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INCOME INEQUALITY AND TAX PROGRESSION*

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Abstract

Asuming that the population size is fixed, this paper attempts to develop necessary and sufficiant conditions for a tax function to be unambiguously inequality reducing.

I. *Introduction*

The relationship between tax progressivity and income inequality has been studied by many authors in recent times. Kakwani (1977) showed that *average progression*, that is, increasing average tax liability will make the post-tax distribution more equitable than the pre-tax distribution according to the Lorenz criterion. Jakobsson (1976) and later on Eichhorn, Funke and Richter (1984) and Thon (1987) showed that the implication is, in fact, an equivalence. Precisely it is proved that the Lorenz curve of income after tax will dominate the one before tax if and only if the tax function is average progressive and weekly incentive preserving, where *weak incentive preservation* (IP) means that the post-tax income is a non-decreasing function of the pre-tax income [see Eichhorn, Funke and Richter (1984)][1]. This result is based on a particular concept of distributive justice which demands invariance of inequality under equiproportionate changes in all incomes. Inequality indices satisfying this property are called relative indices. Another possibility is to assume that inequality incices are of absolute type-they should remain unaltered under equal absolute changes in all incomes. The problem of choice between these two approaches is essentially a matter of value judgement and a discussion on their relative merits and demerits could be endless [see Kolm (1976) and Blackorby and Donaldson (1980)].

The absolute counterpart to the Eichhorn-Funke-Richter (EFR) result is due to Moyes (1988). He showed that *minimal progressivity* (that is, increasingness of the tax liability) along with IP is necessary and sufficient for the tax function to be uniformly equalizing according to

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[1] For demonstrating that an average progressive tax function reduces inequality, Kakwani (1977) implicitly assumed that it is weakly incentive preserving. In fact, in almost all such distributional comparisons, the preservation property is taken as an assumption. See, for example, Jakobsson (1976), Lambert (1989) and Chakravarty (1990).

the absolute Lorenz domination[2]. It is important to note that both the EFR and Moyes results are proved for tax functions that are assumed to be independent of the population size.

In this paper we propose two dominance criteria ($R1$ and $R2$) that are based on a concept of Runciman (1966). Following EFR and Moyes, we then look for the population size independent tax functions that are equivalent to $R1$ and $R2$ respectively. Interestingly, it turns out that the tax function corresponding $R1$ ($R2$) is identical to the Moyes (EFR) function. This establishes, under the assumption of population size independence, the equivalence between $R2$ ($R1$) and the Lorenz (absolute Lorenz) criterion. But we find that if the population size is fixed, then $R1$ and $R2$ are different from the absolute Lorenz and the Lorenz dominations respectively. It will therefore be interesting to see what property a population size dependent tax function must have in order that it becomes inequality reducing. We show that a tax function satisfying *IP* (and depending on the population size) becomes equalizing in the absolute Lorenz sense if and only if it meets *subgroup minimal progressivity*, a weaker concept than minimal progressivity. A similar result is derived for the Lorenz domination case.

The paper is organised as follows. The next section sets up the notation and definitions employed in the paper. Section 3 presents the main findings and Section 4 makes some concluding remarks.

II. *Notation and Definitions*

The set of feasible income distributions in an n-person society is $D^n=\{x\varepsilon R^n \mid x_i>0$ for all $i\}$, where R^n is the n-dimensional Euclidean space and the positive integer $n\geq 2$ is arbitrary. Without loss of generality assume that all income distributions are illfare ranked, that is, for all $x\varepsilon D^n$, $x_1\leq x_2\leq \ldots \leq x_n$. For any $x\varepsilon D^n$, let $\lambda(x)$ be the mean of x.

Given arbitrary p, $q\varepsilon D^n$, we say that p weakly dominates q according to the Lorenz criterion ($p>_{=L} q$ for short) if the Lorenz curve of p lies nowhere below that of q, that is,

$$\sum_{i=1}^{k} p_i / n\lambda(p) \geq \sum_{i=1}^{k} q_i / n\lambda(q) \tag{1}$$

for all k, $1\leq k\leq n$. The relation $p>_{=L} q$ is equivalent to the condition that q is at least as unequal as p by all relative inequality indices that satisfy S-convexity [see Foster (1985), Foster and Shorrocks (1988) and Chakravarty (1990)]. According to S-convexity a rank preserving transfer of income from a rich person to a poor person does not increase inequality. Moyes (1987) showed that q is at least as unequal as p by all absolute S-convex inequality indices if and only if the absolute Lorenz curve of p weakly dominates that of q ($p>_{=LA} q$ for short), that is, if and only if,

$$\sum_{i=1}^{k} (p_i-\lambda(p))/n \geq \sum_{i=1}^{k} (q_i-\lambda(q))/n \tag{2}$$

for all k, $1\leq k\leq n$. We speak of strong domination in (1) and (2) if strict inequality holds for at least one $k<n$.

Next, we present two ranking relations building on an idea of Runciman (1966).

[2] A generalisation of EFR and Moyes' results was developed in Pfingsten (1988).

According to Runciman for any person in a society the extent of relative deprivation arising out of the comparison of his situation with that of a better off person is given by the difference between the latter and the former. In view of this, the ith person's deprivation in comparison with jth person's income x_j, where $x_i \leq x_j$, can be taken as $(x_j - x_i)$. Now, person i is deprived of all incomes higher than x_i. Therefore, the total deprivation felt by person i in comparison with all incomes higher than x_i is $\sum_{j=i+1}^{n}(x_j - x_i)$. Given $p, q \varepsilon D^n$ we say that p weakly dominates q by Runciman criterion 1 ($p >_{=R1} q$ for short) if

$$\sum_{i=k+1}^{n}(q_i - q_k) \geq \sum_{i=k+1}^{n}(p_i - p_k) \tag{3}$$

for all k, $1 \leq k \leq n-1$. Thus, R1 means that for any person the aggregate deprivation under q is at least as large as that under p.

Now, instead of considering simple income differences, as is done in (3), we can look at the utility differences of the form $U(x_j) - U(x_i)$, where U is increasing and concave. Assuming that $U(z) = \log z$, we say that for $p, q \varepsilon D^n$, p weakly dominates q by Runciman criterion 2 ($p >_{=R2} q$ for short) if for all k, $1 \leq k \leq n-1$,

$$\sum_{i=k+1}^{n}(\log q_i - \log q_k) \geq \sum_{i=k+1}^{n}(\log p_i - \log p_k). \tag{4}$$

For strict dominance according to $R1$ ($R2$) we require at least one strict inequality in (3) ((4)).[3]

A taxation scheme is a function f: $D^1 \rightarrow D^1$ that associates a pre-tax income u to a post-tax income $f(u)$; $t(u) = u - f(u)$ is the tax liability. The person with income level u will be called a tax-payer, unaffected or subsidized according as $t(u)$ is positive, zero or negative. For any $x \varepsilon D^n$, we write $f(x)$ for $(f(x_1), \ldots, f(x_n))$.

We now state some properties of income taxation in terms of an arbitrary f: $D^1 \rightarrow D^1$.

(a) Weak Incentive Preservation (IP): f is non-decreasing in pre-tax incomes, that is, for all $u > v > 0$, $f(u) \geq f(v) > 0$.

(b) Weak Average Progression (AP): Average tax rate is non-decreasing in pre-tax income, that is, for all $u > v > 0$, $(u - f(u))/u \geq (v - f(v))/v$.

(c) Weak Minimal Progression (MP): Tax liability is non-decreasing in pre-tax incomes, that is, for all $u > v > 0$, $u - f(u) \geq v - f(v)$.

(d) Weak Subgroup Minimal Progression (SM): Let $n \geq 2$ be given. Then for any partitioning of the population into two groups, the poor and the rich, the average tax liability of the latter should be at least as large as that of the former. That is, given the pre-tax income distribution x, for any k, $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k}(x_i - f(x_i))/k \leq \sum_{i=k+1}^{n}(x_i - f(x_i))/(n-k)$$

(e) Weak Subgroup Admissibility (SA): Let $n \geq 2$ be given. Then the cumulative tax rate of the bottom k $(0 \leq k \leq 1)$ proportion of the population does not exceed the aggregate tax rate. That

[3] For further discussions on the ordering R1, particularly for its social welfare implications, see Chakravarty (1996) and Chakravarty, Chattopadhyay and Majumder (1995).

is, given pre-tax distribution x, for any k, $1 \leq k \leq n$,

$$(n(\lambda(x)-\lambda(f(x)))/n\lambda(x)) \geq (\sum_{i=1}^{k}(x_i-f(x_i))/\sum_{i=1}^{k}x_i).$$

Properties IP and MP were introduced by Fei (1981). It should be noted that if tax liabilities are positive then AP implies MP. Furthermore, MP implies SM, but the converse is not true. However, MP and SM are equivalent in *a* two-person society. In (e) if the tax function satisfies MP, then SA means that the ratio between cumulative taxes and pre-tax incomes is not larger than the ratio between total tax and pre-tax aggregate income, which in turn means that a larger share of tax burden falls on the higher income groups. This is one notion of progressivity suggested by Jakobsson.[4] It should be evident that while the first three properties are independent of the population size n, the last two properties depend explicitly on n. Strong versions of IP, AP, MP, SM and SA can be defined by replacing the weak inequalities in the above definitions by strict inequalities.

III. *The Results*

We will state/prove only weak versions of the different results. Strong versions can be stated/proved under appropriate modifications. We begin the section with the formal presentation of the EFR and Moyes results.

Proposition 1 [Eichhorn, Funke and Richter (1984)].
For all $n \geq 2$, for all $x \varepsilon D^n$, $f(x) >_{=L} x$ if and only if the tax function satisfies IP and AP.

Proposition 2 [Moyes (1988)].
For all $n \geq 2$, for all $x \varepsilon D^n$, $f(x) >_{=LA} x$ if and only if the taxation scheme satisfies IP and MP.

The next result identifies the tax structures that agree with the ranking criteria $R1$ and $R2$ respectively.

Proposition 3
(a) For all $n \geq 2$, for all $x \varepsilon D^n$, $f(x) >_{=R1} x$ if and only if the tax function satisfies IP and MP.
(b) For all $n \geq 2$, for all $x \varepsilon D^n$, $f(x) >_{=R2} x$ if and only if the taxation scheme satisfies IP and AP.

Proof (a) *Sufficiency*: By IP, the post-tax income vector is illfare ranked. Note that MP implies

$$x_i - x_{i-1} \geq f(x_i) - f(x_{i-1}) \tag{5}$$

for all $i=2, 3 \ldots n$. Now

$$\sum_{i=k+1}^{n}(x_i-x_k) = x_{k+1}-x_k+x_{k+2}-x_k+\ldots$$

[4] This requirement has been shown to be equivalent to the condition that liability progression, the elasticity of tax liability with respect to income before tax, should not be smaller than unity at all income levels (see Jakobsson (1976) and Chakravarty (1990).

$$=x_{k+l}-x_k+x_{k+2}-x_{k+l}+x_{k+l}-x_k+\dots$$

$$\geq f(x_{k+l})-f(x_k)+f(x_{k+2})-f(x_{k+l})+f(x_{k+l})-f(x_k)+\dots$$

(from (5))

$$=\sum_{i=k+1}^{n}(f(x_i)-f(x_k)).$$

Hence we have $f(x)\geq_{R1} x$.

Necessity: The structure of this part of the proof is similar to that of EFR and Moyes. Suppose IP is violated. Thus, there exist $0<u<v$ such that $f(u)>f(v)$. Consider $n\geq 2$, such that

$$(n-1)>(v-u)/(f(u)-f(v)). \tag{6}$$

Let $x=(u, u, \dots u, v)$. Then $f(x)=(f(v), f(u), \dots f(u), f(u))$. Consider the aggregate income shortfall of any one of the $(n-1)$ identical poor persons from the rich person in the pre-tax distribution x and that of the poor person from the rich ones in the post-tax distribution $f(x)$. These are given respectively by $(v-u)$ and $(n-1)$ $(f(u)-f(v))$. By the choice of n in (6), we have $(n-1)$ $(f(u)-f(v))>(v-u)$. This contradicts the requirement $f(x)\geqq_{R1} x$.

Suppose now that MP is violated. Then there exist $0<u<v$ such that $v-f(v)<u-f(u)$. Let $x=(u, v)$. Then $f(x)=(f(u), f(v))$. Clearly, $f(v)>f(u)$. The income shortfall of the poor person from the rich one in the distributions x and $f(x)$ are given respectively by $(v-u)$ and $(f(v)-f(u))$. By assumption we have $v-u<f(v)-f(u)$, which contradicts the relation $f(x)\geqq_{R1} x$.

(b) The proof of this part is similar to that of part (a) and hence omitted. □

From propositions 1, 2 and 3 we see that for all $n\geq 2$, $R1$ ($R2$) is equivalent to LA (L). In our next result we show that the Lorenz and the absolute Lorenz dominations differ respectively from the ranking relations $R2$ and $R1$ for a given population size.

Proposition 4 Let $n\geq 2$ be given. Then for arbitrary $x, y\varepsilon D^n$,

(a) $y\geqq_L x$ does not imply $y\geqq_{R2} x$.

(b) $y\geqq_{LA} x$ does not imply $y\geqq_{R1} x$.

proof: This will be done by giving an example. Let $x=(10, 20, 30, 40)$ and $y=(14, 22, 24, 40)$. Then we have $y\geqq_L x$ but not $y\geqq_{R2} x$. Again for the same x, y, the relation $y\geqq_{LA} x$ holds but not $y\geqq_{R1} x$. □

We now try to find out what should be the population size dependent tax schemes which are equivalent for the different inequality criteria. In doing this, we assume, as is done in most cases of distributional comparisons, that the tax function satisfies *IP*.

Proposition 5 Let $n\geq 2$ be given. Suppose that the tax function satisfies *IP*. Then for all $x\varepsilon D^n$, $f(x)\geqq_{\mathrm{LA}} x$ if and only if *SM* holds.

Proof Let $t_i=x_i-f(x_i)$. By *IP* $f(x)$ is illfare ranked.

Clearly, $\sum_{i=1}^{k}(f(x_i)-\lambda(f(x)))\geq\sum_{i=1}^{k}(x_i-\lambda(x))$ can be re-written as

$$\sum_{i=1}^{k}(x_i-t_i-\lambda(x)+\lambda(t))\geq\sum_{i=1}^{k}(x_i-\lambda(x)) \tag{7}$$

where $\lambda(t)$ is the average tax liability. (7), on simplification, gives us

$$\sum_{i=1}^{k} t_i/k \leq \lambda(t) \tag{8}$$

Since $\lambda(t)=(\sum_{i=1}^{k} t_i + \sum_{i=k+1}^{n} t_i)/n$, from (8) it follows that

$$\sum_{i=1}^{k} t_i/k \leq \sum_{i=k+1}^{n} t_i/(n-k) \tag{9}$$

The absolute Lorenz domination implies that the inequality given by (9) is true for all k, $1 \leq k \leq n-1$. Hence $f(x) \geqq_{LA} x$ implies that SM holds. Similarly it can be shown that (9)$\Rightarrow f(x) \geqq_{LA} x$. □

Finally, we have

Proposition 6 Let $n \geq 2$ be given. Suppose that the tax function satisfies IP. Then for all $x \varepsilon D^n$, $f(x) \geqq_L x$ if and only if SA holds.

Proof Let $t_i = x_i - f(x_i)$. By IP $f(x)$ is illfare ranked. Now, $f(x) \geqq_L x$ can be written as

$$\sum_{i=1}^{k} (x_i - t_i)/(\lambda(x)-\lambda(t)) \geq \sum_{i=1}^{k} x_i/\lambda(x) \tag{10}$$

where $1 \leq k \leq n$ and $\lambda(t)$ is the average tax liability. It should be noted that since both $x, f(x)$ εD^n, we have, $\lambda(x)-\lambda(t)=\lambda(f(x))>0$.

From (10) we have

$$\sum_{i=1}^{k} x_i \left[1/(\lambda(x)-\lambda(t))-1/\lambda(x)\right] \geq \sum_{i=1}^{k} t_i /(\lambda(x)-\lambda(t)) \tag{11}$$

for all k, $1 \leq k \leq n$. From (11) it now follows that

$$\lambda(t)/\lambda(x) \geq \sum_{i=1}^{k} t_i / \sum_{i=1}^{k} x_i \qquad \text{for all } k,\ 1 \leq k \leq n. \text{ Hence } SA \text{ holds.}$$

These steps can be retraced back to show the reverse implication. □

IV. *Conclusions*

Kakwani (1977) indicated that an average progressive taxation is uniformly equalizing according to the Lorenz criterion. Later on, many authors, including Jakobsson (1976), Eichhorn et. al. (1984) showed that under certain conditions the above implication turns out to be an equivalence. The absolute version of this result was proved by Moyes (1988). We propose two concepts of domination principles based on Runciman (1966) which lead to identical taxation schemes as proposed by EFR and Moyes under the condition of population size independence and which differ from the Lorenz criteria if the population size is given. We then try to indicate the nature of the equivalent taxation schemes for the Lorenz criteria when the population size is fixed.

Let us now consider the relation $R(U)$ defined as 'For any $x, y \varepsilon D^n$, $y >_{-R(U)} x$ if and only if $\sum_{i=k+1}^{n}(U(x_i)-U(x_k)) \geq \sum_{i=k+1}^{n}(U(y_i)-U(y_k))$ for all k, $1 \leq k \leq n-1$'. It will be interesting to determine the nature of U for which the following result holds: For all $n \geq 2$, for all $x \varepsilon D^n$, $f(x) >_{-R(U)} x$ if and only if $f(x) >_{-L} x$.' Since in this paper our aim was to find the fixed population analogue to the results of Eichhorn, Funke and Richter (1984) and Moyes (1988), we did not try to investigate the issue here. We, however, leave this as a future research program.

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