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TURNPIKE THEORY FOR CAPITAL ACCUMULATION IN A GENERAL STATIONARY MODEL UNDER UNCERTAINTY

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Abstract

A general stationary model of capital accumulation under uncertainty is constructed, and weaker conditions for boundedness of economy and for expansibility of capital stock in production technology are assumed. The existence of a supporting price for the optimal stationary state is proved. By using the supporting price, weakly maximal programs are proved to converge to the optimal stationary state in the weak* topology, in probability, and in the $\mathscr{L}_1$-norm topology. In addition, the almost sure convergence of weakly maximal programs is proved under the assumption of uniform convexity. The "value loss" approach is more effectively used in the proofs.

I. *Introduction*

In this paper we shall construct a general stationary model of capital accumulation under uncertainty, and prove the turupike property of weakly maximal programs of capital accumulation. Stochastic stationary models have been developed in several papers by Dana (1973), Radner (1973), Jeanjean (1974), Evstigneev (1974), and Zilcha (1976-a). The assumtions we will make on the model are more general than those made on their models. In fact, we assume weaker conditions for boundedness of economy and for expansibility of capital stock in production technology.

As preliminary arguments we prove the existence of an optimal statonary state which is the turnpike for capital accumulation, and the existence of a supporting price for the optimal stationary state (Theorems 4. 1 and 4. 2). In our proof of the existence of a supporting price we use the same technique that was applied first by Radner (1973) in a model of stochastic production, and by Jeanjean (1974), Evstigneev (1974), and Zilcha (1976-b) in general stochastic models.

By using the supporting price for the optimal stationary state, we will prove some convergence properties of weakly maximal programs to the optimal stationary state. First, as a basic property of convergence, we prove the convergence in the weak* topology of weakly maximal programs (Theorem 6. 1). Second, we show that weakly maximal programs converge

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to the turnpike in probability and in the $\mathscr{L}_1$-norm topology (Theorem 6. 2 and Corollary). Finally, under the assumption of uniform convexity, the almost sure convergence of weakly maximal programs is proved (Theorem 6. 3).

The turnpike property of optimal programs was proved by Jeanjean (1974) in a model of stochastic production, and by Evstigneev (1974) in a general stochastic model. Our proof is more direct than theirs and the so-called "value loss" is more effectively used in the proof of convergence of weakly maximal programs to the optimal stationary state.

II. *A General Stationary Model*

Let $(\Omega, \mathscr{F}, P)$ be a probability space, where Ω is a set, $\mathscr{F}$ is a σ-field, and P is a probability measure. Each element in Ω is a state of nature and is interpreted as a stream of environments in all past, present, and future periods. The σ-field $\mathscr{F}$ is the set of all possible events. The measure P denotes the probability distribution of states of nature.

Let $N = \{1, 2, 3, \cdots\}$ be a space of time. The information structure is specified by a filtration $\{\mathscr{F}_t \mid t = 0, 1, 2, \cdots\}$, where $\mathscr{F}_t$ is a σ-subfield of $\mathscr{F}$ such that $\mathscr{F}_{t-1} \subset \mathscr{F}_t$ for all $t \in N$. Each $\mathscr{F}_t$ denotes the information about states of nature, which becomes available up to time t.

To describle the possibility of capital stocks and social welfare in the period between time $t-1$ to time t, we use a relation $F_t : \Omega \to R^{2l+1}$, i.e.,

$$\omega \in \Omega \quad \to \quad F_t(\omega) \subset R^{2l+1},$$

where R^{2l+1} denotes a $(2l+1)$–dimensional Euclidean space. For x, y $\in R^l$, and $a \in R$, $(x, y, a) \in F_t(\omega)$ means that, under state ω, capital stock x at time $t-1$ can be transformed into capital stock y at time t, and level a of social welfare can be attained. We assume the measurability of relation F_t, i.e., the graph of F_t defined by

$$G(F_t) = \{(x, y, a, \omega) \mid (x, y, a) \in F_t(\omega)\},$$

is measurable in $\mathscr{B}(R^{2l+1}) \times \mathscr{F}_t$, where $\mathscr{B}(R^{2l+1})$ is the family of all Borel subsets of R^{2l+1}.

Remark 2. 1: Economic models are usually depicted by production sets and utility functions as follows: For each state $\omega \in \Omega$, let $D_t(\omega)$ denote the production set of capital stocks at time t. If $(x, y) \in D_t(\omega)$, capital stock y can be produced at time t from capital stock x at time $t-1$. For $(x, y) \in D_t(\omega)$, let $u_t(x, y, \omega)$ denote the social welfare at time t, which is the maximum utility obtained in transition of capital stock x into capital stock y. Thus, the relation F_t in this paper is derived by defining

$$F_t(\omega) = \{(x, y, a) \mid (x, y) \in D_t(\omega),\, a \leqq u_t(x, y, \omega)\}.$$

We incorporate a stationary structure into the model by making the following two assumptions.

Assumption 1: There is a map $\tau : \Omega \to \Omega$ such that map $\tau : (\Omega, \mathscr{F}_t, P) \to (\Omega, \mathscr{F}_{t-1}, P)$ is

measurability- and measure-preserving for each $t \in N$:
(1) Map τ is one to one and onto, and both τ and its inverse τ^{-1} are measurable.
(2) $P(\tau^{-1}(E)) = P(E)$ for all $E \in \mathscr{F}_{t-1}$.

Assumption 2: For each $t \in N$, $F_t = F_1 \circ \tau^{t-1}$, i.e.,

$$F_t(\omega) = F_1(\tau^{t-1}(\omega)) \qquad \text{for all } \omega \in \Omega,$$

where τ^{t-1} denotes the $(t-1)$–time composite map of τ.

The map τ in Assumption 1 is a time-shifting operator. If each element $\omega \in \Omega$ denote a stream of environments at all periods in time, it may be called a "*history*". Let $\omega \in \Omega$ and $\omega' = \tau(\omega)$. Then, history ω' can be regarded as the exactly same history as history ω, except that everything in history ω happens one period earlier in history ω'.

Assumption 2 means that the possibility of capital stocks and social welfare at each period in time does not depend on time $t \in N$, but only on state $\omega \in \Omega$. From now on, we denote F_1 by F, and $F_t = F \circ \tau^{t-1}$ for each $t \in N$.

Remark 2. 2: Under Assumption 1 we are going to consider the following situation: Let S_t denote the set of possible environments at time t, and Ω be the infinite product of sets S_t, that is,

$$\Omega = \cdots \times S_{-1} \times S_0 \times S_1 \times S_2 \times \cdots.$$

Namely, each state of nature $\omega \in \Omega$ is a stream of environments at all periods. We assume that the set of environments at any period is the same set S, and that $S_t = S$ for all t. Let $\mathscr{A}$ be a σ-field consisting of some subsets of S. For a fixed t, define a subset E of Ω by

$$E = \cdots \times A_{t-2} \times A_{t-1} \times A_t \times S \times S \times \cdots,$$

where $A_s \in \mathscr{A}$ for all $s \leqq t$, and $A_s = S$ for all but finitely many $s \leqq t$. Then, we can regard $\mathscr{F}_t$ as a σ-field generated by sets defined as set E.

For state $\omega \in \Omega$, let ω_t be the t-th cordinate of ω and denote the environment at time t in ω. Then we can define a map, $\omega \to \tau(\omega)$, by

$$\tau(\omega)_{t-1} = \omega_t \qquad \text{for each } \omega \in \Omega \text{ and } t \in N.$$

Let $\omega \in \Omega$ and $\omega' = \tau(\omega)$. Then, $\omega'_{t-1} = \omega_t$ for all $t \in N$, that is, the environment at time t in history ω happens at time $t-1$ in history ω'. Thus, map τ shifts time forward.

III. *Programs of Capital Accumulation*

Let k_0 be an initial stock, which is an $\mathscr{F}_0$-measurable function on Ω to R^{ℓ}. A program of capital accumulation starting from k_0 described by a stochastic process $\{(k_t, u_t) \mid t \in N\}$, where k_t is an $\mathscr{F}_t$-measurable function on Ω to R^{ℓ} and u_t is an $\mathscr{F}_t$-measurable function on Ω to R. The quantities of capital stock and the level of social welfare at time t in state ω are denoted respectively by $k_t(\omega)$ and $u_t(\omega)$.

We assume the boundedness of the economy, and consider only programs starting from essentially bounded initial stocks.

Assumption 3: There are numbers $a^* > 0$ and $b^* > 0$ such that $(x, y, a) \in F(\omega)$ implies $|y| < \max\{b^*, |x|\}$ and $|a| \leqq \max\{a^*, |x|\}$.

Let us denote the set of all essentially bounded measurable functions on $(\Omega, \mathscr{F}_t, P)$ to R^ℓ by $\mathscr{L}_\infty^\ell(\mathscr{F}_t)$. When $\ell = 1$, we write $\mathscr{L}_\infty(\mathscr{F}_t)$ instead of $\mathscr{L}_\infty^1(\mathscr{F}_t)$. For any program $\{(k_t, u_t) | t \in N\}$ starting from k_0, we always assume that $k_0 \in \mathscr{L}_\infty^\ell(\mathscr{F}_0)$. A program $\{(k_t, u_t) | t \in N\}$ starting from k_0 is said to be *feasible* if $(k_{t-1}(\omega), k_t(\omega), u_t(\omega)) \in F_t(\omega)$ a. s. for each $t \in N$.

Lemma 3. 1: For any feasible program $\{(k_t, u_t) | t \in N\}$ from k_0, $k_t \in \mathscr{L}_\infty^\ell(\mathscr{F}_t)$ and $u_t \in \mathscr{L}_\infty(\mathscr{F}_t)$ for all $t \in N$. Moreover, any feasible program is uniformly bounded.

Proof: Since $(k_{t-1}(\omega), k_t(\omega), u_t(\omega)) \in F_t(\omega)$, by Assumption 3 we have $|k_t(\omega)| < \max\{b^*, |k_{t-1}(\omega)|\}$ and $|u_t(\omega)| \leqq \max\{a^*, |k_{t-1}(\omega)|\}$. Therefore, $k_{t-1} \in \mathscr{L}_\infty^\ell(\mathscr{F}_{t-1})$ implies $k_t \in \mathscr{L}_\infty^\ell(\mathscr{F}_t)$ and $u_t \in \mathscr{L}_\infty(\mathscr{F}_t)$. Since $k_0 \in \mathscr{L}_\infty^\ell(\mathscr{F}_0)$, we have $k_t \in \mathscr{L}_\infty^\ell(\mathscr{F}_t)$ and $u_t \in \mathscr{L}_\infty(\mathscr{F}_t)$ for all $t \in N$.

Also, $|k_t|_\infty < \max\{b^*, |k_{t-1}|_\infty\}$ and $|u_t|_\infty \leqq \max\{a^*, |k_{t-1}|_\infty\}$, where $|\cdot|_\infty$ denotes the essential norm. Hence, we have $|k_t|_\infty < \max\{b^*, |k_0|_\infty\}$ and $|u_t|_\infty \leqq \max\{a^*, b^*, |k_0|_\infty\}$, which implies the uniform boundedness of program $\{(k_t, u_t) | t \in N\}$. ■

For any feasible program $\{(k_t, u_t) | t \in N\}$ from k_0, by Lemma 3. 1 the sum of expected utilities obtained up to time T can defined by

$$\textstyle\sum_{t=1}^{T} [\int_\Omega u_t(\omega)\, dP(\omega)].$$

Since the sum of expected utilities may be unbounded as time T goes to $+\infty$, the overtaking criterion, or the catching-up criterion is used to evaluate programs. A feasible program $\{(k_t, u_t) | t \in N\}$ from k_0 is said to be *weakly maximal* if it is not overtaken by any other feasible program starting from the same capital stock, i. e., there is not any feasible program $\{(k'_t, u'_t) | t \in N\}$ from k_0 such that

$$\liminf_{T \to +\infty} [\sum_{t=1}^{T} \int_\Omega u'_t\, dP - \sum_{t=1}^{T} \int_\Omega u_t\, dP] > 0.$$

Also, a feasible program $\{(k_t, u_t) | t \in N\}$ from k_0 is said to be *optimal* if it catches up all other feasible program starting from the same capital stock, i.e., for all feasible program $\{(k'_t, u'_t) | t \in N\}$ from k_0,

$$\limsup_{T \to +\infty} [\sum_{t=1}^{T} \int_\Omega u'_t\, dP - \sum_{t=1}^{T} \int_\Omega u_t\, dP] \leqq 0.$$

If a program catches up all other feasible programs starting from the same capital stock, it is not overtaken by any other feasible program starting from the same capital stock.

Therefore, any optimal program is weakly maximal.

Now, we assume the convexity and continuity of the model.

Assumption 4: The relation $F : \Omega \to R^{2\ell+1}$ is closed- and convex-valued, i.e., $F(\omega)$ is a closed and convex subset of $R^{2\ell+1}$ for all $\omega \in \Omega$.

Let us denote the set of all integrable functions on $(\Omega, \mathscr{F}_t, P)$ to R^ℓ by $\mathscr{L}_1^\ell(\mathscr{F}_t)$. When $\ell = 1$, we write $\mathscr{L}_1(\mathscr{F}_t)$ instead of $\mathscr{L}_1^1(\mathscr{F}_t)$.

Remark 3. 1: For each $f \in \mathscr{L}_\infty^\ell(\mathscr{F}_t)$ a linear function on $\mathscr{L}_1^\ell(\mathscr{F}_t)$ is defined by

$$p \in \mathscr{L}_1^\ell(\mathscr{F}_t) \quad \to \quad \int_\Omega p(\omega) \cdot f(\omega)\, dP(\omega) \in R,$$

and $\mathscr{L}_\infty^\ell(\mathscr{F}_t)$ can be regarded as the set of all norm-continuous linear functions on $\mathscr{L}_1^\ell(\mathscr{F}_t)$ [Dunford & Schwartz (1964), Thm. IV. 8. 5, p. 289]. There is the weakest topology for $\mathscr{L}_1^\ell(\mathscr{F}_t)$ such that the linear functions defined in the above are all continuous. Such a topology for space $\mathscr{L}_1^\ell(\mathscr{F}_t)$ is referred to as the *weak topology*.

On the other hand, for each $p \in \mathscr{L}_1^\ell(\mathscr{F}_t)$, a linear function on $\mathscr{L}_\infty^\ell(\mathscr{F}_t)$ is defined by

$$f \in \mathscr{L}_\infty^\ell(\mathscr{F}_t) \quad \to \quad \int_\Omega p(\omega) \cdot f(\omega)\, dP(\omega) \in R,$$

and it is continuous in the norm topology. There is the weakest topology for $\mathscr{L}_\infty^\ell(\mathscr{F}_t)$ such that the linear functions defined in the above are all continuous. Such a topology for space $\mathscr{L}_\infty^\ell(\mathscr{F}_t)$ is commonly referred to as the *weak* topology*.

Let us define a subset $\mathscr{K}$ of $\mathscr{L}_\infty^\ell(\mathscr{F}_0) \times \mathscr{L}_\infty^{\ell+1}(\mathscr{F}_1)$ by

$$\mathscr{K} = \{(f, g, u) \in \mathscr{L}_\infty^\ell(\mathscr{F}_0) \times \mathscr{L}_\infty^\ell(\mathscr{F}_1) \times \mathscr{L}_\infty(\mathscr{F}_1) \mid \\ (f(\omega), g(\omega), u(\omega)) \in F(\omega) \text{ a. s.}\}.$$

Under Assumption 4, we can prove the following lemma by a standard argument.

Lemma 3. 2: Set $\mathscr{K}$ is a convex and closed subset of $\mathscr{L}_\infty^\ell(\mathscr{F}_0) \times \mathscr{L}_\infty^{\ell+1}(\mathscr{F}_1)$ in the weak* topology.

Proof: The convexity of $\mathscr{K}$ immediately follows from that of $F(\omega)$ for all $\omega \in \Omega$ in Assumption 4, To prove the closedness of $\mathscr{K}$, let (f^n, g^n, u^n) be a net (generalized sequence) in $\mathscr{K}$ converging to a point (f^0, g^0, u^0) in the weak* topology.

Since (f^n, g^n, u^n) can be regarded as a net in $\mathscr{L}_1^\ell(\mathscr{F}_0) \times \mathscr{L}_1^{\ell+1}(\mathscr{F}_1)$, (f^n, g^n, u^n) converges to (f^0, g^0, u^0) in the weak topology. Therefore, there is a sequence $(\underline{f}^n, \underline{g}^n, \underline{u}^n)$ of convex combinations of some elements (f^n, g^n, u^n) converges to (f^0, g^0, u^0) in the $\mathscr{L}_1$-norm topology [Dunford & Schwartz (1964), Cor. V. 3. 14, p. 422]. Without loss of generality, we can assume that $(\underline{f}^n, \underline{g}^n, \underline{u}^n)$ converges to (f^0, g^0, u^0) almost surely. This is because the convergence in the mean implies the convergence in measure, and because any sequence converging in measure has a subsequence converging almost surely [Dunford & Schwartz (1964), Thm. III. 3. 6, p. 122 and Cor. III. 6. 13, p. 150].

Since $F(\omega)$ is convex for all $\omega \in \Omega$, $(\underline{f}^n(\omega), \underline{g}^n(\omega), \underline{u}^n(\omega)) \in F(\omega)$ for all $\omega \in \Omega$. Therefore, since $F(\omega)$ is closed for all $\omega \in \Omega$, it follows that $(f^0(\omega), g^0(\omega), u^0(\omega)) \in F(\omega)$ for all $\omega \in \Omega$, i.e., $(f^0, g^0, u^0) \in \mathscr{K}$. This proves the closedness of $\mathscr{K}$. ■

IV. *Optimal Stationary States and Supporting prices*

A pair of functions $(k, u) \in \mathscr{L}_\infty^l(\mathscr{F}_0) \times \mathscr{L}_\infty(\mathscr{F}_0)$ is called a *stationary state* if $(k, k \circ \tau, u \circ \tau) \in \mathscr{K}$. Let us define the set of all stationary states by

$$\mathscr{S} = \{(k, u) \in \mathscr{L}_\infty^l(\mathscr{F}_0) \times \mathscr{L}_\infty(\mathscr{F}_0) \,|\, (k, k \circ \tau, u \circ \tau) \in \mathscr{K}\}.$$

For a stationary state $(k, u) \in \mathscr{S}$, a feasible program starting from k is defined by $\{(k \circ \tau^t, u \circ \tau^t) \,|\, t \in \mathrm{N}\}$. Such a program is called a *stationary program*. In a stationary program, an identical plan of capital accumulation is repeated forever.

Theorem 4. 1: Under Assumptions 1–4, there exists a stationary state $(k^*, u^*) \in \mathscr{S}$ such that $\int_\Omega u^* \, dP \geqq \int_\Omega u \, dP$ for all $(k, u) \in \mathscr{S}$.

Proof: By Assumption 3 of boundedness, if $(k, k \circ \tau, u \circ \tau) \in \mathscr{K}$, then $|k|_\infty = |k \circ \tau|_\infty < \max\{b^*, |k|_\infty\}$ and $|u|_\infty = |u \circ \tau|_\infty \leqq \max\{a^*, |k|_\infty\}$, where $|\cdot|_\infty$ denotes the essential norm. Hence, we have $|k|_\infty < b^*$, and $|u|_\infty \leqq \max\{a^*, b^*\}$. This proves the boundedness of set $\mathscr{S}$.

By Lemma 3. 2, set $\mathscr{K}$ is weak*-closed. Therefore, we can easily prove that set $\mathscr{S}$ is also weak*-closed. Since set $\mathscr{S}$ is bounded, set $\mathscr{S}$ is weak*-compact [Dunford & Schwartz (1964), Cor. V. 4. 3, p. 424]. Hence, $\int_\Omega u \, dP$ attains the maximnm value at some $(k^*, u^*) \in \mathscr{S}$. ■

The stationary state $(k^*, u^*) \in \mathscr{S}$ in Theorem 4. 1 is called an *optimal stationary state*. In fact, the stationary program $\{(k^* \circ \tau^t, u^* \circ \tau^t) \,|\, t \in N\}$ from k^* can be proved to be an optimal program.

Assumption 5: If $(x, y, a) \in F(\omega)$ and $x \leqq x'$, then $(x', y, a) \in F(\omega)$.

Assumption 6: There is $(\underline{k}, \underline{k}', \underline{u}) \in \mathscr{K}$ such that $\underline{k} \ll \underline{k}' \circ \tau^{-1}$. Here, for $f, g \in \mathscr{L}_\infty^l(\mathscr{F}_0)$, $f \ll g$ means that for some $\varepsilon > 0$, $f(\omega) + \varepsilon \cdot \mathbb{1} \leqq g(\omega)$ a. s., where $\mathbb{1} = (1, \ldots, 1) \in R^l$.

We assume, by Assumption 5, the free disposal of initial capital stocks. By Assumption 6, the existence of an expansible capital stock is assumed. Capital stock $\underline{k}$ in Assumption 6 is expansible to capital stock $\underline{k}'$.

Let $\mathscr{L}_\infty^l(\mathscr{F}_t)^*$ denote the dual space of $\mathscr{L}_\infty^l(\mathscr{F}_t)$, i.e., the set of all norm-continuous linear functions on $\mathscr{L}_\infty^l(\mathscr{F}_t)$ to R.

Lemma 4. 1: For an optimal stationary state (k^*, u^*), there is $\pi^* \in \mathscr{L}_\infty^l(\mathscr{F}_0)^*$ with $\pi^* \geqq 0$ such that

$$\int_\Omega u^* \, dP \geqq \int_\Omega u \, dP - \pi^* \cdot (f - g \circ \tau^{-1})$$

for all $(f, g, u) \in \mathscr{K}$.

Proof: Define a subset of $R \times \mathscr{L}_\infty^l(\mathscr{F}_0)$ by

$$A = \{(a, h) \in R \times \mathscr{L}_\infty^l(\mathscr{F}_0) \mid a < \int_\Omega u\,dP - \int_\Omega u^*\,dP \text{ and } h = f - g \circ \tau^{-1} \text{ for some } (f, g, u) \in \mathscr{K}\}.$$

Then, since set $\mathscr{K}$ is convex, set A is convex. Also, by the optimality of (k^*, u^*), the origin of $R \times \mathscr{L}_\infty^l(\mathscr{F}_0)$ does not belongs to A. By Assumption 5 of free disposal, the norm-interior of A is not empty. Therefore, by a separation theorem [Dunford & Schwartz (1964), Thm. V. 2. 7, p. 417], there exists $(c, -\pi^*) \in R \times \mathscr{L}_\infty^l(\mathscr{F}_0)^*$ with $(c, -\pi^*) \neq 0$ such that $ca - \pi^* \cdot h \leqq 0$ for all $(a, h) \in A$. The shape of set A implies that $c \geqq 0$. Also, Assumption 5 implies that $\pi^* \geqq 0$.

Suppose that $c = 0$, then $\pi^* \cdot (f - g \circ \tau^{-1}) \geqq 0$ for all $(f, g, u) \in \mathscr{K}$. Thus, Assumption 6 implies that $\pi^* = 0$, which contradicts $(c, -\pi^*) \neq 0$. Therefore, $c > 0$, and without loss of generality we can assume that $c = 1$. Hence, $a - \pi^* \cdot h \leqq 0$ for all $(a, h) \in A$, which implies this lemma. ■

The linear continuous function π^* in the above lemma is called a *supporting price* for stationary state (k^*, u^*). Furthermore, for an optimal stationary state (k^*, u^*), we can find a supporting price which belongs to $\mathscr{L}_1^l(\mathscr{F}_0)$ rather than to $\mathscr{L}_\infty^l(\mathscr{F}_0)^*$.

Theorem 4. 2: For an optimal stationary state (k^*, u^*), there is $p^* \in \mathscr{L}_1^l(\mathscr{F}_0)$ with $p^* \geqq 0$ such that

$$\int_\Omega u^*\,dP \geqq \int_\Omega [u - p^* \cdot f + (p^* \circ \tau) \cdot g]\,dP$$

for all $(f, g, u) \in \mathscr{K}$.

Proof: Let $ba^l(\mathscr{F}_0)$ denote the set of all bounded finitely additive l-dimensional vector-valued measures on $\mathscr{F}_0$ that are absolutely continuous with respect to P. Then, $\mathscr{L}_\infty^l(\mathscr{F}_0)^*$ can be identified with $ba^l(\mathscr{F}_0)$ [Dunford & Schwartz (1964), Thm. IV. 8 .16, p. 296]. Therefore, for the linear continuous function π^* in Lemma 4. 1, there is $\nu \in ba^l(\mathscr{F}_0)$ such that

$$\pi^* \cdot f = \int_\Omega f\,d\nu \qquad \text{for all } f \in \mathscr{L}_\infty^l(\mathscr{F}_0).$$

Since $\pi^* \geqq 0$, it follows that $\nu \geqq 0$. Therefore, ν can be decomposed into two parts [Yosida & Hewitt (1952). Thm. 1. 23, p. 52], that is, $\nu = \nu_c + \nu_p$, where ν_c is a non-negative countably additive measure on $\mathscr{F}_0$ which is absolutely continuous with respect to P and ν_p is a non-negative purely finitely additive measure on $\mathscr{F}_0$. Hence, by the Radon-Nikodym theorem, there is a unique $p^* \in \mathscr{L}_1^l(\mathscr{F}_0)$ with $p^* \geqq 0$ such that

$$\int_\Omega f\,d\nu_c = \int_\Omega p^* \cdot f\,dP \qquad \text{for all } f \in \mathscr{L}_\infty^l(\mathscr{F}_0).$$

Moreover, there is a sequence of $\mathscr{F}_0$-measurable sets, $\{A_n\}$, such that $A_n \subset A_{n+1}$, $\nu_p(A_n) = 0$ for all n, and $\lim_{n \to +\infty} P(A_n) = 1$ [Yosida & Hewitt (1952), Thm. 1. 22, p. 52].

Let $(f, g, u) \in \mathscr{K}$ and define f_n, g_n, and u_n by

$$(f_n(\omega), g_n(\omega), u_n(\omega)) = \begin{cases} (f(\omega), g(\omega), u(\omega)) & \omega \in A_n \\ (k^*(\omega), k^*(\tau(\omega)), u^*(\tau(\omega))) & \text{otherwise.} \end{cases}$$

Then, $(f_n, g_n, u_n) \in \mathscr{K}$. Therefore, by Lemma 4. 1 we have

$$\begin{aligned} \int_\Omega u^* \, dP &\geqq \int_\Omega u_n \, dP - \pi^* \cdot (f_n - g_n \circ \tau^{-1}), \\ &= \int_\Omega u_n \, dP - \int_\Omega (f_n - g_n \circ \tau^{-1}) \, d\nu_c - \int_\Omega (f_n - g_n \circ \tau^{-1}) \, d\nu_p, \\ &= \int_\Omega u_n \, dP - \int_\Omega p^* \cdot (f_n - g_n \circ \tau^{-1}) \, dP - \int_{\Omega \setminus A_n} (f_n - g_n \circ \tau^{-1}) \, d\nu_p, \\ &= \int_\Omega u_n \, dP - \int_\Omega [p^* \cdot f_n - (p^* \circ \tau) \cdot g_n] \, dP. \end{aligned}$$

Hence, as $n \to +\infty$, in the limit we have the conclusion of this theorem. ■

Remarks 4. 1: For each $(f, g, u) \in \mathscr{K}$, we can define a value by

$$L(f, g, u) = \int_\Omega [u^* - u + p^* \cdot f - (p^* \circ \tau) \cdot g] \, dP,$$

which is commonly called "*the (expected) value loss*". By Theorem 4. 2, $L(f, g, u) \geqq 0$ for all $(f, g, u) \in \mathscr{K}$.

V. *The Accessibility to Optimal Stationary States*

Let (k^*, u^*) be an optimal stationary state and $k_0 \in \mathscr{L}_\infty^l(\mathscr{F}_0)$ be a capital stock. The optimal stationary state (k^*, u^*) is said to be *accessible* from k_0 if there exist $k_1, u_1 \in \mathscr{L}_\infty^l(\mathscr{F}_1)$ and a number θ with $0 \leqq \theta < 1$ such that $(k_0, k_1, u_1) \in \mathscr{K}$ and $\theta k_0 + (1-\theta) k^* = k_1 \circ \tau^{-1}$.

A feasible program $\{(k_t, u_t) \mid t \in N\}$ from k_0 is said to be *good* if

$$\sum_{t=1}^{\infty} \left[\int_\Omega u_t \, dP - \int_\Omega u^* \, dP \right] > -\infty.$$

Lemma 5. 1: Let $k_0 \in \mathscr{L}_\infty^l(\mathscr{F}_0)$ be a capital stock from which optimal stationary state (k^*, u^*) is accessible. Then, there is a good program from k_0.

Proof: Since (k^*, u^*) is accessible from k_0, there exist k_1, u_1, and θ with $0 \leqq \theta < 1$ such that $(k_0, k_1, u_1) \in \mathscr{K}$ and $\theta k_0 + (1-\theta) k^* = k_1 \circ \tau^{-1}$. For $t \geqq 2$, define $k_t = \theta k_{t-1} \circ \tau + (1-\theta) k^* \circ \tau^t$ *and* $u_t = \theta u_{t-1} \circ \tau + (1-\theta) u^* \circ \tau^t$.

Consider program $\{(k_t, u_t) \mid t \in N\}$ from k_0. For $t \geqq 2$, we have

$$\begin{aligned} &(k_{t-1} \circ \tau^{1-t}, k_t \circ \tau^{1-t}, u_t \circ \tau^{1-t}) \\ &= \theta (k_{t-2} \circ \tau^{2-t}, k_{t-1} \circ \tau^{2-t}, u_{t-1} \circ \tau^{2-t}) + (1-\theta) (k^*, k^* \circ \tau, u^* \circ \tau). \end{aligned}$$

Since $\mathscr{K}$ is convex, by induction, $(k_{t-1} \circ \tau^{1-t}, k_t \circ \tau^{1-t}, u_t \circ \tau^{1-t}) \in \mathscr{K}$ for all $t \geqq 1$, which implies the feasibility of program $\{(k_t, u_t) \mid t \in N\}$.

Moreover, we have

$$\begin{aligned}\Sigma_{t=1}^{\infty}[\int_\Omega u_t\, dP-\int_\Omega u^* dP] &= \Sigma_{t=1}^{\infty}[\int_\Omega \theta\,(u_{t-1}-u^*)\, dP]\\ &= \Sigma_{t=1}^{\infty}[\theta^{t-1}\int_\Omega (u_1-u^*)\, dP]\\ &= \int_\Omega (u_1-u^*)\, dP/(1-\theta) > -\infty.\end{aligned}$$

This implies the goodness of program $\{(k_t, u_t)\,|\,t\in N\}$. ■

Lemma 5. 2: Let $k_0\in\mathscr{L}'_\infty(\mathscr{F}_0)$ be a capital stock from which optimal stationary state (k^*, u^*) is accessible. Then, any weakly maximal program from k_0 is a good program.

Proof: Let $\{(k_t, u_t)\,|\,t\in N\}$ be a weakly maximal program from k_0 and suppose that it were not a good program. Then, by the definition of goodness,

$$\liminf_{T\to+\infty}\sum_{t=1}^{T}[\int_\Omega u_t\, dP-\int_\Omega u^* dP] = -\infty. \tag{5. 1}$$

On the other hand, by Lemma 5. 1, there is a good program from k_0, say $\{(k'_t, u'_t)\,|\,t\in N\}$ such that

$$\Sigma_{t=1}^{\infty}[\int_\Omega u'_t\, dP-\int_\Omega u^* dP] > -\infty. \tag{5. 2}$$

Hence, by (5. 1) and (5. 2), we have

$$\begin{aligned}&\liminf_{T\to+\infty}\sum_{t=1}^{T}[\int_\Omega u'_t\, dP-\int_\Omega u_t\, dP]\\ &=\liminf_{T\to+\infty}\sum_{t=1}^{T}[\int_\Omega (u'_t-u^*)\, dP-\int_\Omega (u_t-u^*)\, dP]\\ &=+\infty.\end{aligned}$$

Thus, program $\{(k'_t, u'_t)\,|\,t\in N\}$ overtakes program $\{(k_t, u_t)\,|\,t\in N\}$, which contradicts the weak maximality of program $\{(k_t, u_t)\,|\,t\in N\}$. ■

VI. *The Convergence of Weakly Maximal Programs to the Turnpike*

The optimal stationary state (k^*, u^*) is called the *turnpike* for capital accumlation. The uniqueness of the turnpike can be easily proved under the following assumption of strict convexity.

Assumption 7: For each $\omega\in\Omega$, $(x, y, a)\in F(\omega)$ and $(x', y', a')\in F(\omega)$ with $x\neq x'$, there exists a number $\delta>0$ such that

$$(\frac{x+x'}{2}, \frac{y+y'}{2}, \frac{a+a'}{2}+\delta)\in F(\omega).$$

By Lemma 3. 1, we know that any feasible program stays in a bounded region. Thus, when we consider the property of feasible programs, without loss of generality, we can make the following assumption of boundedness.

Auxiliary Assumption: Set $\mathscr{K}$ is a bounded subset of $\mathscr{L}_\infty^l(\mathscr{F}_0)\times\mathscr{L}_\infty^{l+1}(\mathscr{F}_1)$. In addition, $F(\omega)$ is a bounded subset of R^{2l+1} *for all* $\omega\in\Omega$.

Since set $\mathscr{K}$ is weak*-closed by Lemma 3. 2, under the Auxiliary Assumption set $\mathscr{K}$ is weak*-compact [Dunford & Schwartz (1964), Cor. V. 4. 3, p. 424].

Lemma 6. 1: For any weak*-open neighborhood U of k^*, there is a number $\delta>0$ such that

$$\int_\Omega u^*\,dP\geqq\int_\Omega[u-p^*\cdot f+(p^*\circ\tau)\cdot g]\,dP+\delta$$

for all $(f,g,u)\in\mathscr{K}$ with $f\in\mathscr{L}_\infty^l(\mathscr{F}_0)\backslash U$.

Proof: Suppose the contrary. There is a sequence $(f^n,g^n,u^n)\in\mathscr{K}$ with $f^n\in\mathscr{L}_\infty^l(\mathscr{F}_0)\backslash U$ such that $\int_\Omega[u^n-p^*\cdot f^n+(p^*\circ\tau)\cdot g^n]\,dP$ converges to $\int_\Omega u^*\,dP$. Since set $\mathscr{K}$ is weak*-compact, we can assume without loss of generality that sequence (f^n,g^n,u^n) converges to a point $(f^0,g^0,u^0)\in\mathscr{K}$ in the weak* topology such that $f^0\neq k^*$. Therefore, we have

$$\int_\Omega u^*\,dP=\int_\Omega[u^0-p^*\cdot f^0+(p^*\circ\tau)\cdot g^0]\,dP. \tag{6. 1}$$

By Assumption 7 of strict convexity, for each $\omega\in\Omega$ there is a number $\rho(\omega)\geqq 0$ such that

$$\left(\frac{k^*(\omega)+f^0(\omega)}{2},\frac{k^*(\tau(\omega))+g^0(\omega)}{2},\frac{u^*(\tau(\omega))+u^0(\omega)}{2}+\rho(\omega)\right)\in F(\omega),$$

where $\rho(\omega)>0$ when $k^*(\omega)\neq f^0(\omega)$. By a measurable selection theorem [Hildenbrand (1974), Thm. 1, p. 54], map, $\omega\rightarrow\rho(\omega)$, can be chosen as an $\mathscr{F}_1$-measurable function, and therefore as an $\mathscr{F}_1$-integrable function. Define functions f', g', and u' by $f'=[k^*+f^0]/2$, $g'=[k^*\circ\tau+g^0]/2$, and $u'=[u^*\circ\tau+u^0]/2+\rho$. Then, $(f',g',u')\in\mathscr{K}$ and, by (6. 1),

$$\begin{aligned}&\int_\Omega[u'-p^*\cdot f'+(p^*\circ\tau)\cdot g']\,dP\\&=(1/2)\int_\Omega[u^*+u^0-p^*\cdot f^0+(p^*\circ\tau)\cdot g^0]\,dP+\int_\Omega\rho\,dP\\&>\int_\Omega u^*\,dP,\end{aligned}$$

which contradicts Theorem 4. 2. ■

Theorem 6. 1: Let $k_0\in\mathscr{L}_\infty^l(\mathscr{F}_0)$ be an initial capital stock from which the turnpike (k^*,u^*) is accessible. Let $\{(k_t,u_t)\,|\,t\in N\}$ be a weakly maximal program from k_0. Then, under Assumptions 1–7, $k_t\circ\tau^{-t}$ converges to k^* in the weak* topology.

Proof: Suppose that $k_t\circ\tau^{-t}$ would not converge to k^* in the weak* topology. Then, there exists a weak*-open neighborhood U of k^* such that $k_t\circ\tau^{-t}\in\mathscr{L}_\infty^l(\mathscr{F}_0)\backslash U$ for infinitely many $t\in N$. Therefore, by Lemma 6. 1, there is a number $\delta>0$ such that

$$\int_\Omega u^*\,dP\geqq\int_\Omega[u_t\circ\tau^{1-t}-p^*\cdot(k_{t-1}\circ\tau^{1-t})+(p^*\circ\tau)\cdot(k_t\circ\tau^{1-t})]\,dP+\delta$$

for infinitely many $t \in N$. Hence, for any T there is n such that

$$\int_\Omega p^* \cdot k_0 \, dP - \int_\Omega p^* \cdot (k_T \circ \tau^{-T}) \, dP - n\delta \geqq \sum_{t=1}^{T} \left[\int_\Omega u_t \, dP - \int_\Omega u^* \, dP \right]$$

and $n \to +\infty$ as $T \to +\infty$. Since k_T stays in a bounded region, the left hand side of the above inequality goes to $-\infty$ as $T \to +\infty$. Therefore, program $\{(k_t, u_t) \mid t \in N\}$ from k_0 is not good, and, by Lemma 5. 2, it is not weakly maximal. This is a contradiction. ■

Corollary: When we regard $\{(k_t \circ \tau^{-t}) \mid t \in N\}$ as a sequence in space $\mathscr{L}_1^l(\mathscr{F}_0)$, then $k_t \circ \tau^{-t}$ converges to k^* in the weak topology.

Proof: The dual space of $\mathscr{L}_1^l(\mathscr{F}_0)$ is smaller than the dual space of $\mathscr{L}_\infty^l(\mathscr{F}_0)$ with the weak* topology. Therefore, the weak* convergence implies the weak convergence. ■

To get a stronger convergence of weakly maximal programs to the turnpike, we have to prove a more general assertion of Lemma 6. 1.

Lemma 6. 2: For any $\varepsilon > 0$ there is a function $\rho \in \mathscr{L}_\infty^l(\mathscr{F}_1)$ with $\rho > 0$ such that

$$\int_\Omega u^* \, dP \geqq \int_\Omega [u - p^* \cdot f + (p^* \circ \tau) \cdot g] \, dP + \int_E \rho \, dP.$$

for all $(f, g, u) \in \mathscr{K}$, where $E = \{\omega \in \Omega \mid |f(\omega) - k^*(\omega)| \geqq \varepsilon\}$.

Proof: Let $\varepsilon > 0$. By the Auxiliary Assumption and Assumption 7, for each $\omega \in \Omega$ there is a number $\rho(\omega) > 0$ such that

$$\left(\frac{x + x'}{2}, \frac{y + y'}{2}, \frac{a + a' + \rho(\omega)}{2}\right) \in F(\omega).$$

for all (x, y, a) and $(x', y', a') \in F(\omega)$ with $|x - x'| \geqq \varepsilon$. By a measurable selection theorem [Hildenbrand (1974), Thm. 1, p. 5], map, $\omega \to \rho(\omega)$, can be taken as an $\mathscr{F}_1$-measurable function, and therefore as an $\mathscr{F}_1$-integrable function.

Let $(f, g, u) \in \mathscr{K}$ and $E = \{\omega \in \Omega \mid |f(\omega) - k^*(\omega)| \geqq \varepsilon\}$.

Suppose the contrary of this lemma, i.e.,

$$\int_\Omega u^* \, dP < \int_\Omega [u - p^* \cdot f + (p^* \circ \tau) \cdot g] \, dP + \int_E \rho \, dP. \tag{6. 2}$$

Define $f' = [f + k^*]/2$ and $g' = [g + k^* \circ \tau]/2$. Also define a function u' by

$$u'(\omega) = \begin{cases} \dfrac{u(\omega) + u^*(\tau(\omega)) + \rho(\omega)}{2} & \text{for } \omega \in E \\[2ex] \dfrac{u(\omega) + u^*(\tau(\omega))}{2} & \text{for } \omega \in \Omega \backslash E. \end{cases}$$

Then, $(f'(\omega), g'(\omega), u'(\omega)) \in F(\omega)$ for all $\omega \in \Omega$, i.e., $(f', g', u') \in \mathscr{K}$. In addition, by (6. 2), we have

$$\begin{aligned}&\int_\Omega [u' - p^* \cdot f' + (p^* \circ \tau) \cdot g']\, dP \\ &= (1/2)\left\{\int_\Omega u^*\, dP + \int_\Omega [u - p^* \cdot f + (p^* \circ \tau) \cdot g]\, dP + \int_E \rho\, dP\right\} \\ &> \int_\Omega u^*\, dP,\end{aligned}$$

which contradicts Theorem 4. 2. ■

Now, we are ready to prove the convergence in probability of weakly maximal programs to the turnpike.

Theorem 6. 2: Let $k_0 \in \mathscr{L}_\infty'(\mathscr{F}_0)$ be an initial capital stock from which the turnpike (k^*, u^*) is accessible. Let $\{(k_t, u_t) | t \in N\}$ be a weakly maximal program from k_0. Then, under Assumptions 1–7, $k_t \circ \tau^{-t}$ converges to k^* in probability. Namely, for any $\varepsilon > 0$,

$$P\{\omega \in \Omega \,|\, |k_t \circ \tau^{-t}(\omega) - k^*(\omega)| \geqq \varepsilon\} \to 0 \qquad \text{as } t \to +\infty.$$

Proof: Suppose that $k_t \circ \tau^{-t}$ would not converge to k^* in probability. Then, there exists $\varepsilon > 0$ and $\delta > 0$ such that $P(E_t) \geqq \delta$ for infinitely many $t \in N$, where

$$E_t = \{\omega \in \Omega \,|\, |k_t \circ \tau^{-t}(\omega) - k^*(\omega)| \geqq \varepsilon\}.$$

By Lemma 6. 2, there is a function $\rho \in \mathscr{L}_\infty(\mathscr{F}_1)$ with $\rho > 0$ such that

$$\begin{aligned}\int_\Omega u^*\, dP \geqq &\int_\Omega [u_t \circ \tau^{1-t} - p^* \cdot (k_{t-1} \circ \tau^{1-t}) + (p^* \circ \tau) \cdot (k_t \circ \tau^{1-t})]\, dP \\ &+ \int_{E_{t-1}} \rho\, dP\end{aligned}$$

for all $t \in N$.

We can easily show there is a number $\gamma > 0$ such that $\int_E \rho\, \mathrm{d}\,\mathrm{P} \geqq \gamma$ for all $E \in \mathscr{F}_0$ with $P(E) \geqq \delta$. Hence, for any T there is n such that

$$\int_\Omega p^* \cdot k_0\, dP - \int_\Omega p^* \cdot (k_T \circ \tau^{-T})\, dP - n\gamma \geqq \sum_{t=1}^{T} \left[\int_\Omega u_t\, dP - \int_\Omega u^*\, dP\right]$$

and $n \to +\infty$ as $T \to +\infty$. Since k_T stays in a bounded region, the above inequality implies that program $\{(k_t, u_t) | t \in N\}$ from k_0 is not good, and by Lemma 5. 2, it is not weakly maximal. This is a contradiction. ■

Corollary: When we consider $\{k_t \circ \tau^{-t} | t \in N\}$ as a sequence in space $\mathscr{L}_1'(\mathscr{F}_0)$, then $k_t \circ \tau^{-t}$ converges to k^* in the norm topology.

Proof: By the Corollary of Theorem 6. 1, $k_t \circ \tau^{-t}$ converges to k^* in the weak topology. Therefore, the convergence in probability implies the convergence in the norm topology [Dunford & Schwartz (1964), Thm. IV. 8. 12, p. 295].

In order to get the almost sure convergence of weakly maximal programs to the turnpike, we need the following assumption of uniform convexity.

Assumption 8: For any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\left(\frac{x+x'}{2}, \frac{y+y'}{2}, \frac{a+a'}{2}+\delta\right) \in F(\omega)$$

for all $\omega \in \Omega$, and $(x, y, a), (x', y', a') \in F(\omega)$ with $|x - x| \geqq \varepsilon$.

Under Assumption 8, we can strengthen Lemma 6. 2 as follows.

Lemma 6. 3: For each $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\int_\Omega u^*\, dP \geqq \int_\Omega [u - p^* \cdot f + (p^* \circ \tau) \cdot g]\, dP + \delta\, P(E)$$

for all $(f, g, u) \in \mathscr{K}$, where $E = \{\omega \in \Omega \mid |f(\omega) - k^*(\omega)| \geqq \varepsilon\}$.

Proof: In the proof of Lemma 6. 2, by Assumption 8 of uniform convexity, we can set $\rho(\omega) = \delta$ for all $\omega \in \Omega$. ■

Theorem 6. 3: Let $k_0 \in \mathscr{L}_\infty'(\mathscr{F}_0)$ be an initial capital stock from which the turnpike (k^*, u^*) is accessible. Let $\{(k_t, u_t) \mid t \in N\}$ be a weakly maximal program from k_0. Then, under Assumptions 1–6 and 8, $k_t \circ \tau^{-t}$ converges to k^* almost surely. Namely, for almost every $\omega \in \Omega$, $k_t \circ \tau^{-t}(\omega)$ converges to $k^*(\omega)$ as $t \to +\infty$.

Proof: For each $\varepsilon > 0$, let us define

$$E_t^\varepsilon = \{\omega \in \Omega \mid |k_t \circ \tau^{-t}(\omega) - k^*(\omega)| \geqq \varepsilon\}, \text{ and}$$

$$D^\varepsilon = \limsup_t E_t^\varepsilon = \cap_{n=1}^\infty \cup_{t>n} E_t^\varepsilon.$$

If $\omega \in \Omega$ and $k_t \circ \tau^{-t}(\omega)$ does not converge to $k^*(\omega)$, then there is a number $\varepsilon > 0$ such that $\omega \in E_t^\varepsilon$ for infinitely many $t \in N$, and therefore $\omega \in D^\varepsilon$. Conversely, $\omega \in D^\varepsilon$ implies that $k_t \circ \tau^{-t}(\omega)$ does not converge to $k^*(\omega)$.

Suppose that $k_t \circ \tau^{-t}$ would not converge to k^*almost surely. Then there is a number $\varepsilon > 0$ such that $P(D^\varepsilon) > 0$. By the Borel-Cantelli lemma [Ross (1983), Prop. 1. 1. 2, p. 3], $\Sigma_{t=1}^\infty P(E_t^\varepsilon) = +\infty$. In fact, if $\Sigma_{t=1}^\infty P(E_t^\varepsilon) < +\infty$, then we have

$$P(D^\varepsilon) = P(\cap_{n=1}^\infty \cup_{t>n} E_t^\varepsilon) = \lim_{n\to\infty} P(\cup_{t>n} E_t^\varepsilon) \leqq \lim_{n\to\infty} \Sigma_{t=n}^\infty P(E_t^\varepsilon) = 0,$$

which is a contradiction.

By Lemma 6. 3, there is a number $\delta > 0$ such that

$$\int_\Omega u^*\, dP \geqq \int_\Omega [u_t \circ \tau^{1-t} - p^* \cdot (k_{t-1} \circ \tau^{1-t}) + (p^* \circ \tau) \cdot (k_t \circ \tau^{1-t})]\, dP + \delta\, P(E_{t-1}^\varepsilon)$$

for all $t \in N$. Hence, for any T,

$$\int_\Omega p^* \cdot k_0\, dP - \int_\Omega p^* \cdot (k_T \circ \tau^{-t})\, dP - \delta\, \Sigma_{t=1}^T P(E_t^\varepsilon) \geqq \Sigma_{t=1}^T \left[\int_\Omega (u^t - u^*)\, dP \right].$$

Since k_T stays in a bounded region, the above inequality implies that program $\{(k_t, u_t) \mid t \in N\}$ from k_0 is not good, and, by Lemma 5. 2, it is not weakly maximal. This is a contradiction. ■

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