions of the matrix of the block-matrix A :

I. *Hierarchy of backward linkages of first and second regions* :

$$A=\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

This hierarchy reflects "pull" economic linkages and appears also in the generalized Gauss-Fourier-Jourdan direct elimination method (Gantmacher, 1959).

II. *Order-replaced hierarchy of backward linkages of second and first regions* :

$$A=\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}+\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}$$

This hierarchy reflects the order-replaced "pull" economic linkages.

III. *Hierarchy of forward linkages of first and second regions* :

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

This hierarchy reflects "push" economic linkages and appears also in the generalized Gauss-Fourier-Jourdan dual elimination method.

IV. *Order-replaced hierarchy of forward linkages of second and first regions* :

$$A=\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}+\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$$

This hierarchy reflects the order-replaced "push" economic linkages.

V. *Hierarchy of isolated region versus the rest of economy* :

$$A=\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

This hierarchy reflects also the extraction method (Strassert, 1968) excluding from the economy backward and forward linkages of the second region. Such a hierarchy appears also in the bordering method of the matrix inversion (Faddeeva, 1959).

VI. *Hierarchy of the rest of economy versus second isolated region* :

$$A=\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}+\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

This decomposition reflects the order-replaced hierarchy of backward and forward linkages of the second region versus rest of economy.

VII. *The hierarchy, of backward and forward linkages of the first region versus rest of economy* :

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$$

VIII. *The order-replaced hierarchy of backward and forward linkages of the first region versus rest of economy* :

$$A=\begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}+\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

IX. *The hierarchy of intra-regional versus inter-regional economic relationships* :

$$A=\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

X. *The hierarchy of inter-regional versus intra-regional economic relationships* :

$$A=\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

XI. *The hierarchy of lower-triangular sub-system vs. interregional linkages of second region* :

$$A=\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$$

XII. *The order-replaced hierarchy of interregional linkages of second region vs. the lower-triangular sub-system* :

$$A=\begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

XIII. *The hierarchy of upper-triangular sub-system vs. interregional linkages of first region* :

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}$$

XIV. *The order-replaced hierarchy of interregional linkages of first region vs. the upper-triangular sub-system* :

$$A=\begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

It is important to stress that the many other hierarchies of sub-systems can be derived by summation of basic block-matrices A_{11}, A_{12}, A_{22} and A_{21}. For example,

$$A=[A_{11}+A_{22}]+A_{21}+A_{12}=\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}+\begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}.$$

We intend to construct the complete hierarchy of the linkages sub-systems later in the paper.

III. *Inner Regional Multipliers*

In this section, the existence of various inverse matrices is postulated. Moreover, all matrix identities can be verified by direct calculation. It should be noted that the historical references to the literature on matrix inversion of the last eighty years are taken mostly from Henderson and Searle (1981) and Miyazawa (1976). This section includes the set of inner regional multipliers, the set of inverse matrices which are the "building blocks" of our classification of the synergetic interactions between the economic sub-systems. These inner multipliers are included in table 1; hereafter, we present some comments on the entries in this table (the bold numbering refers to the corresponding entries in this table).

1. The matrices $B_1=(I-A_{11})^{-1}$ and $B_2=(I-A_{22})^{-1}$ represent the Miyazawa internal matrix multipliers of the first and second regions showing the interindustrial propagation effects within each region, while the matrices, $A_{21}B_1$, B_1A_{12}, $A_{12}B_2$, B_2A_{21} show the induced effects on output or input activities in the two regions (Miyazawa, 1966).

2. The expressions

$$S_1=I-A_{11}-A_{12}B_2A_{21},\quad S_2=I-A_{22}-A_{21}B_1A_{12} \tag{8}$$

are usually referred to as the Schur complements. Their analog first appeared in the following Schur's matrix identity (Schur, 1917):

$$\begin{bmatrix} B_1 & 0 \\ A_{21}B_1 & I \end{bmatrix}\begin{bmatrix} I-A_{11} & -A_{12} \\ -A_{21} & I-A_{11} \end{bmatrix}=\begin{bmatrix} I & -B_1A_{12} \\ 0 & I-A_{22}-A_{21}B_1A_{12} \end{bmatrix} \tag{9}$$

It is interesting to note that the transfer in (9) to inverse matrices gives the Schur-Banachiewicz inverse formula (Banachiewicz, 1937; Ouellette, 1978):

$$\begin{bmatrix} I-A_{11} & -A_{12} \\ -A_{21} & I-A_{11} \end{bmatrix}=\begin{bmatrix} I & D_1A_{12}B_2 \\ 0 & D_2 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ A_{21}B_1 & I \end{bmatrix} \tag{10}$$

This formula provides the basis for our classification of the synergetic interactions between the economic linkages sub-systems (see Table 2).

The inverses, D_1 and D_2 of the Schur complements (8) are referred to as the *Schur inverses* for the first and second regions. They represent the enlarged Leontief inverse for one region revealing the induced economic influence of the other region; i. e., the Schur inverses represent total propagation effects in the first and second regions. Their properties (Jossa, 1940; Duncan,

1944) indicate the additive and multiplicative dependencies between the Schur inverses D_1 and D_2.

3. Miyazawa (1966) introduced left and right external matrix multipliers of the first and second regions, D_{11}^{L}, D_{11}^{R}, D_{22}^{L}, D_{22}^{R}. These multipliers are incorporated in the multiplicative decompositions of the Schur inverses and they represent the total propagation effects in the first and second regions as the products of internal and external regional matrix multipliers (Miyazawa, 1966, 1976; Sonis and Hewings, 1993). These Miyazawa products imply the additive representations of Schur inverses.

4, 5. By introducing the abbreviated Schur inverses, D_{11}, D_{22}, and the left and right induced internal multipliers for the first and second regions, B_1^L, B_1^R, B_2^L, B_2^R, one can obtain the multiplicative decompositions of Schur inverses:

$$D_1=B_1^L D_{11}=D_{11} B_1^R;\quad D_2=B_2^L D_{22}=D_{22} B_2^R \tag{11}$$

and their corresponding additive representations.

6–10. The formulae for this group of multipliers can be obtained by considering the block-matrices:

$$M=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix},\quad N=\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{12} \end{bmatrix},\quad S=\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \tag{12}$$

that represent the backward and forward linkages of the first region, the second region and the interregional relations of both regions.

Miyazawa (1968, 1976) used such a block matrix for the analysis of the interrelationships among various income groups in the process of income formation. In the Miyazawa interpretation, the matrix A_{11} represents the interindustry direct inputs; the matrix A_{21} represents the value-added ratios of household sectors; and the matrix A_{12} represents the coefficients of consumption expenditures. Miyazawa considered the usual Leontief interindustry inverse $B_1=(I-A_{11})^{-1}$; the matrix multiplier $A_{21}B_1$ showing the induced income earned from production activities among industries; the matrix multiplier B_1A_{12} showing the induced production due to endogenous consumption (per unit of income) in each household sector; the matrix multiplier $A_{21}B_1A_{12}$ showing interrelationships among incomes through the process of propagation from consumption expenditures.

The matrix $D_{22}=(I-A_{21}B_1A_{12})^{-1}$ is interpreted as the interrelational income multiplier, and the matrix multiplier $D_{22}A_{21}B_1$ is interpreted as the matrix multiplier of income formation. Further, the following Schur inverse

$$D_1^*=(I-A_{11}-A_{12}A_{21})^{-1} \tag{13}$$

may be referred to as the enlarged Leontief inverse, and the inverses

$$D_{11}^{*L}=(I-B_1A_{12}A_{21})^{-1};\quad D_{11}^{*R}=(I-A_{12}A_{21}B_1)^{-1} \tag{14}$$

are called the left and right subjoined inverse matrix multipliers; they reflect the effects of endogenous changes in each income groups' consumption expenditures.

TABLE 1. INNER REGIONAL MULTIPLIERS AND THEIR PROPERTIES

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1. Internal regional multipliers:

Definitions:

$B_1=(I-A_{11})^{-1}$; $\quad B_2=(I-A_{22})^{-1}$

Properties:

$B_1=I+A_{11}B_1$; $\quad B_2=I+A_{22}B_2$

$A_{11}B_1=B_1A_{11}$; $\quad A_{22}B_2=B_2A_{22}$

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2. Schur complements and Schur inverses:

Definitions:

$S_1=I-A_{11}-A_{12}B_2A_{21}$; $\quad S_2=I-A_{22}-A_{21}B_1A_{12}$

$D_1=(I-A_{11}-A_{12}B_2A_{21})^{-1}$; $\quad D_2=(I-A_{22}-A_{21}B_1A_{12})^{-1}$

Properties:

$D_1=B_1+B_1A_{12}D_2A_{21}B_1$; $\quad D_2=B_2+B_2A_{21}D_1A_{12}B_2$

$D_1A_{12}B_2=B_1A_{12}D_2$; $\quad D_2A_{21}B_1=B_2A_{21}D_1$

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3. Left and right Miyazawa external matrix multipliers:

Definitions:

$D_{11}^L=(I-B_1A_{12}B_2A_{21})^{-1}$; $\quad D_{22}^L=(I-B_2A_{21}B_1A_{12})^{-1}$

$D_{11}^R=(I-A_{12}B_2A_{21}B_1)^{-1}$; $\quad D_{22}^R=(I-A_{21}B_1A_{12}B_2)^{-1}$

Properties:

$D_{11}^L=I+B_1A_{12}D_2A_{21}$; $\quad D_{22}^L=I+B_2A_{21}D_1A_{12}$

$A_{12}B_2A_{21}D_{11}^L=A_{12}D_2A_{21}$; $\quad A_{21}B_1A_{12}D_{22}^L=A_{21}D_1A_{12}$

$D_{11}^R=I+A_{12}D_2A_{21}B_1$; $\quad D_{22}^R=I-A_{21}D_1A_{12}B_2$

$D_{11}^R A_{12}B_2A_{21}=A_{12}D_2A_{21}$; $\quad D_{22}^R A_{21}B_1A_{12}=A_{21}D_1A_{12}$

$D_1=D_{11}^R+A_{11}D_1=D_{11}^L+D_1A_{11}$; $\quad D_2=D_{22}^R+A_{22}D_2=D_{22}^L+D_2A_{22}$

$D_1=B_1D_{11}^R+D_{11}^LB_1$; $\quad D_2=B_2D_{22}^R=D_{22}^LB_2$

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4. Abbreviated Schur inverses:

Definitions:

$D_{11}=(I-A_{12}B_2A_{21})^{-1}$; $\quad D_{22}=(I-A_{21}B_1A_{12})^{-1}$

Properties:

$D_{11}=I+A_{12}B_2A_{21}D_{11}$; $\quad D_{22}=I+A_{21}B_1A_{12}D_{22}$

$A_{12}B_2A_{21}D_{11}=D_{11}A_{12}B_2A_{21}$; $\quad A_{21}B_1A_{12}D_{22}=D_{22}A_{21}B_1A_{12}$

$D_1=D_{11}+D_{11}A_{11}D_1$; $\quad D_2=D_{22}+D_{22}A_{21}D_2$

$D_{11}A_{11}D_1=D_1A_{11}D_{11}$; $\quad D_{22}A_{21}D_2=D_2A_{21}D_{22}$

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5. Left and right induced internal multipliers:

Definitions:

$B_1^L=(I-D_{11}A_{11})^{-1}$; $\quad B_2^L=(I-D_{22}A_{22})^{-1}$

$B_1^R=(I-A_{11}D_{11})^{-1}$; $\quad B_2^R=(I-A_{22}D_{22})^{-1}$

Properties:

$B_1^L=I+D_{11}A_{11}B_1^L$; $\quad B_2^L=I+D_{22}A_{22}B_2^L$

$D_{11}A_{11}B_1^L=B_1^LD_{11}A_{11}$; $\quad D_{22}A_{22}B_2^L=B_2^LD_{22}A_{22}$

$B_1^R=I+A_{11}D_{11}B_1^R$; $\quad B_2^R=I+A_{22}D_{22}B_2^R$

$A_{11}D_{11}B_1^L=B_1^LA_{11}D_{11}$; $\quad A_{22}D_{22}B_2^L=B_2^LA_{22}D_{22}$

$D_1=B_1^R+A_{12}D_2A_{21}B_1$; $\quad D_2=B_2^R+A_{21}D_1A_{12}B_2$

$\quad =B_1^L+B_1A_{12}D_2A_{21}$; $\quad =B_2^L+B_2A_{21}D_1A_{12}$

$D_1=B_1^LD_{11}=D_{11}B_1^R$; $\quad D_2=B_2^LD_{22}=D_{22}B_2^R$

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6. Enlarged Leontief inverses:

Definitions:

$D_1^* = (I - A_{11} - A_{12} A_{21})^{-1}$; $\quad D_2^* = (I - A_{22} - A_{21} A_{12})^{-1}$

Properties:

$D_1^* = B_1 + B_1 A_{12} D_{22} A_{21}$; $\quad D_2^* = B_2 + B_2 A_{21} D_{11} A_{12} B_2$

$D_{22} A_{21} B_1 = A_{21} D_1^*$; $\quad D_2^* A_{21} = B_2 A_{21} D_{11}$

$D_{11} = I + A_{12} D_2^* A_{21}$; $\quad D_{22} = I + A_{21} D_1^* A_{12}$

$D_1^* A_{12} = B_1 A_{12} D_{22}$; $\quad D_{11} A_{12} B_2 = A_{12} D_2^*$

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7. Induced external multipliers:

Definitions:

$D_{11}^* = (I - A_{12} A_{21})^{-1}$; $\quad D_{22}^* = (I - A_{21} A_{12})^{-1}$

Properties:

$D_{11}^* = I + A_{12} D_{22}^* A_{21}$; $\quad D_{22}^* = I + A_{21} D_{11}^* A_{12}$

$D_{11}^* A_{12} = A_{12} D_{22}^*$; $\quad D_{22}^* A_{21} = A_{21} D_{11}^*$

$D_1^* = D_{11}^* + D_{11}^* A_{11} D_1^*$; $\quad D_2^* = D_{22}^* + D_{22}^* A_{22} D_2^*$

$D_{11}^* A_{11} D_1^* = D_1^* A_{11} D_{11}^*$; $\quad D_{22}^* A_{22} D_2^* = D_2^* A_{22} D_{22}^*$

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8. Left and right induced internal multipliers:

Definitions:

$B_1^{*L} = (I - D_{11}^* A_{11})^{-1}$; $\quad B_2^{*L} = (I - D_{22}^* A_{22})^{-1}$

$B_1^{*R} = (I - A_{11} D_{11}^*)^{-1}$; $\quad B_2^{*R} = (I - A_{22} D_{22}^*)^{-1}$

Properties:

$B_1^{*L} = I + D_1^* A_{11}$; $\quad B_2^{*L} = I + D_2^* A_{22}$

$D_1^* A_{11} = D_{11}^* A_{11} B_1^{*L}$; $\quad D_2^* A_{22} = D_{22}^* A_{22} B_2^{*L}$

$B_1^{*R} = I + A_{11} D_1^*$; $\quad B_2^{*R} = I + A_{22} D_2^*$

$A_{11} D_1^* = B_1^{*R} A_{11} D_{11}^*$; $\quad A_{22} D_2^* = B_2^{*R} A_{22} D_{22}^*$

$D_1^* = B_1^{*L} D_{11}^* = D_{11}^* B_1^{*R}$; $\quad D_2^* = B_2^{*L} D_{22}^* = D_{22}^* B_2^{*R}$

$D_1^* = B_1^{*R} + A_{12} D_{22} A_{21} B_1$; $\quad D_2^* = B_2^{*R} + A_{21} D_{11} A_{12} B_2$

$\quad = B_1^{*L} + B_1 A_{12} D_{22} A_{21}$; $\quad = B_2^{*L} + B_2 A_{21} D_{11} A_{12}$

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9. Left and right subjoined inverses:

Definitions:

$D_{11}^{*L} = (I - B_1 A_{12} A_{21})^{-1}$; $\quad D_{22}^{*L} = (I - B_2 A_{21} A_{12})^{-1}$

$D_{11}^{*R} = (I - A_{12} A_{21} B_1)^{-1}$; $\quad D_{22}^{*R} = (I - A_{21} A_{12} B_2)^{-1}$

Properties:

$D_1^* = B_1 D_{11}^{*R} = D_{11}^{*L} B_1$; $\quad D_2^* = B_2 D_{22}^{*R} = D_{22}^{*L} B_2$

$D_1^* = D_{11}^{*R} + A_{11} D_1^* = D_1^{*L} + D_1^* A_{11}$; $\quad D_2^* = D_{22}^{*R} + A_{22} D_2^* = D_{22}^{*L} + D_2^* A_{22}$

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10. Left and right induced subjoined inverses:

Definitions:

$D_{11}^{**L} = [I - D_{11}(A_{11} - A_{12} B_2 A_{22} A_{21})]^{-1}$; $\quad D_{22}^{**L} = [I - D_{22}(A_{22} - A_{21} B_1 A_{11} A_{12})]^{-1}$

$D_{11}^{**R} = [I - (A_{11} - A_{12} B_2 A_{22} A_{21}) D_{11}]^{-1}$; $\quad D_{22}^{**R} = [I - (A_{22} - A_{21} B_1 A_{11} A_{12}) D_{22}]^{-1}$

Properties:

$D_1^* = D_{11}^{**L} D_{11} = D_{11} D_{11}^{**R}$; $\quad D_2^* = D_{22}^{**L} D_{22} = D_{22} D_{22}^{**R}$

$D_1^* = D_{11} + D_1^* D_{11}(A_{11} - A_{12} B_2 A_{22} A_{21})$; $\quad D_2^* = D_{22} + D_2^* D_{22}(A_{22} - A_{21} B_1 A_{11} A_{12})$

The properties of **6–10** can be obtained from **2–4** by putting $A_{22}=0$, $B_2=I$ or $A_{11}=0$, $B_1=I$ or $A_{11}=0$, $B_1=0$ and $A_{22}=0$, $B_2=I$.

IV. *Leontief Block-matrix Inverses and Classification of Outer Left and Right Block-matrix Multipliers.*

Consider the hierarchy of Input-Output sub-systems represented by the decomposition $A=A_1+A_2$. Let us introduce the Leontief block-inverse $L(A)=L=(I-A)^{-1}$ and the Leontief block-inverse $L(A_1)=L_1=(I-A_1)^{-1}$ corresponding to the first sub-system. The outer left and right block-matrix multipliers M_L and M_R are defined by equalities:

$$L=L_1 M_R=M_L L_1 \tag{15}$$

The definition (15) implies immediately that:

$$M_L=L(I-A_1)=(I-L_1 A_2)^{-1} \tag{16}$$

$$M_R=(I-A_1)L=(I-A_2 L_1)^{-1} \tag{17}$$

The calculation of the outer block-multipliers M_L and M_R is based on the particular form of the Leontief block-inverse $L(A)=L$. In this study we will apply the following form of the Leontief block-inverse (Banachiewicz, 1937; Aitken, 1937; Hotelling, 1943; Duncan, 1944; see also Miyazawa, 1976; Sonis and Hewings, 1993):

$$L=\begin{bmatrix} D_1 & D_1 A_{12} B_2 \\ D_2 A_{21} B_1 & D_2 \end{bmatrix} \tag{18}$$

This formula can be verified by direct matrix multiplication, using definitions of the Schur inverses and their properties (see table 1, entries 1 and 2). Further, we will present the application of formulas (16), (17) and (18) to the derivation of a taxonomy of synergetic interactions between regions. All formulas can be verified by direct matrix multiplications using the formulas from the Table 1. The results are presented in the first and second levels of table 2.

TABLE 2. TAXONOMY OF SYNERGETIC INTERACTIONS BETWEEN ECONOMIC SUB-SYSTEM, THEIR OUTER BLOCK-MATRIX MULTIPLIERS AND ADDITA COMPONENTS OF MULTIPLICATIVE AND ADDITIVE DECOMPOSITIONS OF THE LEONTIEF INVERSES

[Each entry consists of three levels: in the first level, a description of the structure and the corresponding form of the A matrix is shown. In the second level, the outer and inner multipliers are presented. On the third level, the additive decompositions of the Leontief block-matrix are shown]

Level 1	*Description*	*Form of the A_1 matrix*
	Outer multipliers	*Inner multipliers*
Level 2	$M_L \qquad L_1=L(A_1) \qquad M_R$	
Level 3	$L=L_1+(M_L-I)L_1=L_1+L_1(M_R-I)$	

I. *Hierarchy of backward linkages of first and second regions*

$$A_1=\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}:$$

$$\begin{bmatrix} I & D_1A_{12}B_2 \\ 0 & D_2 \end{bmatrix} \quad \begin{bmatrix} B_1 & 0 \\ A_{21} & I \end{bmatrix} \quad \begin{bmatrix} D_{11}^R & A_{12}D_2 \\ A_{22}D_2A_{21}B_1 & B_2^R \end{bmatrix} \quad \begin{matrix} D_{11}^R \\ B_2^R \end{matrix}$$

$$L=\begin{bmatrix} B_1 & 0 \\ A_{21}B_1 & I \end{bmatrix}+\begin{bmatrix} B_1A_{12} \\ I-S_2 \end{bmatrix}D_2[A_{21}B_1 \quad I]$$

II. *Order replaced hierarchy of backward linkages*

$$A_1=\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}:$$

$$\begin{bmatrix} D_1 & 0 \\ D_2A_{21}B_1 & I \end{bmatrix} \quad \begin{bmatrix} I & A_{12}B_2 \\ 0 & B_2 \end{bmatrix} \quad \begin{bmatrix} B_1^R & A_{11}B_1A_{12}D_2 \\ A_{21}D_1 & D_{22}^R \end{bmatrix} \quad \begin{matrix} B_1^R \\ D_{22}^R \end{matrix}$$

$$L=\begin{bmatrix} I & A_{12}B_2 \\ 0 & B_2 \end{bmatrix}+\begin{bmatrix} I-S_2 \\ B_2A_{21} \end{bmatrix}D_1[I \quad A_{12}B_2]$$

III. *Hierarchy of forward linkages of first and second regions*

$$A_1=\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}:$$

$$\begin{bmatrix} D_{11}^L & D_1A_{12}B_2A_{22} \\ D_2A_{21} & B_2^L \end{bmatrix} \quad \begin{bmatrix} B_1 & B_1A_{12} \\ 0 & I \end{bmatrix} \quad \begin{bmatrix} I & 0 \\ D_2A_{21}B_1 & D_2 \end{bmatrix} \quad \begin{matrix} D_{11}^L \\ B_2^L \end{matrix}$$

$$L=\begin{bmatrix} B_1 & B_1A_{12} \\ 0 & I \end{bmatrix}+\begin{bmatrix} B_1A_{12} \\ I \end{bmatrix}D_2[A_{21}B_1 \quad I-S_2]$$

IV. *Order replaced hierarchy of forward linkages*

$$A_1=\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}:$$

$$\begin{bmatrix} B_1^L & D_1A_{12} \\ D_2A_{21}B_1A_{11} & D_{22}^L \end{bmatrix} \quad \begin{bmatrix} I & 0 \\ B_2A_{21} & B_2 \end{bmatrix} \quad \begin{bmatrix} D_1 & D_1A_{12}B_2 \\ 0 & I \end{bmatrix} \quad \begin{matrix} B_1^L \\ D_{22}^L \end{matrix}$$

$$L=\begin{bmatrix} I & 0 \\ B_2A_{21} & B_2 \end{bmatrix}+\begin{bmatrix} 1 \\ B_2A_{21} \end{bmatrix}D_1[I-S_2 \quad A_{12}B_2]$$

V. *Hierarchy of isolated region versus the rest of economy* $A_1=\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$:

$$\begin{bmatrix} D_{11}^L & D_1 A_{12} B_2 \\ D_2 A_{21} & D_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{11}^R & A_{12} D_2 \\ D_2 A_{21} B_1 & D_2 \end{bmatrix} \quad D_{11}^R;\ D_{11}^R$$

$$L=\begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix}+\begin{bmatrix} B_1 A_{12} & 0 \\ 0 & I \end{bmatrix} D_2 \begin{bmatrix} I & I \\ I & I-S_2 \end{bmatrix}\begin{bmatrix} A_{21} B_1' & 0 \\ 0 & I \end{bmatrix}$$

VI. *Hierarchy of the rest of economy versus second isolated region* $A_1=\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$:

$$\begin{bmatrix} B_1^L & 0 \\ D_2 A_{21} B_1 A_{11} & I \end{bmatrix} \begin{bmatrix} D_{11} & A_{21} D_2^* \\ D_2^* A_{21} & D_2^* \end{bmatrix} \begin{bmatrix} B_1^R & A_{11} B_1 A_{12} D_2 \\ 0 & I \end{bmatrix} \quad \begin{matrix} B_1^L; B_1^R; D_{11} \\ D_2^* \end{matrix}$$

$$L=\begin{bmatrix} D_{11} & A_{21} D_2^* \\ D_2^* A_{21} & D_2^* \end{bmatrix}+\begin{bmatrix} 1 \\ B_2 A_{21} \end{bmatrix} D_{11} A_{11} D_1 [I \quad A_{12} B_2]$$

VII. *The hierarchy of backward and forward linkages of the first region versus rest of economy* $A_1=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$:

$$\begin{bmatrix} I & D_1 A_{12} B_2 A_{22} \\ 0 & B_2^L \end{bmatrix} \begin{bmatrix} D_1^* & D_1^* A_{12} \\ A_{21} D_1^* & D_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22} D_2 A_{21} B_1 & B_2^R \end{bmatrix} \quad \begin{matrix} D_1^* \\ B_2^L; B_2^R; D_{22} \end{matrix}$$

$$L=\begin{bmatrix} D_1^* & D_1^* A_{12} \\ A_{21} D_1^* & D_{22} \end{bmatrix}+\begin{bmatrix} B_1 A_{12} \\ 0 \end{bmatrix} D_2 D_{22} A_{22} [A_{21} B_1 \quad I]$$

VIII. *The order replaced hierarchy of backward and forward linkages of the first region versus rest of economy* $A_1=\begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$:

$$\begin{bmatrix} D_1 & D_1 A_{12} \\ D_2 A_{21} B_1 & D_{22}^L \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} D_1 & D_1 A_{12} B_2 \\ A_{21} D_1 & D_{22}^R \end{bmatrix} \quad D_{22}^L; D_{22}^R$$

$$L=\begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix}+\begin{bmatrix} I & 0 \\ 0 & B_2 A_{21} \end{bmatrix} D_1 \begin{bmatrix} I-S_1 & I \\ I & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_{12} B_2 \end{bmatrix}$$

IX. *The hierarchy of intra- versus inter- regional relationships* $A_1=\begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$:

$$\begin{bmatrix} D_1 & D_1 A_{12} \\ D_2 A_{21} B_1 & D_{22}^L \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} D_1 & D_1 A_{12} B_2 \\ A_{21} D_1 & D_{22}^R \end{bmatrix} \quad D_{22}^L; D_{22}^R$$

$$L=\begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix}+\begin{bmatrix} I & 0 \\ 0 & B_2 A_{21} \end{bmatrix} D_1 \begin{bmatrix} I-S_1 & I \\ I & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_{12} B_2 \end{bmatrix}$$

X. *The hierarchy of inter versus intra regional relationships*

$$A_1=\begin{bmatrix}0 & A_{12}\\ A_{21} & 0\end{bmatrix}:$$

$$\begin{bmatrix}B_1^L & D_1A_{12}B_2A_{22}\\ D_2A_{21}B_1A_{11} & B_2^L\end{bmatrix}\quad\begin{bmatrix}D_{11}^* & D_{11}^*A_{12}\\ D_{22}^*A_{21} & D_{22}^*\end{bmatrix}\quad\begin{bmatrix}B_1^R & A_{11}D_1A_{12}B_2\\ A_{22}D_2A_{21}B_1 & B_2^R\end{bmatrix}\quad\begin{matrix}D_1^*\\ B_2^L;B_2^R;D_{22}\end{matrix}$$

$$L=\begin{bmatrix}D_{11}^* & D_{11}^*A_{12}\\ D_{22}^*A_{21} & D_{22}^*\end{bmatrix}+\begin{bmatrix}I & B_1A_{12}\\ B_2A_{21} & B_2\end{bmatrix}\begin{bmatrix}D_1A_{11}D_{11}^* & 0\\ 0 & D_2A_{22}D_{22}^*\end{bmatrix}\begin{bmatrix}I & A_{12}\\ A_{21} & I\end{bmatrix}$$

XI. *The hierarchy of lower triangular sub system versus Interregional linkages of second region*

$$A_1=\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}:$$

$$\begin{bmatrix}D_{11}^L & D_1A_{12}\\ D_2A_{21} & D_{22}^L\end{bmatrix}\quad\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}\quad\begin{bmatrix}D_{11}^R & A_{12}D_2\\ A_{21}D_1 & D_{22}^R\end{bmatrix}\quad\begin{matrix}D_{11}^L;D_{11}^R\\ D_{22}^L;D_{22}^R\end{matrix}$$

$$L=\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}+\begin{bmatrix}D_1A_{12}B_2 & 0\\ 0 & D_2A_{21}B_1\end{bmatrix}\begin{bmatrix}A_{21} & I-A_{22}\\ I-A_{11} & A_{12}\end{bmatrix}\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}$$

XII. *The order replaced hierarchy of interregional linkages of second region versus lower triangular sub system*

$$A_1=\begin{bmatrix}0 & A_{12}\\ 0 & 0\end{bmatrix}:$$

$$\begin{bmatrix}D_1 & D_1A_{12}B_2A_{22}\\ D_2A_{21}B_1 & B_2^L\end{bmatrix}\quad\begin{bmatrix}I & A_{12}\\ 0 & I\end{bmatrix}\quad\begin{bmatrix}B_1^R & A_{11}D_1A_{12}B_2\\ D_2A_{21}B_1 & B_2^R\end{bmatrix}\quad\begin{matrix}B_1^R\\ B_2^L\end{matrix}$$

$$L=\begin{bmatrix}I & A_{12}\\ 0 & I\end{bmatrix}+\begin{bmatrix}D_1 & 0\\ 0 & D_2\end{bmatrix}+\begin{bmatrix}I-S_1 & A_{12}B_2-S_1A_{12}\\ A_{21}B_1 & I-S_2\end{bmatrix}$$

XIII. *The hierarchy of upper triangular sub system versus interregional linkages of first region*

$$A_1=\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}:$$

$$\begin{bmatrix}D_{11}^L & 0\\ D_2A_{21} & I\end{bmatrix}\quad\begin{bmatrix}B_1 & B_1A_{12}B_2\\ 0 & B_2\end{bmatrix}\quad\begin{bmatrix}D_{11}^R & 0\\ A_{21}D_1 & D_{22}^R\end{bmatrix}\quad D_{11}^L;D_{11}^R$$

$$L=\begin{bmatrix}B_1 & B_1A_{12}B_2\\ 0 & B_2\end{bmatrix}+\begin{bmatrix}B_1A_{12}\\ I\end{bmatrix}+D_2A_{21}B_1[I\quad A_{12}B_2]$$

XIV. *The order replaced hierarchy of interregional linkages of first region versus upper triangular sub system*

$$A_1=\begin{bmatrix}0 & 0\\ A_{21} & 0\end{bmatrix}:$$

$$\begin{bmatrix}B_1^L & D_1A_{12}B_2\\ D_2A_{21}B_1 & D_2\end{bmatrix}\quad\begin{bmatrix}I & 0\\ A_{21} & I\end{bmatrix}\quad\begin{bmatrix}D_1 & D_1A_{12}B_2\\ A_{22}D_2A_{21}B_1 & B_2^R\end{bmatrix}\quad\begin{matrix}B_1^L\\ B_2^R\end{matrix}$$

$$L=\begin{bmatrix}I & 0\\ A_{21} & I\end{bmatrix}+\begin{bmatrix}D_1 & 0\\ 0 & D_2\end{bmatrix}+\begin{bmatrix}I-S_1 & A_{12}B_2\\ A_{21}B_1-S_2A_{21} & I-S_2\end{bmatrix}$$

V. *Outer Block-decompositions of the Leontief Inverses.*

It is important to note that the form of the Leontief block-inverse that is used in (18) implies the following triple decompositions which separates multiplicatively the effects of intra-regional economic relationships of isolated regional economies $\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, the inter-regional feedback effects $\begin{bmatrix} I-A_{11} & A_{12} \\ A_{21} & I-A_{22} \end{bmatrix}$ and the intra-regional economic dependencies of interacting regions $\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$:

$$\begin{aligned} L &= \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I & A_{12}B_2 \\ A_{21}B_1 & I \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I-A_{11} & A_{12} \\ A_{21} & I-A_{22} \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} I & B_1A_{12} \\ B_2A_{21} & I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I-A_{11} & A_{12} \\ A_{21} & I-A_{22} \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \end{aligned} \tag{19}$$

Equations (19) are the block-matrix analogues of the Miyazawa decompositions of the Schur inverses into the product of external/internal multipliers. The application of different forms of the Miyazawa decompositions from Table 1 provides the further possibilities of construction of another block-matrix analog of (19).

First, the Miyazawa product of external/internal multipliers

$$D_1 = B_1 D_{11}^R = D_{11}^L B_1; \quad D_2 = B_2 D_{22}^R = D_{22}^L B_2 \tag{20}$$

implies the following outer decompositions:

$$\begin{aligned} L &= \begin{bmatrix} D_{11}^L & 0 \\ 0 & D_{22}^L \end{bmatrix} \begin{bmatrix} I & B_1A_{12} \\ B_2A_{21} & I \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I & A_{12}B_2 \\ A_{21}B_1 & I \end{bmatrix} \begin{bmatrix} D_{11}^R & 0 \\ 0 & D_{22}^R \end{bmatrix} \end{aligned} \tag{21}$$

Further, the Miyazawa type products:

$$D_1 = B_1^L D_{11} = D_{11} B_1^R; \quad D_2 = B_2^L D_{22} = D_{22} B \tag{22}$$

and

$$D_1^* = D_{11}^{**L} D_{11} = D_{11} D_{11}^{**R}; \quad D_2^* = D_{22}^{**L} D_{22} = D_{22} D \tag{23}$$

generate another outer decomposition:

$$
\begin{aligned}
L&=\begin{bmatrix} D_{11} & D_1^* A_{12} \\ D_2^* A_{21} & D_{22} \end{bmatrix}\begin{bmatrix} B_1^R & 0 \\ 0 & B_2^R \end{bmatrix}=\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}\begin{bmatrix} I & D_{11}^{**R} A_{12} \\ D_{22}^{**R} A_{21} & I \end{bmatrix}\begin{bmatrix} B_1^R & 0 \\ 0 & B_2^R \end{bmatrix} \\
&=\begin{bmatrix} B_1^L & 0 \\ 0 & B_2^L \end{bmatrix}\begin{bmatrix} D_{11} & A_{12} D_2^* \\ A_{21} D_1^* & D_{22} \end{bmatrix}=\begin{bmatrix} B_1^L & 0 \\ 0 & B_2^L \end{bmatrix}\begin{bmatrix} I & A_{12} D_{22}^{**R} \\ A_{21} D_{11}^{**R} & I \end{bmatrix}\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}
\end{aligned}
\tag{24}
$$

An entirely different type of outer decompositions are generated within the Miyazawa income generation scheme. For the subsystem $M=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$ the Leontief block-inverse has the following outer decomposition which separates the backward and forward linkages effects:

$$
L(M)=\begin{bmatrix} D_1^* & D_1^* A_{12} \\ A_{21} D_1^* & D_{22} \end{bmatrix}=\begin{bmatrix} I & 0 \\ A_{21} & I \end{bmatrix}\begin{bmatrix} D_1^* & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & A_{12} \\ 0 & I \end{bmatrix} \tag{25}
$$

An analogous formulation holds for the subsystem $N=\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$::

$$
L(N)=\begin{bmatrix} D_{11} & A_{12} D_2^* \\ D_2^* A_{21} & D_2^* \end{bmatrix}=\begin{bmatrix} I & A_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & D_2^* \end{bmatrix}\begin{bmatrix} I & 0 \\ A_{21} & I \end{bmatrix} \tag{26}
$$

Equations (22) and (23) represents the analytical basis for the demo-economic input-output system proposed by Madden and Batey (1983).

VI. *Additive hierarchical decompositions of the Leontief block-inverses.*

Consider the hierarchy of input-output sub-systems represented by the decomposition $A=A_1+A_2$ and their Leontief block-inverse $L(A)=L=(I-A)^{-1}$ and the Leontief block-inverse $L(A_1)=L_1=(I-A_1)^{-1}$ corresponding to the first sub-system. The multiplicative decomposition of the Leontief inverse $L=L_1 M_R=M_L L_1$ can be easily converted to the sum:

$$
L=L_1+(M_L-I)L_1=L_1+L_1(M_R-I) \tag{27}
$$

If f is the vector of final demand and x is the vector of gross output, then the decomposition (27) generates the decomposition of gross output into two parts: $x_1=L_1 f$ and the increment $Dx=x-x_1$. Such a decomposition is important for the empirical analysis of the structure of actual gross output. In the third levels of table 2, we present the classification of possible additive decompositions of the Leontief block-inverse for all decompositions of input-output system into the pair-vise hierarchies.

VII. *Evaluation*

While 14 types of pair-wise hierarchies of economic linkages have been developed, it is possible to suggest a typology of categories into which these types may be placed. The following characterization is suggested:

1. backward linkage type (I, II): power of dispersion
2 forward linkage type (III, IV): sensitivity of dispersion
3. intra- and inter- linkages type (IX, X): internal and external dispersion
4. isolated region vs. the rest of the economy interactions style (V, VI, VII, VIII)
5. triangular sub-system vs. the interregional interactions style (XI, XII, XIII, XIV).

By viewing the system of hierarchies of linkages in this fashion, it will be possible to provide new insights into the properties of the structures that are revealed. For example, the types allocated to category 5 reflect structures that are based on order and circulation. Furthermore, these partitioned input-output systems can distinguish among the various types of dispersion (such as 1, 2 and 3) and among the various patterns of interregional interactions (such as 4 and 5). Essentially, the 5 categories and 14 types of pair-wise hierarchies of economic linkages provide the opportunity to select according to the special qualities of each region's activities and for the type of problem at hand; in essence, the option exists for the basis of a typology of economy types based on hierarchical structure.

VIII. *Conclusions*

The theoretical developments provided here set the stage for some important empirical analysis; for example, examinations of the structure of trading relationships among regions within a country (such as Japan) or between nations (for example, within the European Union or the Pacific region). Furthermore, the structure of an economy as revealed by many standard macroeconomic models often masks important differences in the nature of the internal and external interdependencies. As the analysis of structure moves beyond consideration of 2x2 (region versus the rest of the economy) formulations, the notions of hierarchy assume even more importance. In addition, an important additional extension remains to be explored, namely the structure of decompositions when matrices for more than one point in time exist. Here, the potential linkages between structure, hierarchy and decomposition could be integrated with some of the methodology explored by Sonis *et al.* (1996) in the identification of the sources for structural change.

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