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MATRIX SENSITIVITY, ERROR ANALYSIS AND INTERNAL/EXTERNAL MULTIREGIONAL MULTIPLIERS

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Abstract

Sherman and Morrison (1949, 1950) provided essential insights into the properties of sensitivity and error analysis in input-output systems through procedures for specifying the effects of changes in one coefficient or changes in one row or column on the rest of the system. Sonis and Hewings (1989, 1992) extended this work and generalized it to include a whole range of changes in direct inputs using the concept of a field of influence. The present paper considers the matricial changes in which sub-matrices of the matrix of direct coefficients change simultaneously; these sub-matrices may be considered to reflect different economic substructures. In this way, the block-generalization of the Sherman-Morrison and Sonis-Hewings formulae are possible through the development and extension of Miyazawa's (1976) notion of internal and external multi-region multipliers. In this way, the matricial decompositions of the matrix of fields of influence can be established.

I. *Introduction*

The well-known Sherman-Morrison (1950) formula describes changes in the components of the Leontief inverse caused by changes in one component of the matrix of direct inputs in an input-output model. More specifically, let $A=\|a_{ij}\|$ be an nxn matrix of direct input coefficients with an associated Leontief inverse, $B=(I-A)^{-1}=\|b_{ij}\|$. If change, e, occurs only in one place, (i_0,j_0) of the matrix A, then the components of the new Leontief inverse matrix, $B(e)=\|b_{ij}(e)\|$, will reveal the original Sherman-Morrison (1950) coordinate form:

$$b_{ij}(e)=b_{ij}+\frac{b_{ii_0}b_{j_0j}e}{1-b_{j_0i_0}e} \qquad (1)$$

Since 1950, this formula has been used extensively in error and sensitivity input-output analysis (see Bullard and Sebald, 1977, 1988; Sohn, 1986 and Sonis and Hewings, 1991 for a review of some of the earlier work). On the basis of this formula, the first order (direct) field of influence $F[i_0,j_0]$ of the increment $e_{i_0j_0}$ was introduced (Sonis and Hewings, 1989) as the matrix generated by the multiplication of the i_0^{th} column of the Leontief inverse matrix with the j_0^{th} row:

$$F[i_0,j_0]=\begin{pmatrix} b_{1i_0} \\ b_{2i_0} \\ \vdots \\ b_{ni_0} \end{pmatrix}\begin{pmatrix} b_{j_01} & b_{j_02} & \cdots & b_{j_0n} \end{pmatrix} \tag{2}$$

If the change, e, occurs in only one place, then the Leontief inverse, $B(e)$, has the form:

$$B(e)=B+\frac{e}{1-b_{j_0i_0}e}F[i_0,j_0] \tag{3}$$

If the changes occurs in the i_0^{th} row of the matrix A, :

$$\bar{e}_{i_0}=\left(e_{i_01},e_{i_02},\cdots,e_{i_0n}\right)$$

then the following Sherman-Morrison (1949) formula would apply, connecting elements of the new and old Leontief inverses, $B\left(\bar{e}_{i_0}\right)=\left\|b_{ij}\left(\bar{e}_{i_0}\right)\right\|$; $B=\left\|b_{ij}\right\|$:

$$b_{ij}\left(\bar{e}_{i_0}\right)=b_{ij}+\frac{b_{ii_0}\sum_{s=1}^{n}b_{sj}e_{i_0s}}{1-\sum_{s=1}^{n}b_{si_0}e_{i_0s}} \qquad i,j=1,2,\ldots,n \tag{4}$$

In the field of influence form (Sonis and Hewings, 1989), the presentation would be:

$$B\left(\bar{e}_{i_0}\right)=B+\frac{\sum_{s=1}^{n}e_{i_0s}F[i_0,s]}{1-\sum_{s=1}^{n}b_{si_0}e_{i_0s}} \tag{5}$$

A further generalization, when all direct coefficients change, can be found in Sonis and Hewings (1989, 1991, 1992). In the present paper, the approach will be to build upon this prior work, especially equations (3) and (5) for the case of a partitioned input-output system and, finally, to relate this to Miyazawa's (1976) distinction between internal and external multipliers in multiregional input-output systems.

II. *Block-Generalized Sherman-Morrison Formula*

For the case where the matrix, A, represents a multiregional block matrix of direct inputs:

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \tag{6}$$

where the blocks, A_{ij}, are the intra- and inter-regional matrices of direct inputs of the region i into region $j(i,j=1,2,\ldots,n)$. The following Proposition 1 will be proven as follows. Let:

$$E=\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & E_{i_0j_0} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{7}$$

be a block-matrix representing the incremental matrix change in the (i_0,j_0)-block, $A_{i_0j_0}$. Then, the following is true:

$$B(E)=(I-A-E)^{-1}=$$

$$=B+\begin{bmatrix} B_{1i_0} \\ B_{2i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix} \Delta(E)E_{i_0j_0}\left[B_{j_01}, B_{j_02}, \ldots, B_{j_0n}\right] \tag{8}$$

where $\begin{bmatrix} B_{1i_0} \\ B_{2i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix}$ and $\left[B_{j_01}, B_{j_02}, B_{j_0n}\right]$ are the rectangular matrices representing the i_0^{th} block-column and the j_0^{th} block-row of the original Leontief inverse and

$$\Delta(E)=\left(I-E_{i_0j_0}B_{j_0i_0}\right)^{-1} \tag{9}$$

In the block-coordinate form, the following generalization of the Sherman-Morrison formula (1) may be shown as:

$$B_{ij}(E)=B_{ij}+B_{ii_0}\Delta(E)E_{i_0j_0}B_{j_0j} \qquad i,j=1,2,\ldots,n \tag{10}$$

If the matrix, E, of incremental changes has the form of changes represented in the i_0^{th} block-row:

$$E=\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ E_{i_0 1} & \cdots & E_{i_0 j_0} & \cdots & E_{i_0 n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

then Proposition 2 holds:

$$B(E)=B+\begin{bmatrix} B_{1i_0} \\ B_{2i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix} \Delta(E)\left\{I-\sum_{s=1}^{n} E_{i_0 s}\left[B_{s1}, B_{s2}, \ldots, B_{sn}\right]\right\} \tag{12}$$

where:

$$\Delta(E)=\left(I-\sum_{s=1}^{n} E_{i_0 s} B_{s i_0}\right)^{-1} \tag{13}$$

or, in a block-coordinate form:

$$B_{ij}(E)=B_{ij}+B_{ii_0}\Delta(E)\sum_{s=1}^{n} E_{i_0 s} B_{sj} \qquad i,j=1,2,\ldots,n \tag{14}$$

Comparison of (2) and (8) indicates that the transfer from the single economy to the multiregional setting leads to the decomposition of the field of influence into two block-rectangular matrices.

III. *Proofs*

Assume the case where the change occurs in only one block $(i_0 j_0)$. Further assume the following statement to be true:

$$(I-A_1-A_2)^{-1}=G_1G_2=G_1+G_1(G_2-I) \tag{15}$$

where $G_1=(I-A_1)^{-1}$; $G_2=(I-A_2G_1)^{-1}$. (Formula(15) can be justified by direct matrix multiplication; the general formula for Leontief inverses of the type $(I-A_1-A_2-\ldots-A_n)^{-1}$ is provided in Sonis and Hewings, 1993). For matrices, A and E, from (6) and (7), one obtains:

$$B(E)=(I-A-E)^{-1}=B+B(G_2-I) \tag{16}$$

where $B=(I-A)^{-1}$ and $G_2=(I-EB)^{-1}$.

It is easy to see that for $B=\|B_{ij}\|$

$$EB=\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ E_{i_0j_0}B_{j_01} & \cdots & E_{i_0j_0}B_{j_0i_0} & \cdots & E_{i_0n}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

and

$$G_2=(I-EB)^{-1}=\begin{bmatrix} I & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ -E_{i_0j_0}B_{j_01} & \cdots & I-E_{i_0j_0}B_{j_0i_0} & \cdots & -E_{i_0n}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & I \end{bmatrix}^{-1}$$

Consider

$$\Delta(E)=\left(I-E_{i_oj_0}B_{j_0i_0}\right)^{-1} \tag{17}$$

then

$$\Delta(E)\left(I-E_{i_oj_0}B_{j_0i_0}\right)=I$$

and, therefore,

$$\Delta(E)-I=\Delta(E)E_{i_oj_0}B_{j_0i_0} \tag{18}$$

It is possible to check, by direct multiplication, that:

$$G_2=\begin{bmatrix} I & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \Delta(E)E_{i_0j_0}B_{j_01} & \cdots & \Delta(E) & \cdots & \Delta(E)E_{i_0j_0}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & I \end{bmatrix} \tag{19}$$

Further, using (18), one obtains:

$$B(G_2-I)=\begin{bmatrix} B_{11} & \cdots & B_{1i_0} & \cdots & B_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{i_01} & \cdots & B_{i_0j_0} & \cdots & B_{i_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{n1} & \cdots & B_{ni_0} & \cdots & B_{nn} \end{bmatrix}\begin{bmatrix} I & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \Delta(E)E_{i_0j_0}B_{j_01} & \cdots & \Delta(E)-I & \cdots & \Delta(E)E_{i_0j_0}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & I \end{bmatrix}=$$

$$=\begin{bmatrix} B_{1i_0}\Delta(E)E_{i_0j_0}B_{j_01} & \cdots & B_{1i_0}\Delta(E)E_{i_0j_0}B_{j_0i_0} & \cdots & B_{1i_0}\Delta(E)E_{i_0j_0}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{i_0i_0}\Delta(E)E_{i_0j_0}B_{j_01} & \cdots & B_{i_0i_0}\Delta(E)E_{i_0j_0}B_{j_01} & \cdots & B_{i_0i_0}\Delta(E)E_{i_0j_0}B_{j_0n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ B_{ni_0}\Delta(E)E_{i_0j_0}B_{j_01} & \cdots & B_{ni_0}\Delta(E)E_{i_0j_0}B_{j_0i_0} & \cdots & B_{ni_0}\Delta(E)E_{i_0j_0}B_{j_0n} \end{bmatrix}=$$

$$=\begin{bmatrix} B_{1i_0} \\ \vdots \\ B_{i_0i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix}\Delta(E)E_{i_0j_0}\left[B_{j_01},\ldots,B_{j_0i_0},\ldots,B_{j_0n}\right. \tag{20}$$

which provides us with (8) . For the proof of (12) , one can repeat the previous proof, exchanging (7) and (9) by (11) and (13).

IV. *The Relationship of Changes in Gross output in the Multiregional Input-Output System Created by Changes on the Matrix of Intra-Regional Direct Inputs in One Region*

Consider the same multiregional system introduced in section 3 and now assume that the intraregional matrix, A_{jj}, of the j^{th} region has the following incremental change, E_{jj}. In this case, the matrix A , can be represented in the following block structure:

$$A=\begin{pmatrix} A_{jj} & A_{jR} \\ A_{Rj} & A_{RR} \end{pmatrix} \tag{21}$$

where A_{jR}, A_{Rj} are the inter-regional matrices representing direct input connections between region j and the rest of the economy while the matrix A_{RR} represents the direct inputs within the rest of the economy.

The incremental change in direct inputs in the region j is presented by the matrix:

$$E=\begin{pmatrix} E_{jj} & 0 \\ 0 & 0 \end{pmatrix} \tag{22}$$

and the final demand vectors and gross output vectors and their changes may be presented as:

$$f=\begin{pmatrix} f_j \\ f_R \end{pmatrix}; \quad X=\begin{pmatrix} X_j \\ X_R \end{pmatrix}; \quad \Delta X=\begin{pmatrix} \Delta X_j \\ \Delta X_R \end{pmatrix} \tag{23}$$

further indicating the separation of the j^{th} region from the rest of the economy. Now, the Leontief inverses can be presented in similar block form:

$$\begin{aligned} B&=(I-A)^{-1}=\begin{pmatrix} B_{jj} & B_{jR} \\ B_{Rj} & B_{RR} \end{pmatrix} \\ B(E)&=(I-A-E)^{-1}=\begin{pmatrix} B_{jj}(E) & B_{jR}(E) \\ B_{Rj}(E) & B_{RR}(E) \end{pmatrix} \end{aligned} \tag{24}$$

Furthermore:

$$\begin{pmatrix} X_j \\ X_R \end{pmatrix}=\begin{pmatrix} B_{jj} & B_{jR} \\ B_{Rj} & B_{RR} \end{pmatrix}\begin{pmatrix} f_j \\ f_R \end{pmatrix} \tag{25}$$

or

$$\begin{cases} X_j=B_{jj}f_j+B_{jR}f_R \\ X_R=B_{Rj}f_j+B_{RR}f_R \end{cases} \tag{26}$$

By applying (20) to (21) and (22), one has:

$$\Delta X=[B(E)-B]f=\begin{pmatrix} B_{jj}\Delta(E)E_{jj}B_{jj} & B_{jj}\Delta(E)E_{jj}B_{jR} \\ B_{Rj}\Delta(E)E_{jj}B_{jj} & B_{Rj}\Delta(E)E_{jj}B_{jR} \end{pmatrix}\begin{pmatrix} f_j \\ f_R \end{pmatrix} \tag{27}$$

where $\Delta(E)=\left(I-E_{jj}B_{jj}\right)^{-1}$. This implies that:

$$\begin{aligned} \Delta X_j&=B_{jj}\Delta(E)E_{jj}B_{jj}f_j+B_{jj}\Delta(E)E_{jj}B_{jR}f_R= \\ &=B_{jj}\Delta(E)E_{jj}(B_{jj}f_j+B_{jR}f_R)=B_{jj}\Delta(E)E_{jj}X_j \end{aligned} \tag{28}$$

$$\begin{aligned} \Delta X_R&=B_{Rj}\Delta(E)E_{jj}B_{jj}f_j+B_{Rj}\Delta(E)E_{jj}B_{jR}f_R= \\ &=B_{Rj}\Delta(E)E_{jj}(B_{jj}f_j+B_{jR}f_R)=B_{Rj}\Delta(E)E_{jj}X_j \end{aligned} \tag{29}$$

Applying the factorization scheme used in (15) to the A matrix yields:

$$A=\begin{pmatrix} A_{jj} & A_{jR} \\ A_{Rj} & A_{RR} \end{pmatrix}=\begin{pmatrix} A_{jj} & 0 \\ A_{Rj} & 0 \end{pmatrix}+\begin{pmatrix} 0 & A_{jR} \\ 0 & A_{RR} \end{pmatrix} \tag{30}$$

and this provides the following presentation (see Sonis and Hewings, 1993) :

$$B=(I-A)^{-1}=\begin{pmatrix} B_{jj} & B_{jR} \\ B_{Rj} & B_{RR} \end{pmatrix}=\begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{RR} \end{pmatrix}\begin{pmatrix} I & B_j A_{jR} \\ B_R A_{Rj} & I \end{pmatrix}\begin{pmatrix} B_j & 0 \\ 0 & B_R \end{pmatrix} \tag{31}$$

where:

$$B_j=\left(I-A_{jj}\right)^{-1}; \quad B_R=\left(I-A_{RR}\right)^{-1} \tag{32}$$

are the Miyazawa (1976) internal multipliers for the region j and the rest of the economy, R, and:

$$\Delta_{jj}=\left(I-B_j A_{jR} B_R A_{Rj}\right)^{-1}; \quad \Delta_{RR}=\left(I-B_R A_{Rj} B_j A_{jR}\right)^{-1} \tag{33}$$

are the Miyazawa external multipliers for region j and the rest of the economy, R (see also Sonis and Hewings, 1993). After multiplication, one obtains, the Schur-Miyazawa formula (Miyazawa, 1976) :

$$B=\begin{pmatrix} \Delta_{jj} B_j & \Delta_{jj} B_j A_{jR} B_R \\ \Delta_{RR} B_R A_{Rj} B_j & \Delta_{RR} B_R \end{pmatrix}=\begin{pmatrix} \Delta_j & \Delta_j A_{jR} B_R \\ \Delta_R A_{Rj} B_j & \Delta_R \end{pmatrix} \tag{34}$$

where

$$\begin{aligned} \Delta_j &= \Delta_{jj} B_j=\left(I-A_{jj}-A_{jR} B_R A_{Rj}\right)^{-1} \\ \Delta_R &= \Delta_{RR} B_R=\left(I-A_{RR}-A_{Rj} B_j A_{jR}\right)^{-1} \end{aligned} \tag{35}$$

are the internal multiplier of the region j influenced by the rest of the economy and the internal multiplier for the rest of the economy, R, influenced by region j. Moreover,

$$B_j A_{jR} \Delta_R=\Delta_j A_{jR} B_R; \quad B_R A_{Rj} \Delta_j=\Delta_R A_{Rj} B_j \tag{36}$$

Therefore, (28) and (29) may be presented in the form:

$$\Delta X_j=\Delta_j \Delta(E) E_{jj} X_j \tag{37}$$

$$\Delta X_R=\Delta_R A_{Rj} B_j \Delta(E) E_{jj} X_j=B_R A_{Rj} \Delta_j\, \Delta(E) E_{jj} X_j \tag{38}$$

from which it is easy to see that:

$$\Delta_j(E)=\Delta_j \Delta(E)=\left[I-\left(A_{jj}+E_{jj}\right)-A_{jR} B_R A_{Rj}\right]^{-1} \tag{39}$$

is the external multiplier of region j, generated by the interregional change and influenced by the rest of the economy.

Formulae (37) through (39) generate the fundamental relationships between the changes in the components of the gross outputs through the action of the matrix multipliers:

$$\begin{aligned} \Delta X_j &= \Delta_j(E) E_{jj} X_j \\ \Delta X_R &= B_R A_{Rj} \Delta X_j \end{aligned} \tag{40}$$

V. *Change in Gross Output Created by Changes in the Block-Row of Direct Matrix Inputs*

If the matrix of incremental changes (22) is replaced by the matrix:

$$E=\begin{pmatrix} E_{jj} & E_{jR} \\ 0 & 0 \end{pmatrix} \tag{41}$$

then similar reasoning will result in:

$$\begin{cases} \Delta X_j=\Delta_j(E)\left(E_{jj}X_j+E_{jR}X_R\right) \\ \Delta X_R=B_R A_{Rj}\Delta X_j \end{cases} \tag{42}$$

where:

$$\Delta_j(E)=\left[I-\left(A_{jj}+E_{jj}\right)-\left(A_{jR}+E_{jR}\right)B_R A_{Rj}\right]^{-1} \tag{43}$$

is the external multiplier of the region, j, after intra- and inter-regional change, influenced by the rest of the economy.

VI. *Conclusions*

Perhaps, the main contribution of this short paper is in the manner and new form in which two important paradigms in multiregional input-output analysis, fields of influence and Miyazawa's matrix multipliers, are linked through the conceptual and numerical methods of partitioned input-output analysis.

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