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AN EXTENSION OF KRYLOV'S APPROACH TO STOCHASTIC SOLUTIONS: THE SPACE $\mathcal{L}E$

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Abstract

We extend Krylov (1980)'s $\mathcal{L}$-derivative approach to stochastic solutions. His results are based on polynomial growth of the payoff function which is inconvenient for most financial applications. A space called $\mathcal{L}E$, which is the set of Markov diffusions whose exponential moments are finite, is introduced to incorporate exponential growth.

Stochastic solution technique, also known as Feynman-Kac formula, is one of the most frequently used tool for dealing with financial economics problems in continuous time. In spite of its usefulness, however, the regularity conditions required for the application are very delicate and often ignored in the financial economics literature.

In this article, we discuss properties of stochastic solutions to the Cauchy problems of degenerate parabolic partial differential equations (PDE's). Our focus is on the smoothness of stochastic solutions. We follow Krylov (1980)'s $\mathcal{L}$-derivative approach, which gives more suitable results for our task than do standard references such as Friedman (1975), and, Karatzas and Shreve (1987). (Neither of them includes results allowing degeneracy of diffusion coefficients.) However, Krylov's results require a polynomial growth condition which is inconvenient for many financial applications. For instance, option payoffs and HARA utilities have exponential growth if the underlying asset price processes are assumed to be Markov diffusions with bounded drift and dispersion processes.

We extend his results by weakening the polynomial growth condition and assuming the drift and dispersion processes to be bounded. A key concept introduced in the present work is a space called $\mathcal{L}E$, which is the set of Markov diffusions whose exponential moments are finite. Although the proof is parallel to Krylov's approach and many details will be omitted here, we must be careful in verifying the growth conditions since the proof is sensitive to them. For later reference, the results here will be presented in a more general setting than is required in most financial problems. Applications of the ramification in this article can be found in Kuwana (1993, 1995).

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I. *Estimates of moments and exponential transforms*

Let a probability space $(\Omega, \mathfrak{F}, P)$ and a filtration $\{\mathfrak{F}_s\}$ be given. Let $\boldsymbol{\mu}_s(\boldsymbol{x})$ and $\boldsymbol{\xi}_s$ be d-dimensional random vectors and $\mathbf{S}_s(\boldsymbol{x})$ be a $d \times d_1$ random matrix. We assume $\boldsymbol{\mu}_s(\boldsymbol{x})$, $\boldsymbol{\xi}_s$ and $\mathbf{S}_s(\boldsymbol{x})$ are $\mathfrak{F}_s$-progressively measurable. We assume that $\boldsymbol{\mu}_s(\boldsymbol{x})$ and $\mathbf{S}_s(\boldsymbol{x})$ satisfy the following condition.

Condition 1.1. *There exist constants $M \geq 0$ such that for $\forall \boldsymbol{\xi}_s, \forall \boldsymbol{y} \in \mathbb{R}^d, \forall s \in [0,T]$*

(a) $$\|\mathbf{S}_s(\boldsymbol{x}) - \mathbf{S}_s(\boldsymbol{y})\| \leq M\|\boldsymbol{x} - \boldsymbol{y}\| \quad , \quad \|\boldsymbol{\mu}_s(\boldsymbol{x}) - \boldsymbol{\mu}_s(\boldsymbol{y})\| \leq M^2\|\boldsymbol{x} - \boldsymbol{y}\|$$

(b) $$E\left[\int_0^T (\|\boldsymbol{\xi}_s\|^2 + \|\boldsymbol{\mu}_s(\mathbf{0})\|^2 + \|\mathbf{S}_s(\mathbf{0})\|^2) ds\right] < \infty$$

Here $\|\cdot\|$ is the usual matrix norm, i.e. $\|\mathbf{A}\|^2 = \mathbf{trA'A}$.

We consider a process $\{\boldsymbol{X}_s\}_0^T$ in $\mathbb{R}^d$ which satisfies a stochastic integral equation:

(1.1) $$\boldsymbol{X}_s = \boldsymbol{\xi}_s + \int_0^s \boldsymbol{\mu}_\theta(\boldsymbol{X}_\theta) d\theta + \int_0^s \mathbf{S}_\theta(\boldsymbol{X}_\theta) d\boldsymbol{W}_\theta \quad , \quad s \in [0,T],$$

where $\boldsymbol{W}_s$ is a $\mathbb{R}^{d_1}$ dimensional standard Brownian motion on $(\Omega, \mathfrak{F}, P)$. Condition 1.1 insures that the stochastic integral equation (1.1) has a unique strong solution. (See e.g. Krylov (1980) Theorem 2.5.7)

Let $\delta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$, $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, and $g : \mathbb{R}^d \to \mathbb{R}$ be continuous in $\boldsymbol{\xi}_s$ for each t Borel functions. Also, let

$$\varphi_s = \int_0^s \delta(t + \theta, \boldsymbol{X}_\theta) d\theta.$$

Our objective in this article is to analyze the behavior of a payoff function given by

$$v(t, \boldsymbol{x}) = E\left[\int_0^{T-t} f(t + \theta, \boldsymbol{X}_\theta) e^{-\varphi_\theta} d\theta + g(\boldsymbol{X}_{T-t}) e^{-\varphi_{T-t}}\right]$$

when $\boldsymbol{\xi}_s \equiv \boldsymbol{x}$ and $t \in [0,T]$ are given constant and $\boldsymbol{\mu}_s(\boldsymbol{x}_s)$, $\mathbf{S}_s(\boldsymbol{x})$ are nonstochastic.

First of all, we need an estimate of moments of $\boldsymbol{X}_s$. The following lemma is a consequence of Krylov's Lemma 2.5.1. (pp.78–79) and obvious inequalities.

Lemma 1.2. *Suppose there exist nonnegative $\mathfrak{F}_s$-progressively measurable processes α_s and β_s which satisfy*

$$\|\boldsymbol{\mu}_s(\boldsymbol{x})\| \leq \alpha_s + K^2\|\boldsymbol{x}\| \quad , \quad \|\mathbf{S}_s(\boldsymbol{x})\|^2 \leq 2\beta_s^2 + 2K^2\|\boldsymbol{x}\|^2.$$

Then we have for all $q \geq 1$,

$$(E\|\boldsymbol{X}_s - \boldsymbol{\xi}_s\|^{2q})^{1/q} \leq \int_0^s e^{(4qK^2+1)(s-\theta)}\{E\alpha_\theta^{2q}\}^{1/q}d\theta + (4q-2)\int_0^s e^{(4qK^2+1)(s-\theta)}\{E\beta_\theta^{2q}\}^{1/q}d\theta. \tag{1.2}$$

In particular, if $\alpha_s \leq m$ and $\beta_s \leq m$ for all $s \in [0,t]$,

$$(E\|\boldsymbol{X}_s - \boldsymbol{\xi}_s\|^{2q}) \leq (4q-1)^q m^{2q}(4qK^2+1)^{-q}e^{(4q^2K^2+q)t}. \tag{1.3}$$

Remark. Condition 1.1 automatically implies the growth condition in the lemma with $\alpha_s = \|\boldsymbol{\mu}_s(\mathbf{0})\|$, $\beta_s = \|\mathbf{S}_s(\mathbf{0})\|$ and $K = M$. However, we need to let $K \downarrow 0$ later.

When we consider a derivative of a process depending on parameters in §2, we need moments of difference of two processes. Let $\{\tilde{\boldsymbol{X}}_s\}_0^T$ be a process on $(\Omega, \mathfrak{F}, P)$ given by

$$\tilde{\boldsymbol{X}}_s = \tilde{\boldsymbol{\xi}}_s + \int_0^s \tilde{\boldsymbol{\mu}}_\theta(\tilde{\boldsymbol{X}}_\theta)d\theta + \int_0^s \tilde{\mathbf{S}}_\theta(\tilde{\boldsymbol{X}}_\theta)d\boldsymbol{W}_\theta, \quad s \in [0,T],$$

We have a corollary to Lemma 1.2.

Corollary 1.3. *Suppose $\boldsymbol{\mu}$ and $\mathbf{S}$ satisfy Condition 1.1 (a). Then for all $q \geq 1$ and $s \in [0,t]$,*

$$\begin{aligned} E\|\boldsymbol{X}_s - \tilde{\boldsymbol{X}}_s\|^{2q} \leq{}& \left(4^q + N(q,M)t^{q-1}e^{(4q^2M^2+q)t}\right) E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q} \\ &+ 32^q t^{q-1} E\left[\int_0^s e^{(4q^2M^2+q)(\theta-t)}\|\boldsymbol{\mu}_\theta(\tilde{\boldsymbol{X}}_\theta) - \tilde{\boldsymbol{\mu}}_\theta(\tilde{\boldsymbol{X}}_\theta)\|^{2q}d\theta\right] \\ &+ (128q)^q t^{q-1} E\left[\int_0^s e^{(4q^2M^2+q)(\theta-t)}\|\mathbf{S}_\theta(\tilde{\boldsymbol{X}}_\theta) - \tilde{\mathbf{S}}_\theta(\tilde{\boldsymbol{X}}_\theta)\|^{2q}d\theta\right], \end{aligned} \tag{1.4}$$

where

$$N(q,M) = \left((32M^4)^q + (128qM^2)^q\right)(4q^2M^2+q)^{-1}.$$

This is Krylov's Corollary 2.5.5 with more precise coefficients indicating dependency on q. We need these coefficients in order to estimate exponential transforms.

By using Corollary 1.3, we can estimate moments and exponential transforms of $\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta - \tilde{\boldsymbol{X}}_\theta)\|$.

Theorem 1.4. *Suppose Condition 1.1 is satisfied.*

(a) For all $q \geq 1$ and $s \in [0,t]$ we have

$$\begin{aligned} E\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta - \tilde{\boldsymbol{X}}_\theta\|^{2q} \leq{}& 2^{2q-1}E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q} + N_1(s,q,M) \\ &+ N_2(s,q,M)E\int_0^s e^{\lambda(q,M)(\theta-t)}\|\boldsymbol{\mu}_\theta(\tilde{\boldsymbol{\xi}}_s) - \tilde{\boldsymbol{\mu}}_\theta(\tilde{\boldsymbol{\xi}}_s)\|^{2q}d\theta \\ &+ N_2(s,q,M)E\int_0^s e^{\lambda(q,M)(\theta-t)}\|\mathbf{S}_\theta(\tilde{\boldsymbol{\xi}}_s) - \tilde{\mathbf{S}}_\theta(\tilde{\boldsymbol{\xi}}_s)\|^{2q}d\theta, \end{aligned}$$

where

$$\lambda(q, M) = 4q^2M^2 + q$$

$$N_1(s, q, M) = 2^{2q}M^{2q}N_2(s, q, M)\int_0^s e^{\lambda(q,M)(\theta-t)}E\|\boldsymbol{X}_\theta - \boldsymbol{\xi}_s\|^{2q}d\theta$$

$$N_2(s, q, M) = (2^{11q}t^{q-1}q^q + 1)2^{6q-1}e^{2qM^2t}.$$

(b) Further suppose there exists a constant $b > 0$ such that

$$\|\boldsymbol{\mu}_s(\boldsymbol{x})\| + \|\tilde{\boldsymbol{\mu}}_s(\boldsymbol{x})\| \leq b \quad , \quad \|\mathbf{S}_s(\boldsymbol{x})\|^2 + \|\tilde{\mathbf{S}}_s(\boldsymbol{x})\|^2 \leq 2b^2.$$

Then for all $0 < \gamma < 2$, $a > 0$ and $s \in [0, t]$, there exists a constant $N_3(a, b, \gamma) < \infty$ such that

$$E\exp\left[a\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta - \tilde{\boldsymbol{X}}_\theta\|^\gamma\right] \leq N_3(a, b, \gamma)\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{\xi}_\theta - \tilde{\boldsymbol{\xi}}_\theta\|^\gamma)\right]\right\}^{\frac{1}{2}}$$

Proof. Part (a) is Krylov's Theorem 2.5.9 with more precise expression of constants. We sketch how these constants are obtained.

Let $\max\{\sup_{0\leq s\leq T}\|\boldsymbol{\mu}_s(\mathbf{0})\|, \sup_{0\leq s\leq T}\|\mathbf{S}_s(\mathbf{0})\|\} = m$ which is finite by Condition 1.1 (b). Note that by the remark to Lemma 1.2, Condition 1.1 (a) implies the condition needed in Lemma 1.2 with $K = M$, $\alpha_s = \|\boldsymbol{\mu}_s(\mathbf{0})\|$ and $\beta_s = \|\mathbf{S}_s(\mathbf{0})\|$. Then as in the proof of Krylov's Theorem 2.5.9 and by (1.3) of Lemma 1.2, for all $s \in [0, t]$ we have

$$\begin{aligned}&E\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta - \boldsymbol{\xi}_\theta\|^{2q}\\ &\leq 2^{2q+1}e^{2qK^2t}\left\{E\|\boldsymbol{X}_s - \boldsymbol{\xi}_s\|^{2q} + t^{2q-1}E\int_s^t\|\boldsymbol{\mu}_\theta(\mathbf{0})\|^{2q}d\theta\right\}\\ &\leq 2^{2q+1}e^{2qK^2t}\left\{(4q-1)^q m^{2q}(4qK^2+1)^{-q}e^{(4q^2K^2+q)t} + t^{2q}m^{2q}\right\}\end{aligned}\tag{1.5}$$

Here we intentionally wrote the Lipschitz constant as K instead of M. This is because we need to let $K \downarrow 0$ for proving the exponential estimate. It should be noted that when Krylov proved the first inequality of (1.5), the Lipschitz condition was used only to insure the linear growth of $\|\boldsymbol{\mu}_s(\boldsymbol{x})\|$ and $\|\mathbf{S}_s(\boldsymbol{x})\|$.

As in the proof of Krylov's Corollary 2.5.5, let $\boldsymbol{Y}_s = (\boldsymbol{X}_s - \tilde{\boldsymbol{X}}_s) - (\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s)$. Then $\boldsymbol{Y}_x$ satisfies a stochastic integral equation

$$\mathbf{Y}_s = \int_0^s \boldsymbol{\mu}_\theta^\Delta(\boldsymbol{Y}_\theta)d\theta + \int_0^s \mathbf{S}_\theta^\Delta(\boldsymbol{Y}_\theta)d\boldsymbol{W}_\theta$$

where

$$\boldsymbol{\mu}_\theta^\Delta(\boldsymbol{y}) = [\boldsymbol{\mu}_\theta(\boldsymbol{y} + \tilde{\boldsymbol{X}}_\theta + \boldsymbol{\xi}_\theta - \tilde{\boldsymbol{\xi}}_\theta) - \tilde{\boldsymbol{\mu}}_\theta(\tilde{\boldsymbol{\xi}}_\theta)] \quad , \quad \mathbf{S}_\theta^\Delta(\boldsymbol{y}) = [\mathbf{S}_\theta(\boldsymbol{y} + \tilde{\boldsymbol{X}}_\theta + \boldsymbol{\xi}_\theta - \tilde{\boldsymbol{\xi}}_\theta) - \tilde{\mathbf{S}}_\theta(\tilde{\boldsymbol{\xi}}_\theta)].$$

Since $\boldsymbol{\mu}_\theta^\Delta(\boldsymbol{y})$ and $\mathbf{S}_\theta^\Delta(\boldsymbol{y})$ satisfies the Lipschitz condition with constant M, we can apply Corollary 1.3 to $\boldsymbol{Y}_s$. Hence by the inequalities

$$\|\boldsymbol{X}_s - \tilde{\boldsymbol{X}}_s\|^{2q} \leq 2^{2q-1}(\|\boldsymbol{Y}_s\|^{2q} + \|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q}),$$

$$\begin{aligned}\|\boldsymbol{\mu}_s^\Delta(\mathbf{0})\|^{2q} &\leq 2^{2q-1}(\|\boldsymbol{\mu}_s(\tilde{\boldsymbol{X}}_s + \boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s) - \boldsymbol{\mu}_s(\tilde{\boldsymbol{\xi}}_s)\|^{2q} + \|\boldsymbol{\mu}_s(\tilde{\boldsymbol{\xi}}_s) - \tilde{\boldsymbol{\mu}}_s(\tilde{\boldsymbol{\xi}}_s)\|^{2q}) \\ &\leq 2^{2q-1}(M^{2q}\|\boldsymbol{X}_s - \boldsymbol{\xi}_s\|^{2q} + \|\boldsymbol{\mu}_s(\tilde{\boldsymbol{\xi}}_s) - \tilde{\boldsymbol{\mu}}_s(\tilde{\boldsymbol{\xi}}_s)\|^{2q}),\end{aligned}$$

$$\|\mathbf{S}_s^\Delta(\mathbf{0})\|^{2q} \leq 2^{2q-1}(M^{2q}\|\boldsymbol{X}_s - \boldsymbol{\xi}_s\|^{2q} + \|\mathbf{S}_s(\tilde{\boldsymbol{\xi}}_s) - \tilde{\mathbf{S}}_s(\tilde{\boldsymbol{\xi}}_s)\|^{2q}),$$

and by the first inequality of (1.5) and Corollary 1.3, we obtain

$$\begin{aligned}E\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta - \tilde{\boldsymbol{X}}_\theta\|^{2q} &\leq 2^{2q-1}(E\sup_{0\leq\theta\leq s}\|\boldsymbol{Y}_\theta\|^{2q} + E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q}) \\ &\leq 2^{4q}e^{2qM^2t}\left\{E\|\boldsymbol{Y}_s\|^{2q} + t^{2q-1}E\int_s^t \|\boldsymbol{\mu}_\theta^\Delta(\mathbf{0})\|^{2q}d\theta\right\} \\ &\qquad + 2^{2q-1}E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q} \\ &\leq (2^{5q}t^{q-1}+1)2^{4q}e^{2qM^2t}E\int_0^s e^{(4q^2M^2+q)(\theta-t)}\|\boldsymbol{\mu}_\theta^\Delta(\mathbf{0})\|^{2q}d\theta \\ &\qquad + 2^{11q}q^qt^{q-1}e^{2qM^2t}E\int_0^s e^{(4q^2M^2+q)(\theta-t)}\|\mathbf{S}_\theta^\Delta(\mathbf{0})\|^{2q}d\theta \\ &\qquad + 2^{2q-1}E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q} \\ &\leq 2^{2q-1}E\|\boldsymbol{\xi}_s - \tilde{\boldsymbol{\xi}}_s\|^{2q} + N_1(s,q,M) \\ &\qquad + N_2(s,q,M)E\int_0^s e^{\lambda(q,M)(\theta-t)}\|\boldsymbol{\mu}_\theta(\tilde{\boldsymbol{\xi}}_s) - \tilde{\boldsymbol{\mu}}_\theta(\tilde{\boldsymbol{\xi}}_s)\|^{2q}d\theta \\ &\qquad + N_2(s,q,M)E\int_0^s e^{\lambda(q,M)(\theta-t)}\|\mathbf{S}_\theta(\tilde{\boldsymbol{\xi}}_s) - \tilde{\mathbf{S}}_\theta(\tilde{\boldsymbol{\xi}}_s)\|^{2q}d\theta.\end{aligned}$$

Next we prove assertion (b). Note that

$$\|\boldsymbol{\mu}_s^\Delta(\boldsymbol{x})\| \leq 2b \quad , \quad \|\mathbf{S}_s^\Delta(\boldsymbol{x})\| \leq 8b^2.$$

Then we can apply the second inequality of (1.5) to the process $\boldsymbol{Y}_s$ with $m = 3b$. Then for all $K > 0$, we have

$$E\sup_{0\leq\theta\leq s}\|\boldsymbol{Y}_\theta\|^{2q} \leq 2^{2q+1}(3b)^{2q}e^{2qK^2t}\left\{\left(\frac{4q-1}{4qK^2+1}\right)^q e^{(4q^2K^2+q)t} + t^{2q}\right\}.$$

Since the r.h.s. is continuous in K, the inequality still holds if we let $K \downarrow 0$. Then for all $q \geq 1$ and $s \in [0,t]$,

$$E\sup_{0\leq\theta\leq s}\|\boldsymbol{Y}_\theta\|^{2q} \leq 2^{4q+2}(3b)^{2q}e^{q(T-t)}q^q = (K_1(b))^{2q}(2q)^q. \tag{1.6}$$

Here we used $t^{2q} \leq 4^q \exp[qt]$, $q^q \geq 1, q \geq 1$ and obvious inequalities.

Now we proceed as follows. Assume without loss of generality that $2 > \gamma \geq 1$. Then by (1.6), we have

$$
\begin{aligned}
&E\exp\left[a\sup_{0\leq\theta\leq s}\|\boldsymbol{X}_\theta-\tilde{\boldsymbol{X}}_\theta\|^\gamma\right]\\
\leq &E\sup_{0\leq\theta\leq s}\left\{\exp[2^{\gamma-1}a(\|\boldsymbol{Y}_\theta\|^\gamma+\|\boldsymbol{\xi}_\theta-\tilde{\boldsymbol{\xi}}_\theta\|^\gamma)]\right\}\\
\leq &\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{Y}_\theta\|^\gamma\right]\right\}^{\frac{1}{2}}\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{\xi}_\theta-\tilde{\boldsymbol{\xi}}_\theta\|^\gamma)\right]\right\}^{\frac{1}{2}}\\
\leq &\left\{\sum_{n=0}^{\infty}(4a)^nE\sup_{0\leq\theta\leq s}\|\boldsymbol{Y}_\theta\|^{\gamma n}/n!\right\}^{\frac{1}{2}}\\
&\qquad\times\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{\xi}_\theta-\tilde{\boldsymbol{\xi}}_\theta\|^\gamma)\right]\right\}^{\frac{1}{2}}\\
\leq &\left\{\sum_{n=0}^{\infty}(4a)^n\{K_1(b)\}^{\gamma n}(\gamma n)^{\gamma n/2}/n!\right\}^{\frac{1}{2}}\\
&\qquad\times\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{\xi}_\theta-\tilde{\boldsymbol{\xi}}_\theta\|^\gamma)\right]\right\}^{\frac{1}{2}}\\
= &N_3(a,b,\gamma)\left\{E\exp\left[4a\sup_{0\leq\theta\leq s}\|\boldsymbol{\xi}_\theta-\tilde{\boldsymbol{\xi}}_\theta\|^\gamma)\right]\right\}^{\frac{1}{2}}
\end{aligned}
$$

The finiteness of $N_3(a,b,\gamma)$ follows from Stirling's formula. Take n_0 so large that for any $n \geq n_0$, $n! \geq \sqrt{\pi/2}n^{n+1/2}e^{-n}$ and $n^{1-\gamma/2} \geq 8a\{K_1(b)\}^\gamma\gamma^{\gamma/2}$. Then

$$\sum_{n=n_0}^{\infty}(4a)^n\{K_1(b)\}^{\gamma n}(\gamma n)^{\gamma n/2}/n! \leq (2\sqrt{\pi/2})^{-1}\sum_{n=n_0}^{\infty}2^{-n}n^{-1/2} < \infty.$$

Hence we have assertion (b). □

II. $\mathcal{L}$-derivative processes

We define spaces $\mathcal{L}$, $\mathcal{L}B$ $\mathcal{L}E$ and $\mathcal{L}$-derivative process of $\{\boldsymbol{X}_s\}_0^T$ which will be needed to estimate derivatives of $v(t,\boldsymbol{x})$. We write $\boldsymbol{X}_s \in \mathcal{L}$ if for all $q \geq 1$

$$E\int_0^T\|\boldsymbol{X}_s\|^q ds < \infty.$$

We write $\boldsymbol{X}_s \in \mathcal{L}B$ if for all $q \geq 1$

$$E \sup_{0 \leq s \leq T} \|\boldsymbol{X}_s\|^q < \infty.$$

When we deal with exponential growth, it is convenient to consider the space $\mathcal{L}E$ which is the set of all $\{\boldsymbol{X}_s\}_0^T$ satisfying a condition

$$E \sup_{0 \leq s \leq T} \{\exp[a\|\boldsymbol{X}_s\|^\gamma]\} < \infty \quad \text{for all } \ a > 0 \text{ and } 0 < \gamma < 2.$$

It is clear that $\mathcal{L}E \subset \mathcal{L}B \subset \mathcal{L}$.

For a sequence $\boldsymbol{X}_s^1, \ldots, \boldsymbol{X}_s^n, \ldots \in \mathcal{L}$, we write $\mathcal{L}\text{-}\lim_{n\to\infty} \boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ if for all $q \geq 1$

$$\lim_{n\to\infty} E \int_0^T \|\boldsymbol{X}_s^n - \boldsymbol{X}_s^0\|^q ds = 0$$

Similarly, $\mathcal{L}B\text{-}\lim_{n\to\infty} \boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ if for all $q \geq 1$

$$\lim_{n\to\infty} E \sup_{0 \leq s \leq T} \|\boldsymbol{X}_s^n - \boldsymbol{X}_s\|^q = 0,$$

and $\mathcal{L}E\text{-}\lim_{n\to\infty} \boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ if for all $a > 0$ and $0 < \gamma < 2$,

$$\lim_{n\to\infty} E \sup_{0 \leq s \leq T} \{\exp[a\|\boldsymbol{X}_s^n - \boldsymbol{X}_s\|^\gamma]\} = 1.$$

When the process $\boldsymbol{X}_s^{\boldsymbol{p}}$ is parametrized by a real vector parameter $\boldsymbol{p} \in D \subset \mathbb{R}^{d_2}$, we say $\boldsymbol{X}_s^{\boldsymbol{p}}$ is $\mathcal{L}(\mathcal{L}B,\ \mathcal{L}E)$-continuous at $\boldsymbol{p}_0$ if for any sequence $\boldsymbol{p}_n$ such that $\|\boldsymbol{p}_n - \boldsymbol{p}_0\| \to 0$, $\mathcal{L}\text{-}(\mathcal{L}B\text{-}, \mathcal{L}E\text{-})\lim_{n\to\infty} \boldsymbol{X}_s^{\boldsymbol{p}_n} = \boldsymbol{X}_s^{\boldsymbol{p}_0}$. It is clear that $\mathcal{L}E$-continuity $\Rightarrow$ $\mathcal{L}B$-continuity $\Rightarrow$ $\mathcal{L}$-continuity.

For a unit vector $\boldsymbol{l} \in \mathbb{R}^{d_2}$, we say $\boldsymbol{Y}_s^{\boldsymbol{p}} \in \mathcal{L}$ is an $\mathcal{L}(\mathcal{L}B)$-derivative of $\boldsymbol{X}_s^{\boldsymbol{p}}$ in $\boldsymbol{p}$ along a direction $\boldsymbol{l}$ when

$$\boldsymbol{Y}_s^{\boldsymbol{p}} = \mathcal{L}(\mathcal{L}B)\text{-}\frac{\partial}{\partial \boldsymbol{l}} \boldsymbol{X}_s^{\boldsymbol{p}} = \mathcal{L}(\mathcal{L}B)\text{-}\lim_{n\to\infty} r_n^{-1}(\boldsymbol{X}_s^{\boldsymbol{p}_0 + r_n \boldsymbol{l}} - \boldsymbol{X}_s^{\boldsymbol{p}_0})$$

exists for any sequence of real numbers r_n.

The space $\mathcal{L}E$ and $\mathcal{L}E$-convergence are not introduced in Krylov since they are not necessary for proving results with polynomial growth. We do not need the notion of $\mathcal{L}E$-derivative in the following argument. In fact, due to the unavailability of exponential estimates, it is not easy to prove $\mathcal{L}E$-differentiability of the processes we are interested in.

We list elementary properties of $\mathcal{L}$-derivatives. The proofs are are not hard.

Properties of $\mathcal{L}$-derivatives.

(2.1) If $\boldsymbol{X}_s^{\boldsymbol{p}}$ is $\mathcal{L}$-continuously $\mathcal{L}$-differentiable, then $u(\boldsymbol{p}) = E(\boldsymbol{X}_s^{\boldsymbol{p}})$ is continuously differentiable in $\boldsymbol{p}$. $\frac{\partial}{\partial \boldsymbol{l}} E\boldsymbol{X}_s^{\boldsymbol{p}} = E\left(\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}} \boldsymbol{X}_s^{\boldsymbol{p}}\right)$, if the r.h.s. exists.

(2.2) $\mathcal{L}\text{-}\frac{\partial}{\partial l}\boldsymbol{X}_s^{\boldsymbol{p}} = \mathcal{L}B\text{-}\frac{\partial}{\partial l}\boldsymbol{X}_s^{\boldsymbol{p}}$, *if the r.h.s. exists.*
(2.3) The process $\int_0^s \boldsymbol{X}_\theta^{\boldsymbol{p}} d\theta$ *is* $\mathcal{L}B$*-continuous if* $\boldsymbol{X}_s^{\boldsymbol{p}}$ *is* $\mathcal{L}$*-continuous. Also,* $\mathcal{L}B\text{-}\frac{\partial}{\partial l}\int_0^s \boldsymbol{X}_\theta^{\boldsymbol{p}} d\theta = \int_0^s \mathcal{L}\text{-}\frac{\partial}{\partial l}\boldsymbol{X}_\theta^{\boldsymbol{p}} d\theta$ *holds.*
(2.4) Let $\boldsymbol{W}_s$ *be a* d_1*-dimensional Brownian motion on* $(\Omega, \mathfrak{F}, P)$ *and* $\mathbf{S}_s^{\boldsymbol{p}}$ *be* $d \times d_1$ *matrix which is* $\mathfrak{F}_s$*-progressively measurable. If* $\mathbf{S}_s^{\boldsymbol{p}}$ *is* $\mathcal{L}$*-continuous, then* $\int_0^s \mathbf{S}_\theta^{\boldsymbol{p}} d\boldsymbol{W}_\theta$ *is* $\mathcal{L}B$*-continuous. Further, if* $\mathbf{S}_s^{\boldsymbol{p}}$ *is* $\mathcal{L}$*-differentiable, then* $\int_0^s \mathbf{S}_\theta^{\boldsymbol{p}} d\boldsymbol{W}_\theta$ *is* $\mathcal{L}B$*-differentiable and* $\mathcal{L}B\text{-}\frac{\partial}{\partial l}\int_0^s \mathbf{S}_\theta^{\boldsymbol{p}} d\boldsymbol{W}_\theta = \int_0^s \mathcal{L}\text{-}\frac{\partial}{\partial l}\mathbf{S}_\theta^{\boldsymbol{p}} d\boldsymbol{W}_\theta$.
(2.5) Let $\boldsymbol{X}_s^{\boldsymbol{p}}$ *be* $\mathcal{L}B$*-continuous at* $\boldsymbol{p}$ *and continuous in* s*. Suppose* $\tau^{\boldsymbol{p}}$ *is a random variable taking values in* $[0,T]$ *and continuous in probability at* $\boldsymbol{p}$*. Then* $\boldsymbol{X}_{\tau^{\boldsymbol{p}}}^{\boldsymbol{p}}$ *is* $\mathcal{L}$*-continuous at* $\boldsymbol{p}$*.*

In order to prove $\mathcal{L}$-continuity and $\mathcal{L}$-differentiability of composite processes, we need the following lemmas.

Lemma 2.1. *Let* $f_s(\boldsymbol{x})$ *be random variables defined for* $s \in [0,T]$ *and* $\boldsymbol{x} \in \mathbb{R}^d$*. Suppose either*
(a) Let a sequence of processes $\boldsymbol{X}_s^1, \cdots, \boldsymbol{X}_s^n, \cdots \in \mathcal{L}$ *converges in* $\mathcal{L}$ *to* $\boldsymbol{X}_s^0$*. Suppose there exist* $K, a > 0$ *such that* $|f_s(\boldsymbol{x})| \le K(1 + \|\boldsymbol{x}\|^a)$ *for all* $s \in [0,T]$ *and* $\boldsymbol{x} \in \mathbb{R}^d$*,*
or,
(b) Let a sequence of processes $\boldsymbol{X}_s^1, \cdots, \boldsymbol{X}_s^n, \cdots \in \mathcal{L}E$ *converges in* $\mathcal{L}E$ *to* $\boldsymbol{X}_s^0$*. Suppose there exist* $K, a > 0$ *and* $0 < \gamma < 2$ *such that* $|f_s(\boldsymbol{x})| \le K(1 + \exp[a\|\boldsymbol{x}\|^\gamma])$*,*
is satisfied.
Then we have $\mathcal{L}\text{-}\lim f_s(\boldsymbol{X}_s^n) = f_s(\boldsymbol{X}_s^0)$*.*

Proof. Assertion (a) is Krylov's Lemma 2.7.6. Assertion (b) can be proved in a similar manner. We outline the proof here. First by an inequality $|f_s(\boldsymbol{x})|^q \le 2^{q-1}K^q(1 + \exp[aq\|\boldsymbol{x}\|^\gamma]$, we note that $f_s(\boldsymbol{X}_s^n)$ is in $\mathcal{L}$ for all n. Write $h_s(\boldsymbol{Z}_s^n) = f_s(\boldsymbol{X}_s^n) - f_s(\boldsymbol{X}_s^0)$ with $\boldsymbol{Z}_s^n = \boldsymbol{X}_s^n - \boldsymbol{X}_s^0$ and $h_s(\boldsymbol{z}) = f_s(\boldsymbol{z} + \boldsymbol{X}_s^0) - f_s(\boldsymbol{X}_s^0)$. Since $E\int_0^T \|\boldsymbol{Z}_\theta^n\| d\theta \to 0$, we have $\boldsymbol{Z}_s^n \to \boldsymbol{0}$ in measure $dP \times dt$. Thus by Lemma 2.7.5 of Krylov, we have $h_s(\boldsymbol{Z}_s^n) \to 0$ in measure $dP \times dt$. Define a sequence of bounded processes $g_s^n = |h_s(\boldsymbol{Z}_s^n)|(1 + |h_s(\boldsymbol{Z}_s^n)|)^{-1}$. Then $g_s^n \to 0$ in measure $dP \times dt$ and thus for any $q \ge 1$,

$$E\int_0^T \|g_\theta^n\|^{2q} d\theta \to 0.$$

Since $\mathcal{L}E\text{-}\lim_{n\to\infty} \boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ and $\boldsymbol{X}_s^0 \in \mathcal{L}E$,

$$\sup_n E \sup_{0\le\theta\le T} e^{aq\|\boldsymbol{X}_\theta^n\|^\gamma} \le \left\{\sup_n E \sup_{0\le\theta\le T} e^{4aq\|\boldsymbol{X}_\theta^n - \boldsymbol{X}_\theta^0\|^\gamma}\right\}^{\frac{1}{2}} \left\{E \sup_{0\le\theta\le T}\left\{e^{4aq\|\boldsymbol{X}_\theta^0\|^\gamma}\right\}\right\}^{\frac{1}{2}} < \infty,$$

and thus we have

$$\sup_n E \sup_{0\le\theta\le T} \{|h_\theta(\boldsymbol{X}_\theta^n)|^q\} < \infty.$$

Hence we derive

$$E\int_0^T |h(\boldsymbol{Z}_\theta^n)|^q d\theta \le \left\{E\int_0^T |g_\theta^n|^{2q} d\theta\right\}^{\frac{1}{2}} \left\{2^{2q-1}TE \sup_{0\le\theta\le T}\{1 + |h_\theta(\boldsymbol{X}_\theta^n))|^{2q}\}\right\}^{\frac{1}{2}} \to 0,$$

which shows $\mathcal{L}\text{-}\lim_{n\to\infty} f_s(\boldsymbol{X}_s^n) = f_s(\boldsymbol{X}_s^0)$. $\square$

Lemma 2.2. *Suppose either condition (a) or (b) of Lemma 2.1 is satisfied. Let a d-dimensional process $\boldsymbol{X}_s^n(u), u \in [0,1]$ be continuous in u and $\|\boldsymbol{X}_s^n(u) - \boldsymbol{X}_s^0\| \le \|\boldsymbol{X}_s^n - \boldsymbol{X}_s^0\|$. Then*

$$\mathcal{L}\text{-}\lim_{n\to\infty} \int_0^1 f_s(\boldsymbol{X}_s^n(u))du = f_s(\boldsymbol{X}_s^0).$$

Proof. By assumption, it is easy to see that $\boldsymbol{X}_s^n(u) \in \mathcal{L}$ (or $\mathcal{L}E$), and $\mathcal{L}(\mathcal{L}E)\text{-}\boldsymbol{X}_s^n(u) = \boldsymbol{X}_s^0$ for each u. Therefore by Lemma 2.1,

$$E\int_0^T |f_\theta(\boldsymbol{X}_\theta^n(u)) - f_\theta(\boldsymbol{X}_\theta^0)|^q d\theta \to 0$$

as $n \to 0$ for all $u \in [0,1]$ and $q \ge 1$. Hence by Fubini's theorem, Hölder's inequality and dominated convergence, we have for all $q \ge 1$,

$$\begin{aligned} E\int_0^T \left|\int_0^1 f_\theta(\boldsymbol{X}_\theta^n(u)du - f_\theta(\boldsymbol{X}_\theta^0)\right|^q d\theta &\le E\int_0^T \int_0^1 |f_\theta(\boldsymbol{X}_\theta^n(u)) - f_\theta(\boldsymbol{X}_\theta^0)|^q du d\theta \\ &\le \int_0^1 E\int_0^T |f_\theta(\boldsymbol{X}_\theta^n(u)) - f_\theta(\boldsymbol{X}_\theta^0)|^q du d\theta \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This completes the proof. $\square$

By using above lemmas, we can prove the following result on $\mathcal{L}$-continuity and $\mathcal{L}$-differentiability of composite processes.

Theorem 2.3. *Suppose either condition*
(a) $\boldsymbol{X}_s^p$ is $\mathcal{L}$-continuous and n times $\mathcal{L}$-continuously $\mathcal{L}$-differentiable. $f_s(\boldsymbol{x})$ is n times continuously differentiable in $\boldsymbol{x}$. Further, the absolute value of each derivative including $f_s(\boldsymbol{x})$ itself does not exceed $K(1 + \|\boldsymbol{x}\|^a)$ for some $K, a > 0$,
or,
(b) $\boldsymbol{X}_s^p$ is $\mathcal{L}E$-continuous and n times $\mathcal{L}$-continuously $\mathcal{L}$-differentiable. $f_s(\boldsymbol{x})$ is n times continuously differentiable in $\boldsymbol{x}$. Further, the absolute value of each derivative including $f_s(\boldsymbol{x})$ itself does not exceed $K(1 + \exp[a\|\boldsymbol{x}\|^\gamma])$ for some $K, a > 0$ and $0 < \gamma < 2$,
Then $f_s(\boldsymbol{X}_s^p)$ is also n times $\mathcal{L}$-continuously $\mathcal{L}$-differentiable. In particular, we have for a unit vector $\boldsymbol{l} \in R^d$,

$$\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}} f_s(\boldsymbol{X}_s^p) = f_{s(\boldsymbol{Y}_s^p)}(\boldsymbol{X}_s^p)\|\boldsymbol{Y}_s^p\|, \tag{2.6}$$

$$\mathcal{L}\text{-}\frac{\partial^2}{\partial \boldsymbol{l}^2} f_s(\boldsymbol{X}_s^p) = f_{s(\boldsymbol{Z}_s^p)}(\boldsymbol{X}_s^p)\|\boldsymbol{Z}_s^p\| + f_{s(\boldsymbol{Y}_s^p)(\boldsymbol{Y}_s^p)}(\boldsymbol{X}_s^p)\|\boldsymbol{Y}_s^p\|^2, \tag{2.7}$$

where $\boldsymbol{Y}_s^{\boldsymbol{p}} = \mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{\boldsymbol{p}}$, $\boldsymbol{Z}_s^{\boldsymbol{p}} = \mathcal{L}\text{-}\frac{\partial^2}{\partial \boldsymbol{l}^2}\boldsymbol{X}_s^{\boldsymbol{p}}$ and $f_{s(\boldsymbol{y})}(\boldsymbol{x}) = \sum_{j=1}^{d} \frac{\partial f_s(\boldsymbol{x})}{\partial x_j} \cdot y_j \|\boldsymbol{y}\|^{-1}$ (if $\boldsymbol{y} \neq 0$) or 0 (if $\boldsymbol{y} = 0$).

Proof. Part (a) is Krylov's Theorem 2.7.9. As we have done so far, part (b) can be proved similarly. For $\boldsymbol{p}_0, \boldsymbol{l} \in \mathbb{R}^d$ and $u \in [0,1]$, let

$$\boldsymbol{X}_s^{(n)}(u) = u\boldsymbol{X}_s^{\boldsymbol{p}_0+r_n\boldsymbol{l}} + (1-u)\boldsymbol{X}_s^{\boldsymbol{p}_0} \quad , \quad \boldsymbol{Y}_s^{(n)} = r_n^{-1}(\boldsymbol{X}_s^{\boldsymbol{p}+r_n\boldsymbol{l}} - \boldsymbol{X}_s^{\boldsymbol{p}})$$

Then

$$\begin{aligned} r_n^{-1}(f_s(\boldsymbol{X}_s^{\boldsymbol{p}_0+r_n\boldsymbol{l}}) - f_s(\boldsymbol{X}_s^{\boldsymbol{p}_0})) &= r_n^{-1}\int_0^1 \frac{\partial}{\partial u} f_s(\boldsymbol{X}_s^{(n)}(u))du \\ &= \sum_{j=1}^{d} Y_{s,j}^{(n)} \int_0^1 f_{s,x_j}(\boldsymbol{X}_s^{(n)}(u))du \end{aligned}$$

where $Y_{s,j}^{(n)}$ is the j-th component of $\boldsymbol{Y}_s^{(n)}$ and $f_{s,x_j}(\boldsymbol{x}) = \frac{\partial f_s(\boldsymbol{x})}{\partial x_j}, (x_1, \cdots, x_j, \cdots, x_n) = \boldsymbol{x}$. Since $\|\boldsymbol{X}_s^{(n)}(u) - \boldsymbol{X}_s^{\boldsymbol{p}_0}\| \leq \|\boldsymbol{X}_s^{\boldsymbol{p}_0+\boldsymbol{l}/n} - \boldsymbol{X}_s^{\boldsymbol{p}_0}\|$ for all $u \in [0,1]$, by Lemma 2.2 (b), we have $\mathcal{L}\text{-}\lim_{n\to\infty}\int_0^1 f_{s,x_j}(\boldsymbol{X}_s^{(n)}(u))du = f_{s,x_j}(\boldsymbol{X}_s^{\boldsymbol{p}_0})$. Note that $f_{s,x_j}(\boldsymbol{X}_s^{\boldsymbol{p}_0})$ is $\mathcal{L}$-continuous by Lemma 2.1 (b). Hence from Lemma 2.1 (a) applied to the function $g(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}'\boldsymbol{y}$, we conclude

$$\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{\boldsymbol{p}_0} = \mathcal{L}\text{-}\lim_{n\to\infty} r_n^{-1}(f_s(\boldsymbol{X}_s^{\boldsymbol{p}_0+r_n\boldsymbol{l}}) - f_s(\boldsymbol{X}_s^{\boldsymbol{p}_0})) = f_{s(\boldsymbol{Y}_s^{\boldsymbol{p}_0})}(\boldsymbol{X}_s^{\boldsymbol{p}_0})\|\boldsymbol{Y}_s^{\boldsymbol{p}_0}\|,$$

where $\boldsymbol{Y}_s^{\boldsymbol{p}_0} = \mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{\boldsymbol{p}_0}$. $\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{\boldsymbol{p}_0}$ is $\mathcal{L}$-continuous since the product of $\mathcal{L}$-continuous process is also $\mathcal{L}$-continuous.

Properties of higher order $\mathcal{L}$-derivative processes can be proved similarly with induction. □

When we prove continuity of payoff functions with respect to time parameter in the next section, we need the following lemmas.

Lemma 2.4. *Let $h_s^n(\boldsymbol{x})$ be nonstochastic, $h_s^n(\boldsymbol{x}) \to 0$ as $n \to \infty$ and there exists $K_R > 0$ such that*

$$\sup_{\|\boldsymbol{x}-\boldsymbol{y}\|<R} |h_s^n(\boldsymbol{x}) - h_s^n(\boldsymbol{y})| \leq K_R\|\boldsymbol{x}-\boldsymbol{y}\| \tag{2.8}$$

for all n and $R > 0$. Suppose either
(a) $\mathcal{L}\text{-}\lim_{n\to\infty}\boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ and for all $q \geq 1$ $\sup_n E\int_0^T \|\boldsymbol{X}_\theta^n\|^q d\theta < \infty$. Further, there exist constants $K, a > 0$ such that $|h_s^n(\boldsymbol{x})| \leq K(1+\|\boldsymbol{x}\|^a)$ for all n,
or,
(b) $\mathcal{L}E\text{-}\lim_{n\to\infty}\boldsymbol{X}_s^n = \boldsymbol{X}_s^0$ and $\sup_n E\sup_{0\leq\theta\leq T}\exp[\alpha\|\boldsymbol{X}_\theta^n\|^\beta] < \infty$ for all $\alpha > 0$, $0 < \beta < 2$. Further, there exist constants $K, a > 0$ and $0 < \gamma < 2$ such that $|h_s^n(\boldsymbol{x})| \leq K(1+e^{a\|\boldsymbol{x}\|^\gamma})$

for all n.
Then $\mathcal{L}\text{-}\lim_{n\to\infty} h_s^n(\boldsymbol{X}_s^n) = 0$.

By using Lemma 2.7.16 of Krylov, the proof can be done in the same way as we proved Lemma 2.1.

Corollary 2.5. *Let* $f_s^{\boldsymbol{p}}(\boldsymbol{x})$ *be continuous in* $\boldsymbol{p} \in D$ *and* $\boldsymbol{x} \in \mathbb{R}^d$. *Also, there exists constant* $K_R > 0$ *such that*

$$\sup_{\|\boldsymbol{x}-\boldsymbol{y}\|<R} |f_s^{\boldsymbol{p}}(\boldsymbol{x}) - f_s^{\boldsymbol{p}}(\boldsymbol{y})| \le K_R\|\boldsymbol{x}-\boldsymbol{y}\| \tag{2.9}$$

for all $\boldsymbol{p} \in D$ *and* $R > 0$. *Suppose either*
(a) $\boldsymbol{X}_s^{\boldsymbol{p}}$ *is* $\mathcal{L}$*-continuous and for all* $q \ge 1$ *and* $\boldsymbol{p} \in D$, *there exists* $\delta > 0$ *such that* $\sup_{\|\boldsymbol{p}\|<\delta} E\int_0^T \|\boldsymbol{X}_\theta^{\boldsymbol{p}}\|^q d\theta < \infty$. *Further, there exist constants* $K, a > 0$ *such that* $|f_s^{\boldsymbol{p}}(\boldsymbol{x})| \le K(1 + \|\boldsymbol{x}\|^a)$ *for all* $\boldsymbol{p} \in D$, *(*K, a *may depend on* $\boldsymbol{p}$.*)*
or,
(b) $\boldsymbol{X}_s^{\boldsymbol{p}}$ *is* $\mathcal{L}E$*-continuous and* $\sup_{\|\boldsymbol{p}\|<\delta} E\sup_{0\le\theta\le T}\exp[\alpha\|\boldsymbol{X}_\theta^{\boldsymbol{p}}\|^\beta] < \infty$ *for all* $\alpha > 0$ *and* $0 < \beta < 2$. *Further, there exist constants* $K, a > 0$ *and* $0 < \gamma < 2$ *such that* $|f_s^{\boldsymbol{p}}(\boldsymbol{x})| \le K(1 + e^{a\|\boldsymbol{x}\|^\gamma})$ *for all* $\boldsymbol{p} \in D$. *(*K, a, γ *may depend on* $\boldsymbol{p}$.*)*
Then $f_s^{\boldsymbol{p}}(\boldsymbol{X}_s^{\boldsymbol{p}})$ *is* $\mathcal{L}$*-continuous.*

Proof. For a given $\boldsymbol{l} \in \mathbb{R}^d$, take any sequence $r_n \downarrow 0$ and apply Lemma 2.4 with $h_s^n(\boldsymbol{x}) = f_s^{\boldsymbol{p}+r_n\boldsymbol{l}}(\boldsymbol{x}) - f_s^{\boldsymbol{p}}(\boldsymbol{x})$. It is easy to see the conditions of Lemma 2.4 are satisfied. □

III. *Properties of stochastic solutions to parabolic PDE's*

In this section, we derive a result on stochastic solutions. First, we need a lemma on taking $\mathcal{L}$-limit in stochastic integral equations.

Lemma 3.1. *Let*

$$\boldsymbol{X}_s^n = \boldsymbol{\xi}_s^n + \int_0^s \boldsymbol{\mu}_\theta^n(\boldsymbol{X}_\theta^n)d\theta + \int_0^s \mathbf{S}_\theta^n(\boldsymbol{X}_\theta^n)d\boldsymbol{W}_\theta, \quad n = 0, 1, \cdots$$

where $\boldsymbol{\mu}_\theta^n(\boldsymbol{x})$ *and* $\mathbf{S}_\theta^n(\boldsymbol{x})$ *satisfy Lipschitz condition uniformly in* n . *Suppose* $\boldsymbol{\mu}_s^n(\boldsymbol{x}) \to \boldsymbol{\mu}_s^0(\boldsymbol{x})$ *and* $\mathbf{S}_s^n(\boldsymbol{x}) \to \mathbf{S}_s^0(\boldsymbol{x})$ *in* $\mathcal{L}$. *If* $\boldsymbol{\xi}_s^n \to \boldsymbol{\xi}_s^0$ *in* $\mathcal{L}(\mathcal{L}B)$, *then* $\mathcal{L}(\mathcal{L}B)\text{-}\lim_{n\to\infty}\boldsymbol{X}_s^n = \boldsymbol{X}_s^0$. *Further, if* $\mathcal{L}E\text{-}\lim_{n\to\infty}\boldsymbol{\xi}_s^n = \boldsymbol{\xi}_s^0$ *and* $\boldsymbol{\mu}_\theta^n(\boldsymbol{x})$, $\mathbf{S}_\theta^n(\boldsymbol{x})$ *are bounded, we have* $\mathcal{L}E\text{-}\lim_{n\to\infty}\boldsymbol{X}_s^n = \boldsymbol{X}_s^0$.

Proof. Let $\mathbf{G}_s^n(\boldsymbol{x}) = \mathbf{S}_s^n(\boldsymbol{x}) - \mathbf{S}_s^n(\boldsymbol{0})$. Then $\mathbf{G}_s^n(\boldsymbol{x})$ satisfy the Lipschitz condition and $\|\mathbf{G}_s^n(\boldsymbol{x})\| \le K\|\boldsymbol{x}\|$ with K the Lipschitz constant. Furthermore, $\mathbf{G}_s^n(\boldsymbol{x}) \to \mathbf{G}_s^0(\boldsymbol{x})$ in $\mathcal{L}$. Therefore by Lemma 2.4 (a), $\mathbf{G}_s^n(\boldsymbol{X}_s^0) \to \mathbf{G}_s^0(\boldsymbol{X}_s^0)$ in $\mathcal{L}$. Similarly, we have $\boldsymbol{\mu}_s^n(\boldsymbol{X}_s^0) \to$

$\mu_s^0(X_s^0)$. Now, applying Theorem 1.4 (b) to X_s^n and X_s^0, we see $X_s^n \to X_s^0$ in $\mathcal{L}(\mathcal{L}B)$ if $\xi_s^n \to \xi_s^0$ in $\mathcal{L}(\mathcal{L}B)$. In order to prove $\mathcal{L}E$-convergence, we only need to apply Theorem 1.4 directly. □

We consider processes $\{X_s^p\}_0^T$ and $\{X_s^{t,x}\}_0^T$ defined by stochastic integral equations:

$$X_s^p = \xi_s(p) + \int_0^s \mu_\theta(p, X_\theta^p)d\theta + \int_0^s \mathrm{S}_\theta(p, X_\theta^p)dW_\theta \quad , \quad s \in [0,T],$$

$$X_s^{t,x} = x + \int_0^s \mu(t+\theta, X_\theta^{t,x})d\theta + \int_0^s \mathrm{S}(t+\theta, X_\theta^{t,x})dW_\theta \quad , \quad s \in [0,T].$$

Here for each $p \in D$ and $x \in \mathbb{R}^d$, $\mu_s(p,x) \in \mathcal{L}$ and $\xi_s(p) \in \mathcal{L}$ for all $p \in D$ and $x \in \mathbb{R}^d$ are $\mathfrak{F}_s$-progressively measurable d-dimensional vector processes and $\mathrm{S}_s(p,x) \in \mathcal{L}$ is a $\mathfrak{F}_s$-progressively measurable $d \times d_1$ random matrix. $p \in D \subset \mathbb{R}^{d_2}$ is a parameter. Also, $\mu(t+\theta, x)$ is a d-dimensional nonrandom vector and $\mathrm{S}(t+\theta,x)$ is a $d \times d_1$ nonrandom matrix. We assume that $\xi_\theta(p)$, $\mu_\theta(p,x)$, $\mu(t+\theta,x)$, $\mathrm{S}(t+\theta,x)$ and $\mathrm{S}_\theta(p,x)$ satisfy Condition 1.1 for all $p \in D$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. Then by Lemma 1.2, the process X_s^p and $X_s^{t,x}$ are in $\mathcal{L}$ for all $p \in D$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. We have a corollary to Lemma 3.1 gives sufficient conditions for $\mathcal{L}$, $\mathcal{L}B$ and $\mathcal{L}E$-continuity of X_s^p. The proof is easy.

Corollary 3.2. *Suppose $\mu_s(p,x)$ and $\mathrm{S}_s(p,x)$ are $\mathcal{L}$-continuous. If $\xi_\theta(p)$ is $\mathcal{L}(\mathcal{L}B)$-continuous, then X_s^p is $\mathcal{L}(\mathcal{L}B)$-continuous. Further if $\{\|\mu_s(p,x)\| + \|\mathrm{S}_s(p,x)\|\}$ is bounded for all $p \in D$ and $\xi_s(p)$ is $\mathcal{L}E$-continuous, then X_s^p is $\mathcal{L}E$-continuous.*

We show $\mathcal{L}$-differentiability of the process X_s^p. It should be noted again that we do not need $\mathcal{L}E$-differentiability. The next theorem is Krylov's 2.8.4. It should be noted again that we do not need a result for $\mathcal{L}E$-differentiability.

Theorem 3.3. *Suppose that $\mu_s(p,x)$ and $\mathrm{S}_s(p,x)$ are i-times continuously differentiable in (p,x), and each derivative does not exceed $K(1+\|x\|)^m$. Then X_s^p is i-times $\mathcal{L}$-continuously differentiable and*

$$\begin{aligned} Y_s^p &= \mathcal{L}\text{-}\frac{\partial}{\partial l}X_s^p \\ &= \mathcal{L}\text{-}\frac{\partial}{\partial l}\xi_s^p + \int_0^s \left\{\mu_{\theta,(l),*}(p, X_\theta^p) + \mu_{\theta,*,(Y_\theta^p)}(p, X_\theta^p)\|Y_\theta^p\|\right\} d\theta \\ &\quad + \int_0^s \left\{\mathrm{S}_{\theta,(l),*}(p, X_\theta^p) + \mathrm{S}_{\theta,*,(Y_\theta^p)}(p, X_\theta^p)\|Y_\theta^p\|\right\} dW_\theta \end{aligned} \tag{3.1}$$

where $\mu_{s,*,(x)}(p,x)$ and $\mu_{s,(l),*}$ denote directional derivatives of $\mu_s(p,x)$ with respect to x and p respectively. (Similarly for $\mathrm{S}_{s,*,(x)}(p,x)$ and $\mathrm{S}_{s,(l),*}$.) Also, for $l, r \in \mathbb{R}^{d_2}$, we have an expression of second $\mathcal{L}$-derivative as

$$\begin{aligned} Z_s^p &= \mathcal{L}\text{-}\frac{\partial}{\partial r}\left(\mathcal{L}\text{-}\frac{\partial}{\partial l}X_s^p\right) \\ &= \zeta_s^p + \int_0^s \mu_{\theta,*,(Z_\theta^p)}(p, X_\theta^p)\|Z_\theta^p\|d\theta + \int_0^s \mathrm{S}_{(Z_\theta^p)}(p, X_\theta^p)\|Z_\theta^p\|dW_\theta \end{aligned} \tag{3.2}$$

where

$$\zeta_s^{p} = \mathcal{L}\text{-}\frac{\partial}{\partial r}\left[\mathcal{L}\text{-}\frac{\partial}{\partial l}\xi_s(p) + \int_0^s \mu_{\theta,(l),*}(p, X_\theta^{p})d\theta + \int_0^s \mathrm{S}_{\theta,(l),*}(p, X_\theta^{p})dW_\theta\right]$$
$$+\sum_{j=1}^{d}\int_0^s (Y_\theta^{p})_j\left\{\frac{\partial}{\partial x_j}\mu_{\theta,(r),*}(p, X_\theta^{p}) + \frac{\partial}{\partial x_j}\mu_{\theta,*,(Y_\theta^{p})}(p, X_\theta^{p})\|Y_\theta^{p}\|\right\}d\theta$$
$$+\sum_{j=1}^{d}\int_0^s (Y_\theta^{p})_j\left\{\frac{\partial}{\partial x_j}\mathrm{S}_{\theta,(r),*}(p, X_\theta^{p}) + \frac{\partial}{\partial x_j}\mathrm{S}_{\theta,*,(Y_\theta^{p})}(p, X_\theta^{p})\|Y_\theta^{p}\|\right\}dW_\theta.$$

Here $(Y_\theta^{p})_j$ is the j-th element of Y_θ^{p}.

Proof. We just outline the proof. Take a sequence of real numbers $r_n \downarrow 0$ and $l \in \mathbb{R}^{d_2}$. Define processes

$$X_s^{p,n}(u) = uX^{p+r_n l} + (1-u)X_s^{p}, \quad u \in [0,1].$$
$$Y_s^{p,n} = r_n^{-1}(X_s^{p+r_n l} - X_s^{p}),$$

it is not hard to see that $Y_s^{p,n}$ can be expressed as

$$Y_s^{p,n} = \eta_s^{p,n} + \int_0^s \nu_\theta^n(p, Y_\theta^{p,n})d\theta + \int_0^s \mathrm{Q}_\theta^n(p, Y_\theta^{p,n})dW_\theta,$$

where

$$\eta_s^{p,n} = r_n^{-1}(\xi_s(p+r_n l) - \xi_s(p) + \int_0^s\left[\int_0^1 \mu_{\theta,(l),*}(p+ur_n l, X_\theta^{p,n}(u))du\right]d\theta$$
$$+\int_0^s\left[\int_0^1 \mathrm{S}_{\theta,(l),*}(p+ur_n l, X_\theta^{p,n}(u))du\right]dW_\theta$$
$$\nu_s^n(p,x) = \int_0^1 \|x\|\mu_{\theta,*,(x)}(p+ur_n l, X_s^{p,n}(u))du$$
$$\mathrm{Q}_s^n(p,x) = \int_0^1 \|x\|\mathrm{S}_{\theta,*,(x)}(p+ur_n l, X_s^{p,n}(u))du.$$

By applying Lemma 2.2 to the process $(p+ur_n l, X_s^{p,n}(u))$ and from Lemma 3.1, we see that

$$\mathcal{L}\text{-}\lim_{n\to\infty}\nu_s^n(p,x) = \|x\|\mu_{\theta,*,(x)}(p, X_s^{p}(u))$$
$$\mathcal{L}\text{-}\lim_{n\to\infty}\mathrm{Q}_s^n(p,x) = \|x\|\mathrm{S}_{\theta,*,(x)}(p, X_s^{p}(u))$$
$$\mathcal{L}\text{-}\lim_{n\to\infty}\eta_s^{p,n} = \mathcal{L}\text{-}\frac{\partial}{\partial l}\xi_s(p) + \int_0^s \mu_{\theta,(l),*}(p, X_\theta^{p,n})d\theta + \int_0^s \mathrm{S}_{\theta,(l),*}(p, X_\theta^{p,n})dW_\theta$$

Therefore by using Lemma 3.1 again, we obtain expression (3.1). In order to see $\mathcal{L}$-continuity of $\boldsymbol{Y}_s^{\boldsymbol{p}}$, we only need to apply Theorem 2.3. Higher order derivatives can be proved similarly by induction. (3.2) can be obtained by the $\mathcal{L}$-differentiation rule of composite functions (Theorem 2.3) and the interchangeability of $\mathcal{L}$-differential and Riemann integral or stochastic integral (properties 2.3 and 2.4.) $\square$

When $\boldsymbol{p} \equiv (t, \boldsymbol{x})$, $\boldsymbol{\xi}_s(\boldsymbol{p}) \equiv \boldsymbol{x}$, $\boldsymbol{\mu}_\theta(\boldsymbol{p}, \boldsymbol{y}) \equiv \boldsymbol{\mu}(t+\theta, \boldsymbol{y})$, and $\mathbf{S}_\theta(\boldsymbol{p}, \boldsymbol{y}) \equiv \mathbf{S}(t+\theta, \boldsymbol{y})$, the $\mathcal{L}$-derivative of the process $\boldsymbol{X}_s^{t,\boldsymbol{x}} = \boldsymbol{x} + \int_0^s \boldsymbol{\mu}(t+\theta, \boldsymbol{X}_s^{t,\boldsymbol{x}})d\theta + \int_0^s \mathbf{S}(t+\theta, \boldsymbol{X}_s^{t,\boldsymbol{x}})d\boldsymbol{W}_\theta$ w.r.t. x_j has a simpler expression:

$$
(3.3) \qquad \mathcal{L}\text{-}\frac{\partial}{\partial x_j}\boldsymbol{X}_s^{t,\boldsymbol{x}} = \boldsymbol{Y}_{s,j}^{t,\boldsymbol{x}} = \boldsymbol{e}^j + \sum_{k=1}^{d}\int_0^s (\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}})_k \boldsymbol{\mu}_k(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})d\theta + \sum_{k=1}^{d}\int_0^s (\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}})_k \mathbf{S}_k(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})d\boldsymbol{W}_\theta, j = 1, \cdots, d,
$$

where $\mathcal{L}\text{-}\frac{\partial}{\partial x_j}\boldsymbol{X}_s^{t,\boldsymbol{x}}$ is an abbreviation of $\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{t,\boldsymbol{x}}$ with $\boldsymbol{l}$ a $(d+1)$-dimensional unit vector whose $(j+1)$-th element is 1, $\boldsymbol{e}^j \in \mathbb{R}^d$ is the d-dimensional unit vector whose j-th element is 1, $(\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}})_k$ is the k-th element of $\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}}$, $\boldsymbol{\mu}_k(t+\theta, \boldsymbol{y}) = \frac{\partial}{\partial y_k}\boldsymbol{\mu}(t+\theta, \boldsymbol{y})$ and $\mathbf{S}_k(t+\theta, \boldsymbol{y}) = \frac{\partial}{\partial y_k}\mathbf{S}(t+\theta, \boldsymbol{y})$. It is emphasized that the parameter in consideration is $(t, \boldsymbol{x})$. We do not need $\mathcal{L}$-derivative w.r.t. t in the following argument. Similarly, for the second derivative, we have

$$
\begin{aligned}
\mathcal{L}\text{-}\frac{\partial^2}{\partial x_i \partial x_j}\boldsymbol{X}_s^{t,\boldsymbol{x}} = \boldsymbol{Z}_{s,ij}^{t,\boldsymbol{x}} &= \sum_{k=1}^{d}\int_0^s \boldsymbol{\mu}_k(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})(\boldsymbol{Z}_{\theta,ij}^{t,\boldsymbol{x}})_k d\theta \\
(3.4) \qquad &+ \sum_{k=1}^{d}\int_0^s \mathbf{S}_k(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})(\boldsymbol{Z}_{\theta,ij}^{t,\boldsymbol{x}})_k d\boldsymbol{W}_\theta \\
&+ \sum_{k=1}^{d}\sum_{l=1}^{d}\int_0^s (\boldsymbol{Y}_{\theta,i}^{t,\boldsymbol{x}})_k (\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}})_l \boldsymbol{\mu}_{kl}(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})d\theta \\
&+ \sum_{k=1}^{d}\sum_{l=1}^{d}\int_0^s (\boldsymbol{Y}_{\theta,i}^{t,\boldsymbol{x}})_k (\boldsymbol{Y}_{\theta,j}^{t,\boldsymbol{x}})_l \mathbf{S}_{kl}(t+\theta, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})d\boldsymbol{W}_\theta.
\end{aligned}
$$

Here $\boldsymbol{\mu}_{kl} = \frac{\partial^2}{\partial y_k \partial y_l}\boldsymbol{\mu}(t+\theta, \boldsymbol{y})$, $\mathbf{S}_{kl} = \frac{\partial^2}{\partial y_k \partial y_l}\mathbf{S}(t+\theta, \boldsymbol{y})$.

The estimates of moments of $\mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_s^{t,\boldsymbol{x}}$ can be obtained by applying Theorem 1.4. to (3.3). Since $\boldsymbol{\mu}(t+s, \boldsymbol{y})$ and $\mathbf{S}(t+s, \boldsymbol{y})$ satisfy the Lipschitz condition, $\frac{\partial}{\partial y_j}\boldsymbol{\mu}(t+s, \boldsymbol{y})$ $\frac{\partial}{\partial y_j}\mathbf{S}(t+s, \boldsymbol{y})$ are bounded. Thus $\boldsymbol{\mu}_k(t+s, \boldsymbol{y})(\boldsymbol{y})_k$ and $\mathbf{S}_k(t+s, \boldsymbol{y})(\boldsymbol{y})_k$ also satisfy the Lipschitz condition. Hence from Theorem 1.4 (a), we have for any $q \geq 1$,

$$
E \sup_{0 \leq \theta \leq s}\left\| \mathcal{L}\text{-}\frac{\partial}{\partial \boldsymbol{l}}\boldsymbol{X}_\theta^{t,\boldsymbol{x}} \right\|^{2q} \leq N(s, K, q)\|\boldsymbol{x}\|^{2q}.
$$

We summarize the results on $\mathcal{L}$-derivative of $\boldsymbol{X}_s^{t,\boldsymbol{x}}$ in the following Corollary:

Corollary 3.4. *Suppose the Lipschitz conditions on $\boldsymbol{\mu}(t,\boldsymbol{y})$ and $\mathrm{S}(t,\boldsymbol{y})$ are satisfied. Then $\boldsymbol{X}_s^{t,\boldsymbol{x}}$ is $\mathcal{L}B$-continuous for the parameter $(t,\boldsymbol{x})$. Further, if $\boldsymbol{\mu}(t,\boldsymbol{y})$ and $\mathrm{S}(t,\boldsymbol{y})$ are bounded, then $\boldsymbol{X}_s^{t,\boldsymbol{x}}$ is $\mathcal{L}E$-continuous for the parameter $(t,\boldsymbol{x})$. If $\boldsymbol{\mu}(t,\boldsymbol{y})$ and $\mathrm{S}(t,\boldsymbol{y})$ are i-times continuously differentiable in $\boldsymbol{y}$ and the derivatives does not exceed $K(1+\|\boldsymbol{y}\|^a)$ for some $K.a>0$, then $\boldsymbol{X}_s^{t,\boldsymbol{x}}$ is $\mathcal{L}B$-continuously (in $t,\boldsymbol{x}$) $\mathcal{L}B$-differentiable in $\boldsymbol{x}$.*

By combining the results obtained so far, we derive a smoothness result for the payoff function:

$$v(t,\boldsymbol{x}) = E\left[\int_0^{T-t} f(t+\theta,\boldsymbol{X}_\theta^{t,\boldsymbol{x}})e^{-\varphi_\theta}d\theta + g(\boldsymbol{X}_{T-t}^{t,\boldsymbol{x}})e^{-\varphi_{T-t}}\right]$$

with

$$\varphi_s = \int_0^s \delta(t+\theta,\boldsymbol{X}_\theta^{t,\boldsymbol{x}})d\theta.$$

and $\delta(t,\boldsymbol{x})$ a nonnegative function.

Theorem 3.5. *Let $\boldsymbol{\mu}(t,\boldsymbol{x})$ and $\mathrm{S}(t,\boldsymbol{x})$ satisfy the Lipschitz condition. Further assume that $\boldsymbol{\mu}(t,\boldsymbol{x})$ and $\mathrm{S}(t,\boldsymbol{x})$ are twice continuously (in $t,\boldsymbol{x}$) differentiable in $\boldsymbol{x}$ and the derivatives including the functions does not exceed $K(1+\|\boldsymbol{x}\|^a)$ for some $K,a>0$. Further suppose $f(t,\boldsymbol{x})$ and $g(\boldsymbol{x})$ are continuous in $t,\boldsymbol{x}$ and twice continuously (in $t,\boldsymbol{x}$) differentiable in $\boldsymbol{x}$. Then we have the following assertions:*
(a) If the derivatives of $f(t,\boldsymbol{x})$ and $g(\boldsymbol{x})$ does not exceed exceed $K(1+\|\boldsymbol{x}\|^a)$, then $v(t,\boldsymbol{x})$ is continuous in $t,\boldsymbol{x}$ and twice continuously (in $t,\boldsymbol{x}$) differentiable in $\boldsymbol{x}$. Moreover, there exists a constant $M>0$ such that $|v(t,\boldsymbol{x})| \le M(1+\|\boldsymbol{x}\|^a)$.
(b) If the derivatives of $f(t,\boldsymbol{x})$ and $g(\boldsymbol{x})$ does not exceed exceed $K(1+e^{a\|\boldsymbol{x}\|^\gamma}), 0<\gamma<2$, and $\boldsymbol{\mu}(t,\boldsymbol{x})$ and $\mathrm{S}(t,\boldsymbol{x})$ are bounded, then $v(t,\boldsymbol{x})$ is continuous in $t,\boldsymbol{x}$ and twice continuously (in $t,\boldsymbol{x}$) differentiable in $\boldsymbol{x}$. Moreover, there exists a constant $M>0$ such that $|v(t,\boldsymbol{x})| \le M(1+e^{4a\|\boldsymbol{x}\|^\gamma})$.

Proof. We consider a process $(\rho_s^t, \boldsymbol{X}_s^{t,\boldsymbol{x}})$ where $\rho_s^t = t+s$ which is $\mathcal{L}$ or $\mathcal{L}E$-continuous corresponding to the conditions (a) or (b). By the property (2.3) and Theorem 2.3, $\int_0^s \delta(\rho_\nu^t, \boldsymbol{X}_\nu^{t,\boldsymbol{x}})d\nu$ is $\mathcal{L}B$-continuous. Due to the nonnegativity of the function $\delta(t,\boldsymbol{x})$ and by Theorem 2.3 applied to the function $h(x) = e^{-x}1_{\{x\ge 0\}}$, $e^{-\int_0^s \delta(\rho_\nu^t, \boldsymbol{X}_\nu^{t,x})d\nu}$ is $\mathcal{L}$-continuous. Further, by the properties (2.3), (2.5), Theorem 2.4 and Corollary 3.4 applied to the process $(\rho_s^t, \boldsymbol{X}_s^{t,\boldsymbol{x}})$, we see that the process

$$U_s^{t,\boldsymbol{x}} = \begin{cases} 0 & 0\le s\le t \\ \displaystyle\int_0^{s-t} f(\rho_\theta^t, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})e^{-\int_0^\theta \delta(\rho_\nu^t, \boldsymbol{X}_\nu^{t,x})d\nu}d\theta & t<s\le T \end{cases}$$

is $\mathcal{L}B$-continuous at $t,\boldsymbol{x}$. Hence from the property (2.1), $E\Big[\int_0^{T-t} f(\rho_\theta^t, \boldsymbol{X}_\theta^{t,\boldsymbol{x}})$

$e^{-\int_0^\theta \delta(\rho_\nu^t, X_\nu^{t,x})d\nu} d\theta \Big]$ is continuous in t, x. A similar argument shows the continuity of $E\left[g(X_{T-t}^{t,x})\ e^{-\int_0^{T-t} \delta(\rho_\nu^t, X_\nu^{t,x})d\nu}\right]$. Therefore $v(t,x)$ is continuous in t, x.

Continuity (in t, x) of derivatives of $v(t,x)$ w.r.t. x can be proved in a similar manner by $\mathcal{L}$-differentiation rule (Theorem 2.3) and Corollary 3.4. Also, from Theorem 1.4, it is easy to see that $v(t,x)$ satisfies the polynomial or exponential growth conditions. $\square$

We define operators L for $u(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ as

$$Lu(t,x) = \frac{\partial u}{\partial t} + \frac{1}{2}\mathbf{tr}\Sigma(t,x)\frac{\partial^2 u}{\partial x \partial x'} + \mu(t,x)'\frac{\partial u}{\partial x} - \delta(t,x)u$$

where $\Sigma(t,x) = \mathbf{S}(t,x)\mathbf{S}(t,x)'$.

The next theorem shows the existence and uniqueness of the solution to the Cauchy problem of linear parabolic PDE.

Theorem 3.6. *Suppose the conditions of Theorem 3.5 are satisfied. Then $v(t,x)$ is a unique $C^{1,2}([0,T] \times \mathbb{R}^d)$ stochastic solution to the Cauchy problem*

$$Lv(t,x) = 0$$
$$v(T,x) = g(x).$$

Existence and uniqueness are proved for the polynomial growth case by using Markov property of $X_s^{t,x}$ in Krylov's Theorem 2.9.10. In order to show the exponential growth case, we only need to mimic his proof by replacing polynomial growth with exponential growth. Continuity of the first derivative in t follows from Theorem 3.5 and the relation $Lv = 0$.

It should be noted that Friedman(1975) and Karatzas & Shreve assume uniform ellipticity for the existence part. Theorem 3.6 does not require this condition. (see also Remark 2.9.11 of Krylov.)

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