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RECURSIVE UTILITY: DISCRETE TIME THEORY*

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I. *Introduction*

Most of the modern literature on capital theory and optimal growth has proceeded on the assumption that preferences are represented by a functional which is additive over time and discounts future rewards at a constant rate. Recent research in the study of preference orders and utility functions has led to advances in intertemporal allocation theory on the basis of weaker hypotheses. The class of recursive utility functions has been proposed as a generalization of the additive utility family. The recursive utility functions share many of the important characteristics of the additive class. Notably, recursive utility functions enjoy a time consistency property that permits dynamic programming analysis of optimal growth and competitive equilibrium models. The purpose of this paper is to survey the discrete time theory of recursive utility functions and their applications in optimal growth theory.

Recursive utility involves flexible time preference. In contrast, the rigid time preference of the common additively separable utility functions may yield results that seem strange in ordinary circumstances. A consumer facing a fixed interest rate will try either to save without limit, or to borrow without limit, except in the knife-edge case where the rate of impatience equals the interest rate. This problem is especially severe when there are heterogeneous households. Unless all of the households have the same discount factor, the most patient household ends up with all the capital in the long-run, while all other households consume nothing, using their labor income to service their debt.[1] The constant discount rate hypothesis also creates problems for the calculation of welfare losses from capital income taxation. The after-tax return to capital is always the pure rate of time preference. The capital tax is therefore completely shifted to labor in the long-run. As a result, the welfare cost of the tax is higher than it would be if some adjustment of the after-tax rate of return could occur.

Recursive utility escapes these dilemmas by allowing impatience to depend on the path of consumption. The assumptions made on utility allowing for variable time preference imply a weak separability between present and future consumption. This leads to a representation of the utility function in terms of an aggregator function expressing current utility of

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[1] Becker (1980) demonstrated a variant of this result for the case of a borrowing constraint. The relatively impatient households consume their wage income. This result was verification of a conjecture of Ramsey (1928).

a consumption path as a function of current consumption and the future utility derived from the remaining periods consumption. In this way, recursive utility recalls the two-period model of Fisher (1930).

Time additive separable utility has dominated research in economic dynamics owing to the mathematical simplification derived from that functional form. The economic plausibility of additive utility was pushed aside in the interest of obtaining insights and directions for further research. The body of work surveyed in this paper represents the latest efforts at studying dynamic optimization models with intertemporally dependent preferences as embodied in the recursive utility hypotheses. The foundations of recursive utility theory were set by Koopmans and his collaborators during the 1960's and early 1970's. We re-examine this work in light of new developments in general equilibrium theory with infinite dimensional commodity spaces. We focus on the optimal growth model as the paradigm for dynamic models with brief mention of general competitive analysis for exchange economies as another illustration of the methods used to analyze the implications of a recursive utility specification of preferences. Part II of the paper examines the structure of commodity spaces with a countable infinity of goods and reviews the properties of recursive utility functions. Part III explores the variety of notions of impatience and myopia. The aggregator as the primitive expression of the preference order is taken up in Part IV. The dynamic properties exhibited by optimal growth paths with recursive utility objectives is presented in Part V. Concluding comments are found in Part VI and an appendix devoted to mathematical properties of weighted contraction mappings completes the paper.

II. *Recursive Utility and Intertemporal Preferences*

1. *Introduction*

The development of the recursive utility representation of an intertemporal preference order is most easily cast in a world with a countable infinity of time periods, $t=1,2,\ldots,T,\ldots$ where there is one all-purpose good which may be either consumed or accumulated. The description of preference orderings exhibiting recursive separability as well various other features, e.g. myopia or impatience, is presented as a refinement of standard axioms governing preference relations in an infinite dimensional commodity space setup.

Let c_t denote consumption in period t and let k_t denote the capital stock accumulated during period t, to be used in production for period $t+1$. The initial capital stock is k_0. The sequences of consumption levels $C=\{c_t\}_{t=1}^{\infty}$, and of capital stocks $K=\{k_t\}_{t=1}^{\infty}$, are elements of $\mathbb{R}^{\infty}$, the space of all real-valued sequences. For the remainder of this part as well as Part III, the focus will be on the possible consumption sequences; capital will reappear in Part IV.

This section focuses on recursive utility as an abstract preference order. Accordingly, we examine the properties of commodity and price spaces in Section Two, with examples in Section Three. Section Four sets forth the relevant facts about representation of preferences. Finally, Section Five introduces the Koopmans' Axioms, which imply that resursive utility takes a special form.

2. *The Commodity-Price Duality*

The commodity space, $\mathscr{C}$, is a subspace of $\mathbb{R}^\infty$. Elements of $\mathscr{C}$ are denoted by C, X, and Y. The space $\mathscr{C}$ may be chosen as a proper subspace of $\mathbb{R}^\infty$. The case $\mathscr{C}=\ell^\infty$, the space of all bounded real-valued sequences, is a popular example. The space $\mathbb{R}^\infty$ has a natural order property: For elements X and Y of $\mathbb{R}^\infty$, define $\geq$ by $X\geq Y$ if and only if $x_t\geq y_t$ for all t. The partial order relation $\leq$ may be used to define the set $\mathbb{R}^\infty_+$, the positive cone of $\mathbb{R}^\infty$, by the relation $\mathbb{R}^\infty_+=\{X\in\mathbb{R}^\infty: X\geq 0\}$, where 0 is the zero vector. The space $\mathscr{C}$ inherits the order structure determined by the partial order $\geq$; $\mathscr{C}_+$ denotes the positive cone of $\mathscr{C}$. For $X\in\mathbb{R}^\infty$, let $|X|=\{|x_t|\}_{t=1}^\infty$ denote the absolute value of X. Define the projection operator π and shift opetaror by $\pi C=c_1$ and $SC=(c_2,c_3,\ldots)$ for $C\in\mathbb{R}^\infty$. The operator $\pi^N C=(c_1,c_2,\ldots,c_N)$ denotes the projection of C onto the first N coordinate factor spaces. The N^{th} iterate of the shift operator, S^N, is defined by $S^N C=(c_{N+1},c_{N+2},\ldots)$.

The space $\mathbb{R}^\infty$ is an example of a *Riesz space* (or *vector lattice*). That is, for every pair of vectors X and Y, the supremum (least upper bound) and infimum (greatest lower bound) of the set $\{X,Y\}$ exist in $\mathbb{R}^\infty$. In standard lattice notation, $X\vee Y=\sup\{X,Y\}$ and $X\wedge Y=\inf\{X,Y\}$.[2] We will require $\mathscr{C}$ to be a Riesz subspace of $\mathbb{R}^\infty$, that is, whenever X and $Y\in\mathscr{C}$, the elements $X\vee Y$ and $X\wedge Y$ both belong to $\mathscr{C}$.

An important class of commodity spaces may be defined given a vector $\omega=(\omega_1,\omega_2,\ldots)\in\mathbb{R}^\infty$ as follows: let

$$A_\omega=\{X\in\mathbb{R}^\infty: |X|\leq\lambda|\omega| \text{ for some } \lambda\geq 0\},$$

where λ is a scalar and the notation $|X|\leq|Y|$ means $|x_t|\leq|y_t|$ for all t. The set A_ω is the *principal ideal generated by* ω. A_ω is a Riesz subspace of $\mathbb{R}^\infty$. Notice that for $\omega=(1,1,\ldots)$ that $A_\omega=\ell^\infty$. In applications of recursive utility models, the natural commodity space will typically be a principal ideal. The particular application will determine the choice of ω. For instance, in an exchange economy, ω would be the aggregate or social endowment vector. In the Ramsey optimal growth setting, ω would be the path of pure accumulation generated by iteration of production function with seed k_0.

The open-ended horizon characteristic of dynamic economic models means that there are several Hausdorff linear topologies available for $\mathscr{C}$ in contrast to the finite horizon case. Thus several dual spaces are also available in the infinite dimensional framework. The choice of a topology for $\mathscr{C}$ as well as a selection of the dual space has important economic consequences. Koopmans (1960) first observed that the continuity hypothesis maintained on an agent's preference order contained an implicit behavioral assumption about myopia. The properties of the dual space show up in the representation of prices realized in a perfect foresight equilibrium or in support of an optimum allocation.[3] The representation of the price system links to questions about the possibility of bubbles in an equilibrium configuration.[4]

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[2] For a general discussion of Riesz spaces with economic applications, see Aliprantis, Brown and Burkinshaw (1989).

[3] Debreu (1954) first cast equilibrium models in terms of a commodity-price dual pair of linear spaces.

[4] See Gilles (1989) and Gilles and LeRoy (1989).

We will equip $\mathscr{E}$ with a linear topology τ compatible with the algebraic and lattice structure of the space. A subset A of a Riesz space $\mathscr{E}$ is said to be a *solid set* whenever $|Y| \leq |X|$ and $X \in A$ imply $Y \in A$. In this paper, the topologies will always be locally convex-solid: the topology is locally convex and has a base at zero consisting of solid sets.

RIESZ DUAL SYSTEM. A Riesz dual system $(\mathscr{E}, \mathscr{E}')$ is a dual pair of linear spaces such that

(1) $\mathscr{E}$ is a Riesz space;
(2) $\mathscr{E}'$ is an ideal of the order dual $\mathscr{E}^{\sim}$ separating the points of $\mathscr{E}$;
(3) the duality function $\langle \cdot, \cdot \rangle$ is the natural one given by the evaluation $\langle X, P \rangle = P(X) \equiv PX$ for all $X \in \mathscr{E}$ and all $P \in \mathscr{E}'$.

In the economie framework, $\mathscr{E}$ is the commodity space and $\mathscr{E}'$ is the price space. The evaluation $\langle \cdot, P \rangle$ defines a linear functional on $\mathscr{E}$ interpreted as a price system. The assumption that $(\mathscr{E}, \tau)$ is a locally conbex-solid topology implies the dual system $\langle \mathscr{E}, \mathscr{E}' \rangle$ is a Riesz dual system where $\mathscr{E}'$ is the τ-dual of $\mathscr{E}$.[5] Given $\langle \mathscr{E}, \mathscr{E}' \rangle$, denote by $\sigma(\mathscr{E}, \mathscr{E}')$ the *weak topology* and $\tau(\mathscr{E}, \mathscr{E}')$ the *Mackey topology* of the dual pair $\langle \mathscr{E}, \mathscr{E}' \rangle$.

For the Riesz space $\mathscr{E}$, any set of the form

$$[X, Y] = \{Z \in \mathscr{E} : X \leq Z \leq Y\}$$

is called an *order interval* of $\mathscr{E}$. The Riesz dual system $\langle \mathscr{E}, \mathscr{E}' \rangle$ is *symmetric* whenever every order interval of E is $\sigma(\mathscr{E}, \mathscr{E}')$-compact.[6]

3. *Examples of Commodity Spaces*

The Riesz dual system $(\mathbb{R}^\infty, c_{00})$, where c_{00} is the Riesz space of all eventually zero sequences, is a symmetric Riesz dual system. The evaluation is defined by the formula

$$\langle X, P \rangle = \sum_{t=1}^{T} p_t x_t$$

where $P = \{p_t\}$ and $p_t = 0$ for $t > T$. The space $\mathbb{R}^\infty$ is a Fréchet space, i.e., it is a complete metrisable locally convex linear topological space. The Fréchet metric d_F is defined by

$$d_F(X, Y) = \sum_{t=1}^{\infty} \frac{1}{2^t} \frac{|x_t - y_t|}{1 + |x_t - y_t|}.$$

The space $(\mathbb{R}^\infty, d_F)$ is also a separable metric space. Moreover, the Fréchet topology is equivalent to the $\sigma(\mathbb{R}^\infty, c_{00})$-topology, the $\tau(\mathbb{R}^\infty, c_{00})$-topology, as well as the product topology on $\mathbb{R}^\infty$ viewed as the product of countably many copies of $\mathbb{R}$. Convergence of sequences in $(\mathbb{R}^\infty, d_F)$ is coordinatewise.[7]

The Riesz space $\mathscr{E} = \ell^\infty$ underlies two interesting commodity-price dual pairs. First,

[5] See Aliprantis, Brown and Burkinshaw (1989, pp. 99 and 101).

[6] There are several equivalent properties for symmetric Riesz dual systems. See Aliprantis, Brown and Burkinshaw (1989, p. 102).

[7] These facts about $\mathbb{R}^\infty$ may be found in Liusternik and Sobolev (1961). General properties of Fréchet spaces may be found in Robertson and Robertson (1973).

consider the dual pair $\langle \ell^\infty, ba\rangle$, where ba is the space of bounded additive set functions (or charges) on the positive integers. If ℓ^∞ is endowed with the supremum norm topology, then ba is the norm dual of ℓ^∞. The $\|\cdot\|_\infty$ norm of X is defined by the formula

$$\|X\|_\infty = \sup_t |x_t|,$$

and the space $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space, i.e., it is a complete normed linear space. The spaces ℓ^∞ and ba are paired by means of the bilinear form $\langle X, \Pi\rangle$ defined by

$$\langle X, \Pi\rangle = \int X(t)d\Pi.$$

If $\Pi \in ba$ is a countably additive charge, then the Radon-Nikodym Theorem [Dunford and Schwartz (1957, p. 181)] implies there exists a unique $P \in \ell^1$, such that

$$\langle X, \Pi\rangle = \sum_{t=1}^{\infty} p_t x_t$$

for each $X \in \ell^\infty$. The countably additive elements of ba have natural price interpretations. However, there are continuous linear functionals on $(\ell^\infty, \|\cdot\|_\infty)$ which are not identifiable with price systems in this fashion.[8] A charge failing to have a countably additive part is known as a *pure charge* (*purely finitely additive measure*).[9] In general, a charge may be uniquely decomposed into a maximally countably additive measure and a pure charge; this is the Yosida-Hewitt Decomposition Theorem (Bhaskara Rao and Bhaskara Rao, 1983, p. 241). Put differently, ba is the direct sum of ca, the space of countably additive measures, and pch, the space of pure charges. The space $(\ell^\infty, \|\cdot\|_\infty)$ is also a Banach lattice or complete normed Riesz space. Aa such, the $\|\cdot\|_\infty$-topology is locally convex-solid and (ℓ^∞, ba) is a Riesz dual system.

The second interesting pairing of ℓ^∞ with a dual space is the $\langle \ell^\infty, \ell^1\rangle$ specification, where ℓ^1 is the space of all sequences X for which the norm

$$\|X\|_1 = \sum_{t=1}^{\infty} |x_t|$$

is finite. The evaluation $\langle X, P\rangle$ is the natural one defined by

$$\langle X, P\rangle = \sum_{t=1}^{\infty} p_t x_t.$$

This pair constitutes a symmetric Riesz dual system.

The pair$\langle \ell^\infty, \ell^1\rangle$ has several interesting topologies. For example, the $\tau(\ell^\infty, \ell^1)$-topology and the weak*-topology, $\sigma^*(\ell^\infty, \ell^1)$, have been used in economic applications.[10] Another important topology is the *strict topology* defined by seminorms of the form

[8] Banach limits are an example.
[9] See Bhaskara Rao and Bhaskara Rao (1983) for details on the properties of ba.
[10] Aloaglu's Theorem implies that the order intervals of ℓ^∞ are weak*-compact.

$$||X||_\gamma = \sup_t |x_t \gamma_t|,$$

where $\lim_t \gamma_t = 0$. Conway (1967) proved that ℓ^∞ endowed with the strict topology is a strong Mackey space.[11] Consequently, the strict topology is the finest locally convex Hausdorff topology for the dual pair $\langle \ell^\infty, \ell^1 \rangle$.

A third class of examples is based on consideration of the principal ideal A_ω defined for $\omega = (\alpha, \alpha^2, \ldots)$ when $\alpha \geq 1$. The case $\alpha = 1$ corresponds to the situation where $A_\omega = \ell^\infty$. Let $\beta \geq \alpha$ and define the *β-weighted ℓ^∞ norm* (or *β-norm*), $||\cdot||_\beta$, by

$$||X||_\beta = \sup_t |x_t / \beta^t|.$$

The *β-topology* on A_ω is the norm topology induced by the β-norm. If $\alpha = \beta$, then $||\cdot||_\alpha$ is called the *α-norm* and the corresponding topology is the *α-topology*. The space A_ω endowed with the α-norm is an *AM-space with unit*.[12] The α-normed space A_ω is lattice isometric to ℓ^∞: the mapping $\xi : A_\omega \to \ell^\infty$, defined by

$$\xi(x_1, x_2, \ldots, x_t, \ldots) = \left(\frac{x_1}{\alpha}, \frac{x_2}{\alpha^2} \cdot \ldots, \frac{x_t}{\alpha^t}, \ldots \right)$$

is a linear isometry. The α-norm dual of A_ω is denoted by A'_ω; clearly A'_ω is isometric to *ba* (written $A'_\omega \cong ba$). For $\beta > \alpha$, the β-norm is a lattice norm and the β-topology is a locally convex-solid topology on A_ω.[13] The normed Riesz space $(A_\omega, ||\cdot||_\beta)$, $\beta > \alpha$, may be embedded lattice isometrically in c_0 (endowed with the supremum norm). Here c_0 is the space of real-valued sequences convergent to 0. Indeed the mapping $\psi : A_\omega \to c_0$ defined by

$$\psi(x_1, x_2, \ldots, x_t, \ldots) = \left(\frac{x_1}{\beta}, \frac{x_2}{\beta^2}, \ldots, \frac{x_t}{\beta^t}, \ldots \right)$$

is a lattice isometry. Moreover, $\psi(A_\omega)$ is dense in c_0.[14] It follows that the norm completion of $\psi(A_\omega)$ is c_0 and the norm completion of $(A_\omega, ||\cdot||_\beta)$ is lattice isometric to c_0. Let A^*_ω denote the β-norm dual of A_ω. Since A^*_ω and the dual of the norm completion of $(A_\omega, ||\cdot||_\beta)$ coincide, A^*_ω is lattice isometric to $\ell^1 \cong c_0^*$. Finally, note that the β-norm dual ($\beta > \alpha$) is one of the seminorms used to define the strict and Mackey topologies on ℓ^∞, so the β-topology is weaker than the strict (or Mackey) topology on ℓ^∞.

We will frequently use a family of weighted norms and weighted ℓ^∞ spaces. Let $\Theta = \{\theta_t\}$ be a sequence of strictly positive numbers. Define the (ℓ^∞) Θ-norm by $|X|_\theta = \sup_t |X_t| / \theta_t$. The associated weighted ℓ^∞ space is $\ell^\infty(\Theta) = \{X : |X|_\theta \text{ is finite}\}$.[15] Consider the mapping V defined by $(VX)_t = x_t / \theta_t$. Clearly V is an isometry of $\ell^\infty(\Theta)$ onto ℓ^∞. Since ℓ^∞ is a Banach space, so is $\ell^\infty(\Theta)$.

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[11] The Mackey topology is the topology of uniform convergence on the weak*-compact, convex, circled (balanced) subsets of ℓ^1. The strong Mackey topology is the topology of uniform convergence on the weak*-compact subsets of ℓ^1.

[12] The lattice norm $||X||_\infty = \inf \{\lambda > 0 : |X| \leq \lambda \omega\}$ coincides with the α-norm.

[13] Recall that all the lattice norms turning A_ω into a Banach lattice are equivalent [see Aliprantis and Burkinshaw (1985, p. 170)]. The space $(A_\omega, ||\cdot||_\beta)$ is a normed Riesz space but not a Banach lattice when $\beta > \alpha$.

[14] $c_0 \supset \psi(A_\omega) \supset c_{00}$ and c_{00} is a supremum norm dense Riesz subspace of c_0.

[15] This is the Riesz ideal of $\mathbb{R}^\infty$ generated by Θ, and $|X|_\theta$ is the associated lattice norm.

We will mainly focus on the case $\theta_t=\beta^{t-1}$ for some $\beta\geq 1$. These norms can be thought of as having the discount factor $1/\beta$ built in.[16] In this case, the positive orthant of $\ell^\infty(\Theta)$ will be denoted by $\ell^\infty_+(\beta)$ and the Θ-norm by $|X|_\beta$. The sets $\ell^\infty_+(\beta)$ play an important role throughout this paper.

4. *Preference Orders*

A dual pair $\langle \mathscr{E}, \mathscr{E}' \rangle$ is chosen for the commodity-price duality. We require $\langle \mathscr{E}, \mathscr{E}' \rangle$ to be a Riesz dual system with $\mathscr{E}$ a Riesz subspace of $\mathbb{R}^\infty$. The preference relation of the planning agent is denoted by $\succsim$. The planning agent might be a central planner as in optimal growth theory or an infinitely lived household in an intertemporal market setup. It is assumed

PREFERENCE RELATION.

(U1) $\succsim$ is a reflexive, complete, and transitive binary relation;

(U2) $\succsim$ is a monotone relation, i.e. $X\geq Y$ in $\mathscr{E}_+$ implies $X\succsim Y$;

(U3) $\succsim$ is a convex relation, i.e. the set $\{Y\in \mathscr{E}_+ : Y\succsim X\}$ is convex for each $X\in \mathscr{E}_+$.

Properties (U1)–(U3) of the preference relation are based on the algebraic structure of $\mathscr{E}$. Topological considerations will be introduced shortly in the form of continuity hypotheses. The derived strict preference relation $>$ is defined by $X>Y$ if $X\succsim Y$ and not $Y\succsim X$. The indifference relation $\sim$ is defined by $X\sim Y$ if $X\succsim Y$ and $Y\succsim X$ hold.

Endow $\mathscr{E}$ with a linear topology τ. Continuity of the preference relation says roughly that programs that are close to one another are ranked similarly with respect to other profiles. More formally:

(U4) $\succsim$ is a continuous relation, i.e. the sets $\{Y\in \mathscr{E}_+ : X\succsim Y\}$ and $\{Y\in \mathscr{E}_+ : X\precsim Y\}$ are τ-closed.

The continuity hypothesis is fundamental. The variety of alternative topologies for $\mathscr{E}$ raises the question of whether or not there are behavioral implications implicit in the choice of a particular topology for $\mathscr{E}$. We will take up many of these issues in Part III.

It is worth noting that there are examples of preference orders which fail the continuity test on all of $\mathbb{R}^\infty$. Consider the dual pairing $\langle\mathbb{R}^\infty, c_{00}\rangle$ where $\mathbb{R}^\infty$ has the Fréchet metric topology. The preference order

$$X\succsim Y \text{ if and only if } \inf_t x_t \geq \inf_t y_t.$$

is known as the *maximin order*.[17] It is not continuous in the Fréchet distance, although $\{Y\in \mathscr{E}_+ : Y\succsim X\}$ is d_F-closed. The problem is that $\{Y\in \mathscr{E}_+ : X\succsim Y\}$ need not be a d_F-closed set.

Consider the following relation on $\mathbb{R}^\infty_+$:

$$X\succsim Y \text{ if and only if } \sum_{t=1}^{\infty} \delta^{t-1} x_t^\eta \geq \sum_{t=1}^{\infty} \delta^{t-1} y_t^\eta$$

[16] Topologies of this type have been used by Chichilinsky and Kalman (1980) and Dechert and Nishimura (1980) to study optimal paths.

[17] This preference relation was introduced by Rawls (1971) and is also known as the *Rawlsian criterion.*

where $\delta,\eta\in(0,1)$ and $\delta\alpha^{\eta}<1$. In this example, α is a parameter representing the growth rate of capital in the optimal accumulation model or the growth rate of the endowment in an exchange economy. Beals and Koopmans (1969) showed this preference relation is not continuous at the origin of $\mathbb{R}_{+}^{\infty}$ in the d_F metric. However, it will turn out that this relation is continuous on a subset of the commodity space consistent with the feasible allocations. This fact is, in part, the source of our interest in the principal ideal A_{ω}.

Continuity of the preference relation is important for establishing the existence of optimal allocations.[18] Continuity of $\succsim$ also is critical in demonstrating the existence of a utility function which carries the properties of the preference relation in a convenient analytical format. Since most of optimal growth theory is cast in the framework of a utility representation of the planner's preference order, it is natural to present conditions sufficient for the existence of a continuous representation of $\succsim$.

A function $U: \chi\to\mathbb{R}$ is a *utility function* representing $\succsim$ on a set $\chi\subseteq\mathscr{E}_{+}$ provided

$$X\succsim Y \text{ if and only if } U(X)\geq U(Y).$$

The following theorem of Debreu (1954,1959) gives one answer to the representation problem.

SEPARABLE REPRESENTATION THEOREM. *Suppose a preference relation $\succsim$ satisfies (U1) and (U4) on a separable connected set χ. Then $\succsim$ is represented by a continuous utility function U.*

The Separable Representation Theorm applies to preference orders on $\chi=\mathbb{R}_{+}^{\infty}$ where χ is given the relative product topology inherited from $\mathbb{R}^{\infty}$. As $\mathbb{R}^{\infty}$ is a separable space, a continuous utility function exists given (U1) and (U4). Another application occurs in the case $\mathscr{E}=A_{\omega}$ equipped with the β-topology ($\beta>\alpha$): c_0 is a separable Banach space. Set $A(\omega)=\{X\in\mathbb{R}^{\infty}: |X|\leq\omega\}$ and note $A_{\omega}=\cup_{\lambda>0}\lambda A(\omega)$.

LEMMA 1. *Let $\alpha<\beta$ and $\omega_t=\alpha^t$. The β-topology and the relative product topology coincide on $\lambda A(\omega)$ for each $\lambda>0$.*

PROOF. Clearly the β-topology is stronger than the product topology on A_{ω}. Suppose $C^n\to C$ in the product topology. Given $\varepsilon>0$, choose M such that $\lambda(\alpha/\beta)^M<\varepsilon$. Since $C^n\to C$, there is an N with $\sup\{|c_t^n-c_t|/\beta^t: t\leq M\}<\varepsilon$ for $n>N$, and $C^n\to C$ in the β-topology. But then $||C^n-C||_{\beta}<2\varepsilon$ for $n>N$, and $C^n\to C$ in the β-topology. The two topologies are identical. $\square$

It is crucial that $\beta>\alpha$. Majumdar (1975) gives an example illustrating way norm-bounded feasible sets are not compact in the norm topology. The same sort of problem occur here if $\beta=\alpha$. In fact, $\beta=\alpha=1$ is precisely Majumdar's case.

The coincidence of the relative product and β-topologies for $\beta>\alpha$ on the set $\chi=A_{\omega}^{+}$ implies χ is a separable and pathwise-connected subset of a normed linear space. Therefore, the Separable Representation Theorem yields a β-continuous utility function for each β-continuous preference relation.

[18] Clearly, only the set $\{Y\in\mathscr{E}_{+}: Y\succsim X\}$ need be τ-closed for the existence of optimal allocations in the presence of a τ-compact constraint set.

As a topological space, $(\ell^\infty, \|\cdot\|_\infty)$ is not separable. Thus, even if $\succsim$ is $\|\cdot\|_\infty$-continuous on ℓ^∞, we cannot apply the Separable Representation Theorem in order to represent $\succsim$ by a continuous utility function. However, Mas-Colell (1986) exploited the order structure of $\mathscr{C}$ and the monotonicity axiom (U2) in the manner of Kannai (1970) to deduce the existence of a continuous utility representation of $\succsim$ on a portion of the program space.

MONOTONE REPRESENTATION THEOREM. *Suppose a preference relation $\succsim$ satisfies (U1), (U2), and (U4) on a non-empty order interval $\chi \subset \mathscr{C}_+$. Then there is a continuous utility function $U: \chi \to [0,1]$ representing the preference relation $\succsim$.*[19]

PROOF. Let $\chi = [A,B]$ where $A,B \in \mathscr{C}_+$. Since $B \geq A$, (U2) implies $B \succsim A$. If $A \succsim B$, there is nothing to prove. So, let $B \succ A$. Consider the set

$$J = \{\theta A + (1-\theta)B\colon\ 0 \leq \theta \leq 1\}.$$

Since $\theta \to \theta A + (1-\theta)B$ is a homeomorphism from $[0,1]$ to J, the separable representation theorem yields a continuous $f: J \to [0,1]$ such that $f(A)=0$, $f(B)=1$, and $f(X) \geq f(Y)$ whenever $X, Y \in J$ and $X \succsim Y$. For any $X \in \chi$ let $v(X)$ be such that $v(X) \in J$ and $v(X) \sim X$. Because J is connected and a subset of the union of the closed sets $\{Y\colon Y \succsim X\}$ and $\{Y\colon X \succsim Y\}$, such a $v(X)$ must exist. Now set $U(X) = f(v(X))$. Obviously, U is a utility function. Since U is onto $[0,1]$, given any $t \in [0,1]$, there exists an $X \in \chi$ such that $U(X) = t$. The sets

$$U^{-1}[t, \infty) = \{Y\colon Y \succsim X\}$$

$$U^{-1}(-\infty, t] = \{Y\colon X \succsim Y\}$$

are closed by (U4). Therefore U is continuous. $\square$

The Monotone Representation Theorem handles many of the examples we shall discuss. For $\omega = (1,1,\ldots)$, the Monotone Representation Theorem applies to the Beals-Koopmans example when $\chi = [0,\omega]$ and A_ω has the α-topology. In the optimal growth model, the order interval is defined by taking ω to be the path of pure accumulation with seed k_0. If there is a maximum sustainable stock $b > 0$ and the production function is stationary, then the order interval $[0,\omega]$ with $\omega_t = b$ works. The Monotone Representation Theorem implies that a preference order defined over $\mathscr{C}_+$ may only admit a continuous utility function on $[0,\omega]$. This order interval is sufficiently "large" enough to contain *all* economically relevant consumption programs. Paths offering consumption $C \notin [0,\omega]$ cannot be realized by any feasible plan of accumulation given the initial stocks. As such, those programs may be ignored for the theory developed to analyze the existence and characterization of optimal allocations. Finally, we remark that the order property of $\mathscr{C}$ was the critical structural feature of the choice space used to obtain the representation of the preference relation.

The existence of support prices or a competitive equilibrium price system in infinite dimensional commodity spaces is another point where finite horizon results do not readily carry over to the infinite horizon models. The familiar separation theorem argument proving the Second Fundamental Welfare Theorem for a discrete time finite horizon model relies on the fact that the positive cone of a finite dimensional Euclidean space has a non-

[19] The proof is a slight repackaging of Mas-Colell (1986, p 1044).

empty norm interior. Debreu (1954) showed the Second Fundamental Welfare Theorem could be demonstrated for economies modeled on infinite dimensional spaces with this property. The positive cones of the spaces $(\ell^\infty,||\cdot||_\infty)$ and $(A_\omega,||\cdot||_\alpha)$ have non-empty interiors; Debreu's theorem applies to these cases.[20] Unfortunately, the positive cones of the spaces $(\mathbb{R}^\infty,d_F)$ and $(A_\omega,||\cdot||_\beta)$, $\beta>\alpha$, have empty interiors. The interior of ℓ^∞_+ is also empty in the $\tau(\ell^\infty,\ell^1)$-topology. Nearly thirty years after the publication of Debreu's paper, Mas-Colell (1986) gave a demonstration of the Second Fundamental Welfare Theorem for spaces whose positive cone, $\mathscr{E}_+$, has an empty interior. He required $\mathscr{E}$ to be a topological Riesz space. He introduced a new restriction on preference orders, called properness, which restricted marginal rates of substitution in a manner allowing use of the separation theorem from convex analysis to deduce the welfare theorem. A detailed survey of the welfare theorems for infinite dimensional commodity space models may be found in Becker (1991a). We briefly review the notion of proper preferences in order to discuss an application of myopic utility functions in an exchange economy setting and to contrast properness to other restrictions on the marginal rates of substitution in Part III on impatience and discounting.

Let $\mathscr{E}$ be a Riesz space equipped with a locally convex linear topology τ and $\succsim$ a preference relation on $\mathscr{E}_+$.[21]

UNIFORM PROPERNESS. We say that $\succsim$ is a *uniformly τ-proper preference relation* on $\mathscr{E}_+$ if there exists a non-empty τ-open convex cone Γ such that

a) $\Gamma\cap(-\mathscr{E}_+)\neq\phi$; and
b) $(X+\Gamma)\cap\{Y: Y\succ X\}=\phi$ for all $X\in\mathscr{E}_+$.

A uniformly proper preference relation bounds marginal rates of substitution and is intimately linked to the possibility of supporting a weakly preferred set by a continuous linear functional. Formally:

REMARK. If $\succsim$ is a convex τ-proper preference relation on $\mathscr{E}_+$, then there exists a $P\in\mathscr{E}'$, $P\neq 0$, such that $\langle X,P\rangle\leq\langle Z,P\rangle$ for all $Z\in\{Y: Y\succsim X\}$.

For preference relations defined on either a space $\mathscr{E}_+$ with non-empty τ-interior or $\mathscr{E}$ is an *AM*-space with unit, then uniform properness is a strengthening of the monotonicity axiom.[22]

Preferences on $(A_\omega,||\cdot||_\alpha)$ and $(\ell^\infty,||\cdot||_\infty)$ are uniformly norm-proper if $\succsim$ is strictly monotone as in axiom (U2′) below.[23]

MONOTONICITY.

(U2′) $\succsim$ is a strictly monotone relation, i.e. $X\geq Y$, $X\neq Y$, in $\mathscr{E}_+$ implies $X\succ Y$.

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[20] Majumdar (1975) derives support prices for the one sector optimal growth model in discrete time using a separation argument in the spirit of Debreu's paper for the commodity space $(\ell^\infty,||\cdot||_\infty)$.

[21] To simplify our exposition, we take the Properness Characterization Theorem due originally to Mas-Colell (1986) and presented in Aliprantis, Brown and Burkinshaw (1989, p. 117) as the basis for our definition of properness.

[22] See Aliprantis, Brown and Burkinshaw (1989, p. 118) and Mas-Colell (1986, p. 1043).

[23] The maximum order is a monotone, but not strictly monotone preference relation.

Preference relations defined on $\mathbb{R}_{+}^{\infty}$ are not, in general, uniformly proper. In fact, there are no utility functions on $\mathbb{R}_{+}^{\infty}$ which are strictly monotone, quasi-concave, d_F-continuous, and uniformly d_F-proper.[24] This means that the preference relation defined by the utility function

$$U(C)=\sum_{t=1}^{\infty}\delta^{t-1}u(c_t), \tag{1}$$

is not uniformly d_F-proper on $\mathbb{R}_{+}^{\infty}$ for $0<\delta<1$ and u bounded, strictly increasing, and strictly concave on $[0,\infty)$. The parameter δ is called the *discount factor* and $\delta^{-1}-1=\rho$ is called the *pure rate of time preference*. The function u is known as the *one-period reward* or *felicity* function.[25] In order to guarantee convergence in the series (1), we temporarily assume u is bounded. Since this U is an important member of the recursive utility class, the properness condition would seem to have limited applicability in the intertemporal framework. However, the following result from Becker (1991b) illustrates the potential for the β-topology to yield a positive result.

PROPOSITION 1. *If U is strictly monotone, β-norm continuous utility function on A_{ω}^{+} for $\omega_t=\alpha^t$, then the underlying preference order $\succsim$ is a uniformly α-norm proper preference relation,*

PROOF. Since the β-norm is a lattice norm on A_ω, the β-topology is coarser than the α-topology; it follows that the preference is α-norm continuous. Hence a β-norm continuous utility function is continuous in the α-topology. Let $\mathscr{V}$ be the open unit ball in A_ω, i.e. $\mathscr{V}=\{X\in A_\omega: \|X\|_\alpha<1\}$. Let Γ be the cone generated by the set $(-\omega+\mathscr{V})$. Clearly, $\Gamma\subset A_{\omega}^{-}$, the negative cone of A_ω. Notice that $Y\in\Gamma$ implies $Y<0$ and $U(X+Y)<U(X)$. Hence $(X+\Gamma)\cap\{Z\in A_\omega: U(Z)\geq U(X)\}=\phi$. □

It is useful to record the fact that $\succsim$ is uniformly proper in the α-topology in the case where u obeys a classical Inada condition at zero in (1). For example, set $u(c)=\arctan(\sqrt{c})$. A result due to Back (1988) shows examples of utility functions of the form (1) obeying the Inada condition at zero would fail to be uniformly proper in the $\tau(\ell^\infty,\ell^1)$-topology.

5. *Recursive Utility: The Koopmans Axioms*

Theories of intertemporal decision making further specialize the axiom system (U1)–(U4) to capture the essential role of time in the preference order. The purpose of this section is to present an axiomatic basis for a *recursive utility* representation of $\succsim$ to exist. Before developing this axiomatic structure, it is worthwhile to introduce the recursive utility construct through three examples.

A familiar objective in optimal growth theory is the *time-additive separable (TAS)* utility function (1). The TAS form has two interesting properties. First, the marginal rate of substitution between any pair of adjacent dates depends only on consumption at those dates. Formally,

[24] See Aliprantis, Brown and Burkinshaw (1989, p. 174).

[25] u is usually assumed to be twice continuously differentiable on $(0,\infty)$.

$$MRS_{t,t+1}(C_{con})=\frac{u'(c_t)}{\delta u'(c_{t+1})}.$$

In particular, for constant consumption profiles denoted by $C=C_{con}(c_t=c$ for all $t)$,

$$MRS_{t,t+1}(C_{con})=\delta^{-1}, \tag{2}$$

which is independent of C_{con}.

The second property is that U is *recursive.* The behavior embodied in the TAS specification of U has a self-referential property: namely, the behavior of the planner over the infinite time horizon $t=1,2,...$ is guided by the behavior of that agent over the tail horizon $t=T+1,T+2,...$ (for each T) hidden inside the original horizon.[26] For the TAS functional, recursivity means the objective from time $T+1$ to $+\infty$ has the same form as the objective starting at $T=0$ (except for some time shifts in consumption dates). Formally, (1) may be rewritten as

$$\sum_{t=1}^{\infty}\delta^{t-1}u(c_t)=\sum_{t=1}^{T}\delta^{t-1}u(c_t)+\delta^T\sum_{t=1}^{\infty}\delta^{t-1}u(c_{t+T}). \tag{3}$$

For $T=0$, (3) coincides with (1).

An important implication of the recursive structure found in the TAS utility specification is that intertemporal planning in a stationary environment is time consistent in the sense first used by Strotz (1955). If the planner is free to revise his decisions at some time $T>0$, then his decisions at T will depend on the past only through accumulated assets defining the current magnitude of the state variable. Decisions at T will not depend *directly* on past consumption patterns. Indeed, recursive utility is the intertemporal analog of weak separability of future consumption from present consumption as formulated in standard finite horizon demand theory.[27]

The TAS objective functional has been criticized by various authors, dating back to Fisher (1930), on grounds that the pure tare of time preference should not be independent of the size and shape of the consumption profile. When preference are time-additive, this independence is a direct consequence of the strong separability property embodies in (2). Hicks (1965, p. 261) argued against the TAS formulation on grounds that successive consumption units should exhibit a strong complementarity between them. The force of the Hicksian critique is that the amount of consumption in period 1 the planner would be willing to give up to increase consumption in period T should also depend in some way on the planned consumption in adjacent periods (e.g. periods $T-1$ and $T+1$). The additivity hypothesis denies this connection. In essence, Hicks argued that the potential for smoothing consumption in the presence of complementarity between priods is lost in the acceptance that felicities are independent as found in the TAS specification of utility. Lucas and Stokey (1984) argued that the only basis for studying the TAS case is its analytic tractability.

Koopmans (1960) laid the foundation for eliminating both deficiencies of the TAS

[26] We are transposing Gleick's (1987, p. 179) characterization of recursive structure into an economic context.

[27] Consult Blackorby, Primont, and Russell (1978) for details on finite horizon discrete time demand theory.

functional by introducing recursive preferences. The recursive utility class is designed to introduce a degree of generality that is consistent with Fisher's time preference views, offers time consistent optimal planning, and preserves much of the analytical convenience of the TAS case.

The second example of a recursive utility functional is based on an uncertain lifetime model. Let p_t denote the probability that the agent dies at time t and let $q_t=1-p_t$ denote the survival probability at t. Let $U_0=U(0_{con})$ be the utility of the agent if dead. Epstein (1983) showed that

$$U(C)=-\sum_{t=1}^{\infty}\exp[-\sum_{\tau=1}^{t}u(c_\tau)] \tag{4}$$

is a von-Neumann-Morgenstein index.[28] We may use (4) to calculate the expected utility of a consumption stream C, $EU(C)$. Assuming the probability of death is independent of t, a routine calculation yields the expression

$$EU(C)=pU_0-(1-pU_0)\sum_{t=1}^{\infty}\exp[-\sum_{\tau=1}^{t}[u(c_\tau)-\log(1-p)]]. \tag{5}$$

Equation (5) may be further transformed into the equivalent representation

$$EU^*(C)=-\sum_{t=1}^{\infty}\exp[-\sum_{\tau=1}^{\infty}[u(c_\tau)-\log(1-p)]]. \tag{6}$$

The expected utility function EU^* converts the preferences of the agent in the stochastic lifetime problem into the equivalent deterministic payoff functional (6). The utility functional embodies in (6) is a member of the recursive class. Indeed, following the derivation of (3) in the case (6) leads to[29]

$$\begin{aligned}-\sum_{t=1}^{\infty}\exp[-\sum_{\tau=1}^{T}u(c_\tau)]=&-\sum_{t=1}^{T}\exp[-\sum_{\tau=1}^{t}u(c_\tau)]\\&-\sum_{t=1}^{T}\exp[-\sum_{\tau=1}^{t}u(c_\tau)]\sum_{t=T+1}^{\infty}\exp[-\sum_{\tau=T+1}^{t}u(c_\tau)],\end{aligned}$$

which is a time-shifted version of (4). We call this the (EH) utility function after the continuous time form introduced by Epstein and Hynes (1983).[30]

The third example is based on the maximum functional defined by

$$U(C)=\inf_t c_t. \tag{7}$$

Clearly,

$$U(C)=\inf_t c_t=\inf\{c_1, \inf_{t\geq 2} c_t\}=\inf\{c_1,c_2,\dots,c_T, \inf_{t\geq T+1} c_t\}. \tag{8}$$

[28] See Epstein and Hynes (1983) for a deterministic account of this functional in continuous time.
[29] We assume $p=0$ for simplicity.
[30] The EH utility function is closely related to the continuous time Uzawa (1968) functional.

This computation shows the maximum functional also enjoys a recursive structure in common with the TAS and EH cases.

Given a preference order $\succsim$ on χ, a utility representation, U, of $\succsim$ is a *recursive utility function* if there is a real-valued function u defined on $\mathbb{R}_+$ and a real-valued function W defined on $u(\mathbb{R}_+)\times U(\chi)$ such that

$$U(C)=W(u(c_1),U(SC)). \tag{9}$$

The function W is called the *aggregator* and equation (9) is called *Koopman's equation.* We refer to u as the *felicity* function. A recursive utility function expresses the weak separability of the future from the present.[31] Fisher's two-period conception of an agent contemplating current consumption and future utility may be modeled by recursive utility functions.

The three examples introduced above fall under the recursive utility definition. For the TAS case, the aggregator is $W(C,y)=u(c)+\delta y$. The EH functional has $W(c,y)=-(1+y)\exp(-u(c))$ as an aggregator. The maximin aggregator is $W(c,y)=\min\{c,y\}$.

We turn to describing sufficient conditions for $\succsim$ to exhibit a recursive utility representation. The axiom system employed differs slightly from those in Koopmans (1960). Let χ be a path connected subspace of $\mathscr{S}_+$. The notation (z,X) denotes the sequence $(z,x_1, x_2,\ldots)$. We assume the shift operator S defined on $\mathscr{S}$ is continuous as is the embedding $z\mapsto(z,X)$ of X into $\mathscr{S}$ for each X. The projection of $\mathscr{S}_+$ onto the first coordinate subspace is denoted $\pi\mathscr{S}_+$. The Koopmans axioms are:

KOOPMANS AXIOMS.

(K1) $\succsim$ is a stationary relation, i.e. $(z,X)\succsim(z,X')$ if and only if $X\succsim X'$ for all $z\in\pi\mathscr{S}_+$;
(K2) $\succsim$ exhibits limited independence, i.e. for all z,z',X, and $X',(z,X)\succsim(z',X)$ if and only if $(z,X')\succsim(z',X')$;
(K3) $\succsim$ is a sensitive relation, i.e. there is an $X\in\mathscr{S}_+$ and a $z,z'\in\pi\mathscr{S}_+$ with $(z',X)\succsim(z,X)$.

Axiom (K1) states the preference order is independent of calendar time. Axiom (K2) says that preferences between present consumption alternatives are independent of future consumption and that preferences over future consumption streams are independent of the level of consumption. The combination of axioms (K1) and (K2) implies that merely postponing a decision between two programs will not alter the rank order. The time inconsistency problem raised by Strotz (1955) does not arise when preferences are stationary and satisfy limited independence. Axiom (K3) rules out complete indifference between levels of current consumption. It ensures that the preference order is non-trivial. Note that sensitivity follows from strict monotonicity (U2').

Koopmans' (1960) result is that (K1)–(K3) are sufficient for $\succsim$ to have a recursive utility representation. In such cases, we say that utility is *recursively separable.*

RECURSIVE REPRESENTATION THEOREM. *Suppose a preference relation $\succsim$ satisfies (U1)–(U4) and (K1)–(K3) on a path-connected set $\chi\subseteq\mathscr{S}_+$. If $\succsim$ has a continuous utility repre-*

[31] We use standard terminology from demand theory, c.f. Blackorby, Primont, and Russell (1978).

sentation U, then there are continuous functions u and W with $U(C)=W(u(\pi C),U(SC))$. Further, W is non-decreasing in both u and U.

PROOF. Take $C_0 \in \chi$ and define $u(z)=U(z,C_0)$. Note that $u(z)=u(y)$ is equivalent to $U(z,C_0)=U(y,C_0)$ which is in turn equivalent to $U(z,C)=U(y,C)$ for all C by (K2). Since U depends only on $u(z)$, there is a function F with $U(z,C)=F(u(z),C)$.

Now if $U(C)=U(C'), U(z,C)=U(z,C')$ for all z by (K1). Thus F depends only on $U(C)$ and there is a function W with $U(z,C)=W(u(z),U(C))$.

To show W is non-decreasing, consider z and z' with $u=u(z)\geq u(z')=u'$. Then $U(z,C_0)\geq U(z',C_0)$ so $U(z,C)\geq U(z',C)$ for all C. Applying the definitions to the last inequality yields $W(u,U(C))\geq W(u',U(C))$. A similar argument shows that W is non-decreasing in U. Continuity follows from the following lemma. □

LEMMA 2. *If U is continuous, then u and W are continuous.*

PROOF. Since $u(z)=U(z,C_0)$, u is continuous.

We next show W is separately continuous in u and U. Fix $u=u(z)$ and let $U_n \uparrow U$. Take C_1 and C with $U(C_1)=U_1$ and $U(C)=U$. Let $\{C(t)\}$ be a path from C_1 to C. Since χ is a path-connected space, $C(t)\in\chi$. Clearly $U(\{C(t)\})\supseteq[U_1,U]$ since $U(\{C(t)\})$ is connected. Take t_n with $U(C(t_n))=U_n$. Let t^* be any cluster point of $\{t_n\}$. As $C(t_n)\rightarrow C(t^*)$, we have $W(u,U_n)=U(z,C(t_n))\rightarrow U(z,C(t^*))=W(z,U)$. This also applies to $U_n \downarrow U$, so W is continuous in U for fixed u. Similarly, W is continuous in u for fixed U.

Now let $(u_n,U_n)\rightarrow(u,U)$. Define $v_n=\sup_{m>n}u_m, V_n=\sup_{m>n}U_m, y_n=\inf_{m>n}u_m$, and $Y_n=\inf_{m>n}U_m$, so

$$W(v_n,\, V_n)\geq W(u_n,\, U_n)\geq W(y_n,\, Y_n).$$

Fix m, then for $n\geq m$:

$$W(v_m,\, V_n)\geq W(v_n,\, V_n)\geq W(u,\, U).$$

By letting $n\rightarrow\infty$ and using the continuity of W in U, we see that

$$W(v_m,\, U)\geq \lim_n W(v_n,\, V_n)\geq W(u,\, C).$$

Note that $\lim W(v_n,V_n)$ exists since $\{W(v_n,V_n)\}$ is a non-increasing sequence. Now let $m\rightarrow\infty$ to get the result. Using a similar argument on (y_n,Y_n) completes the proof. □

RRMARK. If monotonicity axiom (U2) is strengthened to (U2′), then W is strictly increasing in future utility.

Koopmans (1960) and Koopmans, Diamond, and Williamson (1964) assume $\succsim$ is continuous with respect to the supremum norm topology. However, as noted subsequently in Koopmans (1972b), this type of uniform continuity is not required for the Recursive Representation Theorem.

The Recursive Representation Theorem assumes $\succsim$ has a continuous utility representation. If χ is an order interval and (U2) is strengthened to (U2′), then the Monotone Representation Theorem implies utility is recursive.

COROLLARY. *If χ is an order interval in $\mathscr{E}_+$ and (U1), (U2'), (U3), (U4), and (K1)-(K3) hold for a preference order $\succsim$, then $\succsim$ has a recursive utility representation.*

If χ is a separable space, then this is also holds by the Separable Representation Theorem. For example, consider the space $\ell^\infty(\alpha)$ endowed with the $\|\cdot\|_\beta$-norm, and having $\beta>\alpha$. This is a separable space and pathwise-connected normed linear space. Therefore, any preference order on $\ell^\infty(\alpha)$ which is β-continuous, stationary, obeys the limited independence, and sensitivity axioms must have a recursive representation.

The role of Koopmans' sensitivity axiom is to insure a non-trivial representation of U in terms of the aggregator. If sensitivity fails, then utility functions such as $U(C)=\Lambda(C)$ on $\chi=\ell^\infty$ are admissible where Λ is a Banach limit.[32]

BANACH LIMIT. A Banach limit is a linear functional such that:
(B1) $\Lambda(C)\geq 0$ if $C\geq 0$;
(B2) $\Lambda(C)=\Lambda(S^N C)$ for $N=1,2,\ldots$);
(B3) $\liminf_t c_t \leq \Lambda(C) \leq \limsup_t c_t$.

An example of a utility function which can be extended to a Banach limit on ℓ^∞_+ is the average consumption function

$$U(C)=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}c_t.$$

Banach limit utility functions yield $W(u,U)=U$ by property (B2) and are not really represented by the aggregator: any Banach limit satisfies this recursive relation.[33]

Koopmans' Recursive Representation Theorem says that a recursively separable utility function determines a unique recursive aggregator W. Streufert (1990) investigated the converse: does W uniquely determine U?[34] This uniqueness problem can be cast as an investigation of the uniqueness of a solution U to Koopmans' equation given W. Streufert introduces the notions of biconvergence and tail-sensitivity to provide an affirmative answer to this issue. Biconvergence of U is defined for a fixed order interval $[0,\omega]\subseteq\mathscr{E}_+$, where ω has strictly positive components. Biconvergence requires both that a "poor" consumption profile cannot be preferred to a program eventually offering a "tail" of ω (upper convergence) and a "good" consumption profile cannot be preferred by another eventually offering 0 consumption in the tail (lower convergence). The biconvergence property of U is invariant to continuous monotonic transformations of U. The basic intuition for tail insensitivity is similar. The difference between the two concepts is that tail sensitivity is an ordinal property of the utility scale. Streufert proved that under the biconvergence hypothesis, U is the unique "admissible" solution to Koopmans' equation in W. His converse proposition says that if U is not biconvergent, then Koopmans' equation has multiple "admissible" solutions. In this sense, biconvergence is the weakest condition de-

[32] A Banach limit is an example of a pure charge on ℓ^∞.
[33] Looking ahead to Part IV, a Banach limit utility function has $\delta\beta^\gamma=1$ whereas we require $\delta\beta^\gamma<1$ for the aggregator framework.
[34] The aggregator was first taken as a primitive for preferences on ℓ^∞_+ in Lucas and Stokey (1984). The subsequent analysis of Boyd (1990) extended their results to larger sequence spaces. Lucas and Stokey as well as Boyd attacked the uniqueness problem. Their approach is the heart of Part IV.

livering a unique solution to the Koopmans equation in W.

Fix an order interval $[0,\omega]\subseteq \mathscr{E}_+$. The choice $\omega=(\alpha,\alpha^2,\ldots)$, $\alpha\geq 1$, is possible. We assume $\omega_t>0$ for each t, so $\omega>0$. The function U is said to be *upper convergent* over $[0,\omega]$ if for every $C\in[0,\omega]$

$$\lim_{T\to\infty} U(\pi^T C, S^T\omega)=U(C).$$

The limit always exists since U is monotone and $(\pi^T C,S^T\omega)\geq(\pi^{T+1}C,S^{T+1}\omega)$. The function U is said to be *lower convergent* over $[0,\omega]$ if for every $C\in[0,\omega]$

$$\lim_{T\to\infty} U(\pi^T C, S^T 0)=U(C).$$

The limit always exists since U is monotone and $(\pi^T C,S^T 0)\leq(\pi^{T+1}C,S^{T+1}0)$. The function U is said to be *biconvergent* over $[0,\omega]$ if it is both upper and lower convergent over $[0,\omega]$. Notice that if U is a Banach limit, it is not lower convergent: let $C=(1,1,\ldots)$, then $\Lambda(\pi^T C, S^T 0)=0$ for each T and $\Lambda(C)=1$. We also notice here that 0 has a special role in the definition of biconvergence. In particular, the TAS from with logarithmic felicity cannot be lower convergent on $[0,\omega]$.

Let $U_1:_+\mathscr{E}\to[0,\infty]$ denote a utility representation of $\succsim$. U_1 need not be monotonic or stationary, hence it does not have to equal U—the primitive in Koopmans' Recursive Representation Theorem. Such a function U_1 is a *general* solution to Koopmans' equation (here $u(z)=z$) if there exists a sequence of *subutility* functions $(U_2,U_3,\ldots)$ such that for all $C\in \mathscr{E}_+$ and for all $t\geq 0$:

$$U_t(S^{t-1})=W(c_t,\ U_{t+1}(S^t C)).$$

A general solution U_1 is *admissible* if for all $C\in[0,\omega]$, $U(0)\leq U_1(C)\leq U(\omega)$.

The following example drawn from Streufert (1990) illustrates the need for the admissibility qualification. Let $\omega=(1,1,\ldots)$ and U have the TAS form

$$U(C)=\sum_{t=1}^{\infty} \delta^{t-1}c_t.$$

U is biconvergent over $[0,\omega]$. The aggregator $W(c,U)=c+\delta U$ has the *inadmissible* solution defined by

$$U_t(S^{t-1}C)=U(S^{t-1}C)+\delta^{-t}$$

when $S^{t-1}C\in c_{00}$ and

$$U_t(S^{t-1}C)=U(S^{t-1}C)$$

otherwise. Streufert's first theorem is:[35]

[35] Streufert (1990, p. 81) notes the aggregator is strictly increasing in future utility is weakly increasing, after-period-1 separable, and stationary. The maximum utility function generates an aggregator that is only non-decreasing in future utility. We use a weaker form of limited independence than Streufert. For this reason we require (U2′) instead of (U2).

BICONVERGENCE THEOREM. *If U is biconvergent over* $[0,\omega]$, *and* $\succsim$ *satisfies (U2′), then for any admissible general solution U_1 to Koopmans' euqation,* $U_1=U$ *over* $[0,\omega]$.[36]

PROOF. Choose $C\in[0,\omega]$. Admissibility implies for each $t\geq 1$ that $U_{t+1}(S^tC)\geq U(0)$, otherwise W strictly increasing in future utility would imply

$$\begin{aligned} U_1(\pi^t 0, S^tC) &= W(0, \dots W(0, U_{t+1}(S^tC))\dots) \\ &< W(0, \dots W(0, U(0, 0, \dots))\dots) = U(0), \end{aligned}$$

which would contradict admissibility. Similarly, admissibility implies that for each t,

$$U_{t+1}(S^tC)\leq U(S^t\omega).$$

These two bounds on $U_{t+1}(S^tC)$ imply for each t that:

$$\begin{aligned} U(\pi^t C, S^t\omega) &= W(c_1, \dots W(c_t, U(S^t\omega))\dots) \\ &\geq W(c_1, \dots W(c_t, U_{t+1}(S^tC))\dots) = U_1(C) \\ &\geq W(c_1, \dots W(c_t, U(S^t 0))\dots) = U(\pi^t C, S^t 0). \end{aligned}$$

Biconvergence implies these upper and lower bounds on $U_1(C)$ both converge to $U(C)$. Therefore U and U_1 agree on $[0,\omega]$. □

Streufert also proved a converse to the Biconvergence Theorem.

NON-BICONVERGENCE THEOREM. *Suppose that for every* $C\in[0,\omega]$ *and every period* $t\geq 1$, $U(\pi^t C, [0,S^t\omega])$ *is an interval. Then if U is not biconvergent over* $[0,\omega]$, *there exists an admissible solution U_1 to Koopmans' equation in W such that* $U_1\neq U$ *over* $[0,\omega]$.

PROOF. See Streufert (1990, Theorem B). □

Banach limits provide a family of non-biconvergent utility function examples. Any TAS utility function which is not bounded below on $[0,\omega]$ is not lower convergent, hence it also must fail to enjoy the biconvergence property. The aggregator for these utility functions cannot uniquely determine U over $[0,\omega]$. The question of uniqueness is further discussed in Part IV where the focus is on the aggregator as the primitive concept.

Koopmans' (1972b) explored the existence of a recursive utility representation of a preference order. He showed that under a slight strengthening of (K2), so that all complementarities between adjacent periods consumption could be excluded, utility must be additive across time periods. Given the stationarity axiom, he concluded that utility took the TAS form. The key to the additive representation is the following axiom:

EXTENDED INDEPENDENCE.

(K2′) $\succsim$ exhibits extended independence; for all z,w,z',w',C, and C'.

$$(z, w, C)\succsim(z', w', C) \text{ if and only if } (z, w, C')\succsim(z', w', C').$$

Extended Independence says that preferences over the first two periods consumption

[36] After Streufert (1990, pp. 83–84).

are independent of consumption from period three onwards. The axiom responsible for the TAS representation is:

COMPLETE INDEPENDENCE.

(K2*) $\succsim$ exhibits complete independence; axioms (K2) and (K2′) hold.

ADDITIVE REPRESENTATION THEOREM. *Let $\mathscr{E}=A_\omega$ for some constant sequence $\omega=\omega_{con}>0$. Endow $\mathscr{E}$ with the topology induced by the lattice norm. Assume $\succsim$ satisfies axioms (U1)–(U4), (K1), (K2*), and (K3) on $\mathscr{E}_+$ with $\omega>0$. Then there is a continuous TAS utility function U representing $\succsim$ on $\mathscr{E}_+$. Moreover U is unique up to a positive linear transformation.*

The proof of Koopmans' Additive Representation Theorem is lengthy. However, the essential idea is to construct U in several steps. First, define a utility function U_T on the set of all programs $C\in\mathscr{E}_+$ having $S^TC=(z_{T+1},z_{T+2},\ldots)$ where Z is a fixed reference program. Consumption paths restricted to this subspace may be ranked by an induced preference order on a subset of $\mathbb{R}^T_+$; standard utility representation theorems for independent factor spaces may be invoked to yield an additive utility function on this subspace.[37] Stationarity implies that utility on $\mathbb{R}^T_+$ has the form

$$U_T(c_1, c_2, \ldots, c_T)=\sum_{t=1}^{T}\delta^{t-1}u(c_t). \tag{10}$$

Koopmans then extends U_T to the subspace of programs which are eventually constant, i.e. $S^TC=(c,c,\ldots)$ for some T. Let χ_{con} denote the space of all eventually constant programs. The tail of any program $C\in\chi_{con}$ is shown to contribute an amount $\delta^Tu(c)/(1-\delta)$ to the utility of a program in (10). Thus $C\in\chi_{con}$ implies

$$\begin{aligned}U(C)&=U(c_1.\ \ldots, c_T, S^{T-1}C)\\ &=u(c_1)+\delta u(c_2)+\ldots+\delta^{T-1}u(c_T)+\frac{\delta^Tu(c)}{1-\delta}.\end{aligned}$$

The function U is unique up to a positive linear transformation. The final step is to show that U may be extended to $\mathscr{E}_+$.[38] An application of the Additive Representation Theorem occurs for the case $\omega_{con}=(1,1,\ldots)$ implying $A_\omega=\ell^\infty$. An interesting open question arises: Does the Additive Representation Theorem hold for a general *AM*-space with unit? For example, can the Additive Representation Theorem be extended to cases of growth in ω as would occur if $\omega=(\alpha,\alpha^2,\ldots)$ and $\alpha>1$?[39]

Variations on the recursive axiom system (K1)–(K3) are possible. Rader (1981) shows that adding homotheticity to the hypotheses of the Additive Representation Theorem implies the felicity function u is homogeneous or logarithmic. The homogeneous case was also conjectured by Hicks (1965). Epstein (1986) introduces the class of implicity

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[37] See Koopmans (1972a), Debreu (1960), and Fishburn (1970).

[38] A careful reading of Koopmans' Proposition 3 (pp. 89–91) shows that his requirement that C be bounded in utility is equivalent to $0\leq C\leq\lambda\omega_{con}$ for some $\lambda>0$. This holds here since $\mathscr{E}$ is the principal ideal generated by ω_{con}. The lattice norm topology coincides with the supremum norm topology utilized by Koopmans.

[39] This case has recently been answered in the affirmative by Dolmas (1991).

additive utility functions as an alternative to the additive class based on the Independence Axiom. In his setup, the Independence Axiom states that the marginal rate of substitution between period t and t' consumption depends on the entire consumption path but only through the value of its lifetime utility $U(C)$.[40] In Epstein's formulation, there is scope for a limited degree of complementarity between adjacent periods consumption. He also weakens the stationarity postulate in order to express the idea that the passage of time does not have an effect on preferences so long as lifetime utility is constant. This means $U(c,C)=U(C)$ where $(c,C)\equiv C'$ is the program defined by $c'_1=c$, $c'_t=c_{t-1}(t\geq 2)$. The resulting utility function has the form

$$U(C)=\sum_{t=1}^{\infty}(\delta(u))^{t-1}g(c_t,u),$$

where $u\equiv U(C)$, $\delta(u)\in(0,1)$, $g(\cdot,u):\mathbb{R}_+\to\mathbb{R}$ is strictly concave, continuously differentiable with a positive derivative, and $g(0,u)=0$.

Several writers have explored the consequences eliminating the Independence Axiom. Majumdar (1975) gave the example

$$U(C)=w(c_1, c_2, \dots, c_T)+\sum_{t=1}^{\infty}\delta(t)v_t(c_1, c_2, \dots, c_t),$$

defined on ℓ^∞ where $\delta(t)\geq 0$ for all t, $\sum\delta(t)=1$, $w:\mathbb{R}_+^T\to\mathbb{R}_+$ is continuous, quasi-concave, non-decreasing in each argument, and $\{v_t\}$ is a sequence of quasi-concave, continuous functions from $\mathbb{R}_+^t$ to $\mathbb{R}_+$, each v_t being strictly increasing in all its arguments and the sequence being uniformly bounded above. There is special significance accorded to consumption in periods $1,\dots,T$ as measured by the w function. Moreover, history counts since the felicity given by c_t at time t depends on the consumption enjoyed in all previous periods. He argues U is $\tau(\ell^\infty,\ell^1)$-continuous under the maintained conditions. Clearly, this U is not representable by an aggregator when w is non-trivial and $v_t=v$ and $\delta(t)=(1-\delta)\delta^{t-1}$.[41]

III. *Impatience, Discounting and Myopia*

An impatient consumer or planner prefers earlier rather than later consumption. The question of discounting versus non-discounting of future consumption as a property of a planner's preference order has been a central theme in capital theory dating to the seminal paper of Ramsey (1928). He argued (p. 543) that discounting was a "practice which is ethically indefensible and arises merely from the weakness of the imagination." It should be recalled that Ramsey also investigated the implications of discounting in his model. Indeed, his heterogeneous agent model operated with different agents distinguished by differences in their subjective discount rates. Ramsey seemed to distinguish the property of discounting for a social planner from the presumption of discounting on the part of private

[40] Epstein defines the marginal rate of substitution in terms of the Gateaux derivatives of U in each coordinate direction. We return to this approach in Part III on impatience.

[41] The factor $(1-\delta)$ arises in order for $\{\delta(t)\}$ to satisfy the normalization $\sum\delta(t)=1$.

agents. Ramsey's view of impatience for consumers was in tune with classical perspectives in capital theory. Various writers [e.g. Böhm-Bawerk (1912), Fisher (1930), and Rae (1934)] advanced the impatience hypothesis. Modern research workers have distinguished several forms of impatience.[42] The terms discounting, time perspective, and myopia have been used in slightly different senses in the literature.

The infinite horizon structure of the choice problem raises problems regarding the presence, degree and forms taken by impatience. We will focus on three aspects of these questions. We discuss the linkage between continuity of the preference order and myopia in Section One. In Section Two, we review Koopmans' notions of impatience and time perspective, following this with a discussion of marginal impatience along a consumption profile in Section Three. Section Four concludes this part with a brief discussion of myopia and the properties of support prices for optimal allocations in an exchange economy.

1. Myopia and the Continuity Axiom

The basic intuition for the link between continuity and impatience may be seen by looking at the definition of continuity for a utility function $U:\mathbb{R}_+^\infty \to \mathbb{R}$ where $\mathbb{R}^\infty$ has the product topology. The function U is continuous in the product topology at $C \in \mathbb{R}_+^\infty$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that the relation $C' \in N(C,\delta)$ implies $|U(C') - U(C)| < \epsilon$. Here $C' \in N(C,\delta)$ means there are $t_1, t_2, \ldots, t_k$ such that $|c'_{t_k} - c_{t_k}| < \delta$ $(k=1,\ldots,K)$. The choice $K=1$ is allowed so U is continuous at C if $C^n \to C$ coordinatewise implies $U(C^n) \to U(C)$. If U is continuous at C in this topology, then U is not sensitive to variations in consumption c_t for t sufficiently large. This is a strong impatience idea: utility is sensitive to changes over finite segments of the planning horizon. For t sufficuently large, the variations in consumption are "discounted" to yield no significant incremental contribution to utility. Total utility is dominated by what happens in only a finite number of periods.

Continuity of U (or the underlying preference order) in the product topology on $\mathscr{C}^+ = \mathbb{R}_+^\infty$ has important economic consequences. We recall Diamond's (1965) Impossibility Theorem. A utility U is *equitable* if for each $C, C' \in \mathscr{C}^+, U(C) \geq U(C')$ if and only if $U(\Pi C) \geq U(\Pi C')$ where Π is the permutation operator mapping $\mathscr{C}$ into $\mathscr{C}$ acting on finitely many components of a vector. Diamond proved that there did not exist an equitable and strictly monotone utility function that is continuous in the product topology. In other words, equity is incompatible with product continuity.[43] You cannot treat all periods equally if you have product continuous preferences. Epstein (1987b) argues that the correct interpretation of Diamond's impossibility result is that the choice of the product topology has strong ethical significance given it precludes the possibility of an equitable preference order.

Svensson (1980) gives a disconnected metric topology for $A(\omega)$ with $\omega = (1,1,\ldots)$ and exhibits a preference ordering that is continuous in it. This preference order is monotonic and equitable. His ordering is based on a generalization of the overtaking criterion, and cannot be represented by a utility function. Campbell (1985) also explored the equity question by introducing a stronger topology than the product topology. His aim was to con-

[42] See Epstein (1987b) for an excellent discussion of impatience.

[43] If (U2) is employed instead of (U2′), then the maximum functional is an equitable utility function. However, it is not lower semi-continuous in the product topology.

struct a topology suitable for application of the classical Weierstrass Theorem promising the existence of maximal elements of a utility function which is upper semi-continuous over a compact constraint set. His topology is a metric topology but it does not turn the commodity space into a topological vector space. He also demonstrates an impossibility theorem: a preference relation $\succsim$ satisfying (U1) is continuous in Campbell's topology if and only if $X \sim Y$ for all programs X, Y. In his setup, continuity is inconsistent with *any* form of the monotonicity axiom.

Diamond's Impossibility Theorem was one of the first indications that the continuity axiom on $\succsim$ in the discrete time finite horizon case carried behavioral implications when translated to the infinite horizon setting. Mathematically speaking, the problem is that the topologies utilized in intertemporal analysis are not identical when there is an open-ended horizon. A finite dimensional vector space admits only one (up to equivalence) Hausdorff linear topology whereas the sequence spaces under consideration here admit several Hausdorff linear topologies. For instance, in the case of ℓ^∞, convergence of a sequence in the sup norm topology implies convergence in the Mackey topology and Mackey convergence implies convergence in the relative product topology inherited from $\mathbb{R}^\infty$. The converse is false: there exist sequences convergent in the product topology which are not convergent in the Mackey topology. Similarly, there exist Mackey convergent sequences which are not convergent in the supnorm topology. Economically speaking, the continuity axiom (U4) takes on a different meaning depending on the choice of a topology for a given commodity space. In the ℓ^∞ case, a product continuous $\succsim$ is Mackey continuous and a Mackey continuous $\succsim$ is sup norm continuous. As before, the converse implications are false. In the product topology case, only finitely many periods really count in determining whether or not two sequences are close to one another. In the Mackey case, there are restrictions on infinitely many coordinates in order to test if two sequences are close to one another.[44]

Bewley (1972) suggested an explicit link between the Mackey topology and impatience. Brown and Lewis (1981), Stroyan (1983), and Raut (1986) formalized myopia concepts as requirements. Their ideas were later subsumed in a general framework offered by Aliprantis, Brown and Burkinshaw (1989). They define myopia in terms of the order structure of the commodity space. We will pursue their approach in order to connect it to preference orders and utility functions typically encountered in capital theory.

Brown and Lewis (1981) focused on the space ℓ^∞. They call $\succsim$ *strongly myopic* if for all X, X', and Y, Y', and $Z \in \mathscr{E}^+$, $X > Y$ implies $X > Y + (\pi^N 0, S^N Z)$ for all N sufficiently large.[45] In words, if Z is pushed far enough into the future, adding it to Y does not change the preference for X over Y. This type of myopia follows from continuity in the topology. However, there is also an order theoretic property that is hidden in the definition of a strongly myopic preference relation. The sequence of consumption programs $\{(\pi^N 0, S^N Z)\}_{N=1}^{\infty}$ is decreasing: $\{(\pi^N 0, S^N)\} \downarrow 0$. The sequence $Y^N = (Y + (\pi^N 0, S^N Z))$ is also a decreasing sequence: $Y^N \downarrow Y$. It follows that $|Y^N - Y| = |(\pi^N 0, S^N Z)| \downarrow 0$ as $N \to \infty$. But this is an ex-

[44] A program Y is in a Mackey neighborhood of X if there is an $\epsilon > 0$ and $\Gamma^k = \{\gamma_{t_k}\}_{t=1}^{\infty} \in c_0$ with $k = 1, \ldots, K$ such that $\sup_k \{\sup_t |\gamma_{t_k}(y_{t_k} - x_{t_k})|\} \leq \epsilon$.

[45] Brown and Lewis (1981) also studied a form of *weak myopia* where the vector Z is a constant sequence in the strong myopia definition.

ample of an order convergent sequence. Brown and Lewis' strong myopia idea can be recast as stating $X \succ Y$ implies $X \succ Y^N$ for N sufficiently large when Y^N is order convergent to Y (written $Y^N \xrightarrow{o} Y$).[46]

Sawyer (1988) considers *upward* and *downward myopia.* Upward myopia coincides with Brown and Lewis' definition of strong myopia. Downward myopia occurs whenever $X \succ Y$ implies for all Z there exists an N such that $(\pi^N X, S^N Z) \succ Y$. Downward myopia says that if Z is pushed for enough into the future, then eventually switching plans from X to Z does not change the preference for X over Y. In particular, if Z offers a lower consumption than X in the distant future, then the preference for X over Y is not reversed since the reductions in consumption are sufficiently postponed. Downward myopia also implies the truncation condition proposed by Prescott and Lucas (1972, p. 417). Their condition follows from downward myopia if $Z=0$. The maximum order clearly fails to satisfy the downward myopia hypotheses of either Sawyer or of Prescott and Lucas. Sawyer calls a preference order *fully myopic* if it is both upward and downward myopic. Clearly downward myopia also contains an order convergent property along the same lines as the Brown and Lewis strong myopia condition. Sawyer ultimately rejects the downward myopia property on grounds that it is implausible. He argues that downward myopia implies *all* future consumption beyond some date would be exchanged for an arbitrarily small first period consumption followed by no consumption into the indefinite future.

The fundamental insight of Aliprantis, Brown and Burkinshaw (1989) is to take order continuity of a utility function as the defining characteristic of myopia. The advantage of this approach is to free myopia from direct topological considerations by basing it solely on the lattice structure of the commodity space.

A utility function U is *order convergent* whenever a net $X^\alpha \xrightarrow{o} X$ in $\mathscr{E}_+$ implies $U(X^\alpha) \xrightarrow{o} U(X)$. An order convergent utility function is said to be a *myopic utility function.* An order convergent utility function is taken as the abstraction of the myopia properties introduced by Brown and Lewis (1981) and their followers. We say U is *τ-myopic* if U is τ-continuous. In general, there exist myopic utility functions which are not τ-myopic on a space E and there are τ-myopic utility functions which are not myopic.[47]

The topology τ for a Riesz dual system is *order continuous* if τ is locally solid and $X^\alpha \xrightarrow{o} X$ implies $X^\alpha \xrightarrow{\tau} X$. Symmetric Riesz dual systems form an important class of spaces with an order continuous topology.[48] Another important example of an order continuous topology arises in the case of an order continuous Fréchet lattice. These are spaces which are Fréchet lattices and have an order continuous topology. A space $\mathscr{E}$ is a *Frechet lattice* if it is a complete materizable locally-convex solid Riesz space. The space $\mathbb{R}^\infty$ is a Feéchet lattice endowed with the d_F-metric; the dual pair $\langle \mathbb{R}^\infty, c_{00} \rangle$ is a symmetric Riesz space and hence the d_F-metric induces as an order continuous topology.

REMARK. If $U: \mathscr{E}_+ \to \mathbb{R}$ is τ-myopic and τ is an order continuous topology, then U is myopic.

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[46] Formally, a net $\{X^\alpha\}$ in a Riesz space E is *order convergent* to some element X, denoted $X^\alpha \xrightarrow{o} X$, whenever there exists another net $\{Y^\alpha\}$ with the same indexed set such that $Y^\alpha \downarrow 0$ and $|X^\alpha - X| \le Y^\alpha$ holds for each α.

[47] Aliprantis, Brown and Burkinshaw (1989, pp. 121–122).

[48] See Aliprantis, Brown and Burkinshaw (1989, p. 102).

PROPOSITION 2. *If $\mathscr{E}$ is a Frechet lattice and U is myopic, then U is τ-myopic.*

PROOF. In a Fréchet lattice every τ-convergent sequence has an order convergent subsequence.[49] $\square$

The following Corollary is an obvious application of the above results.

COROLLARY. *A utility function $U:\mathbb{R}_+^\infty \to \mathbb{R}$ is myopic if and only if it iis d_F-myopic.*

The Corollary implies that the maximim utility function is not myopic on $\mathbb{R}_+^\infty$ since it is not lower semicontinuous. This agrees with economic intuition. Another application yields a corollary to Diamond's Impossibility Theorem: there does not exist an equitable and strictly monotone myopic utility function on $\mathbb{R}_+^\infty$.[50]

Myopic utility functions enjoy a strong continuity property on A_ω.

PROPOSITION 3. *If $U:\mathbb{R}_+^\infty \to \mathbb{R}$ is a myopic utility function, then U is $\|\cdot\|_\infty$-myopic on A_ω.*[51]

PROOF. Let $\omega \in \mathbb{R}_+^\infty$, and let $\{Y^n\}$ be a sequence contained in A_ω such that $\|Y^n - Y\|_\infty \to 0$. Put

$$\epsilon_n = \sup \{\|Y^i - Y\|_\infty : i \geq n\}.$$

Notice that $\epsilon_n \downarrow 0$ and that $|Y^n - Y| \leq \epsilon_n \omega$ for all n. Since $\epsilon_n \omega \downarrow 0$, it follows that $Y^n \xrightarrow{o} Y$, and so by the order continuity of U, we see that U is $\|\cdot\|_\infty$-continuous at Y. $\square$

Proposition 2 implies that every myopic utility function on ℓ_+^∞ is sup norm continuous. A myopic utility function on ℓ_+^∞ implies the underlying preference order satisfies the strong myopia condition proposed by brown and Lewis (1981). The definition of myopia requires that $U(X^\alpha) \xrightarrow{o} U(X)$ for any net $\{X^\alpha\} \xrightarrow{o} X$. The Brown and Lewis strong myopia property only demands order convergence for a specially chosen sequence. A similar comment applies to Streufert's biconvergence criterion for utility functions on $[0,\omega]$. It is clear that myopia implies biconvergence. The truth of the converse implication is open.[52]

2. *Impatience and Time Perspective*

Koopmans (1960) introduced a formal notion of impatience. Given a recursive utility function U with aggregator W and felicity u, a program C meets the *impatience condition* if $u(c_1) > u(c_2)$ implies

$$W(u(c_1), W(u(c_2), U(S^2C))) > W(u(c_2), W(u(c_1), U(S^2C))).$$

Reversing the timing of first and second period felicity from consumption lowers lifetime utility if it places the second (smaller) felicity in the first period. This is impatience over

[49] Aliprantis, Brown and Burkinshaw (1989, pp. 121 and 125).

[50] It seems reasonable to conjecture on the basis of the maximin example that there are no equitable and monotonic myopic utility functions.

[51] This is adapted from Aliprantis, Brown and Burkinshaw (1989, p. 122).

[52] Streufert (1990, p. 83) argues that biconvergence of U is equivalent to product continuity on the space $[0,\omega_1] \times [0,\omega_2] \times \cdots$ where each factor space has the discrete topology.

one period; it can be easily extended to any initial segment of the horizon. Koopmans' definition of impatience is a form of *eventual impatience* since changes in consumption levels over a finite number of periods are reflected in the condition.

The standard TAS form of the utility function satisfies this impatience property as $\delta \in (0,1)$. Koopmans went on to demonstrate that the postulates (U1)–(U4) and (K1)–(K3) imply the existence of "zones of impatience" in the three dimensional payoff space $(u_1, u_2, U \circ S^2)$, where $U \circ S^2(C) = U(S^2C)$. Koopmans found that the limit of the utility of a sequence of programs defined by shifting an arbitrary reference program and the repeated insertion of a fixed N-period consumption segment equals the utility of the program consisting of the infinite repetition of the N-period consumption vector. Koopmans (1960, p. 115) expressed surprise that his notion of impatience arose as an implication of his axiom system since the presumption in the literature dating back at least to Böhm-Bawerk was that impatience was a psychological characteristic of economic agents.[53]

Koopmans, Diamond, and Williamson (1964) explored another notion of time preference which they called *time perspective*. In words, a recursive utility function exhibits weak (strong) time perspective if the difference in the utility levels achieved by two programs does not increase (decrease) if the programs are delayed one period and a common first period consumption is inserted. The use of utility differences in the definition meant that this was a cardinal property of utility whereas the impatience concept was ordinal. However, they did demonstrate the existence of an ordinally equivalent representation of U satisfying the axioms (U1)–(U4) and (K1)–(K3), labelled U^*, such that U^* exhibited the weak impatience property. Sawyer (1988) also investigated impatience properties of the utility function along the lines initiated by Koopmans. He showed the existence of a class of stationary recursive utility functions which are not downwardly myopic but nevertheless exhibited zones of impatience analogous to those found by Koopmans. Streufert (1990) also drew an analogy between time perspective and biconvergence: the utility levels realized in the future from following the paths ω and 0 respectively appear to the observing agent at the beginning of the horizon as though they converge as time passes. Time perspective becomes the economic analog of tunnel vision.

3. *The Norm of Marginal Impatience Conditions*

Any two distinct TAS utility functions which are d_F-continuous on $\mathbb{R}_+^\infty$ are myopic by the Corollary to Proposition 1. Suppose U_1 and U_2 are TAS functions with identical felicity functions but have $\delta_1 > \delta_2$. Both have identical myopia properties but the first has a higher discount factor than the second. Intuition suggests that U_1 discounts the future less than U_2. Put differently, U_2 is more impatient than U_1. The Norm of Marginal Impatience was introduced by Becker, Boyd and Foias (1991) as a refinement of the myopia idea. They were motivated to consider this sharper notion of impatience in order to demonstrate an equilibrium existence theorem for a model with heterogeneous agents having utility functions drawn from the recursive class as well as allowing some non-recursive elements. We present two additional axioms below in order to develop the norm of mar-

[53] Recall, the utility function $\|\cdot\|_\infty$-continuous in this setup.

ginal impatience. The axioms are placed directly on the utility function. For simplicity, we only consider the case $U:\mathbb{R}_+^\infty \to \mathbb{R}$ for U d_F-continuous. We note that a utility function satisfying (U1)–(U4) is quasiconcave. For the remainder of this section we use the stronger monotonicity axiom (U2′) without further mention.

CONCAVITY.

(U5) U is a concave function.

One implication of property (U5) is that the left- and right-hand partial derivatives of the utility function exist. These derivatives are denoted by $U_t^-(C)$ and $U_t^+(C)$ respectively. The right-hand directional derivative of U at C in the direction of $E_t=(0,0,\dots,0,1,0,\dots)$ where 1 is in the t^{th} place, is defined as

$$U_t^+(C)=\lim_{\epsilon\to 0+}\frac{U(C+\epsilon E_t)-U(C)}{\epsilon}.$$

The left-hand partial derivative is defined by substituting $\epsilon\to 0^-$ in the limit. The concavity of U implies $U_t^+(C)\leq U_t^-(C)$. If equality holds, we write $U_t(C)$ for the common value and call this the partial derivative of U at C with respect to the t^{th} coordinate. For technical reason, we also require the following axiom:

DIFFERENTIABILITY.

(U6) The partial derivative U_t of U exists for every t.

We start developing the Norm of Marginal Impatience by fixing a reference program ω which is strictly positive. The order interval $[0,\omega]$ plays a crucial role in the following. We view $[0,\omega]$ as the relevant domain of U in the sense that $[0,\omega]$ strictly contains all feasible allocations. We assume that U is a recursive utility function. If U has a C^1 aggregator, then U_t exists and is found by the formula.

$$U_t(C)=W_2(c_1,\ U(S^1C))W_2(c_2,\ U(S^2C))\dots W_2(c_{t-1},\ U(S^{t-1}C))W_1(c_t,\ U(S^tC)), \qquad (1)$$

where W_1 and W_2 are the partials of W with respect to the first and second coordinates.

The next condition restricts the marginal rates of impatience between adjacent time periods over a portion of the program space. Given $t,t+1$, we define the *marginal rate of impatience at* C, $R_{t,t+1}(C)$, by the relation

$$1+R_{t,t+1}(C)=U_t(C)/U_{t+1}(C).$$ [54]

This definition yields the usual measure of the marginal rate of substitution in adjacent periods along a fixed utility contour.

In the differentiable aggregator case we use (1) to obtain

$$\frac{U_t}{U_{t+1}}=\frac{W_1(c_t,\ U(S^tC))}{W_2(c_t,U(S^tC))W_1(c_{t+1},\ U(S^{t+1}C))}$$

[54] The concavity and strict monotonicity properties of u imply $u_t>0$. Notice that the Rawlsian utility function $v(C)=\inf\{c_t: t=1,2,\dots\}$ violates the strict monotonicity axiom and v_t can be 0.

Note that $R_{t,t+1}(C)=R_{1,2}(S^{t-1}C)$. We will denote $R_{1,2}$ by R. Since S^2C only affects $R_{1,2}$ through $U(S^2C)$, R can alternatively be regarded as a function of c_1,c_2 ,and U_3. The specific condition we impose on utility is given below.

BOUNDED NORM OF MARGINAL IMPATIENCE CONDITION. *There is a* $\delta\in(0,1)$ *such that* $1/\delta=\sup_{t=1,2...}\{1+R_{t,t+1}(C)\colon C\in\Pi_{\tau=1}^{\infty}[0,\omega_t],\ c_{t+1}=c_t\}$.

The supremun of the $1+R_{t,t+1}$, which depends only on the ordinal properties of U, is called the *norm of marginal impatience.* We require this to be uniformly bounded on a subset of the program space containing, in particular, all feasible consumption programs. The TAS case is easily seen to satisfy this condition; the norm of marginal impatience is the reciprocal of the discount factor. The norm of marginal impatience restricts the marginal rate of substitution in a different way than properness. Typically, proper preferences cannot satisfy the Inada condition at 0 whereas this may occur with a bounded norm of marginal impatience.

Many aggregators also satisfy $0<\underline{\delta}\leq W_2\leq\bar{\delta}<1$, a strong version of Koopmans time perspective axiom. In this case, we also have

$$\frac{U_t}{U_{t+1}}\leq\frac{W_t(z,\,U(S^tC))}{\underline{\delta}\,W_1(z,\,U(S^{t+1}C))}. \tag{2}$$

when $c_t=c_{t+1}=z$. If C is a constant program, then the marginal rate of impatience at C is bounded from above by $1/\underline{\delta}$. But other sequences are admitted in the Bounded Norm of Marginal Impatience Condition. Suppose ω is a constant sequence with $\omega_t=w$. The ratio on the right hand side of (2) can blow up only as $z\to 0^+$ since $z\leq w$. This does not happen if there is a number M such that

$$\lim_{z\to 0+}\frac{W_1(z,\,y)}{W_1(z,\,y')}\leq M. \tag{3}$$

for y and y' in the range of the corresponding utility function U. The commonly used aggregators satisfy (3). For the TAS class, $M=1$ will do. The EH utility function and corresponding aggregator satisfies (3). Consequently, the EH utility function satisfies the Bounded Norm of Marginal Impatience Condition.[55] Thus two EH utility functions may be consistent with the same myopia property and possess different norms of marginal impatience.

One implication of the Bounded Norm of Marginal Impatience Condition is recorded below. This result says that the rate of marginal impatience is not increasing over a portion of the program space.

PROPOSITION 4. *Let u be a utility function satisfying the Bounded Norm of Marginal Impatience Condition. Then*

$$\sup_{t=1,2...}\ \{1+R_{t,t+1}(C)\colon C\in\prod_{\tau=1}^{\infty}[0,\,\omega_t],\ c_{t+1}\leq c_t\}=1/\delta.$$

[55] In Part IV, the aggregator is taken as primitive. Many other aggregators satisfy (3). For example, the KDW aggregator will satisfy this restriction.

PROOF. Let $C \in [0,\omega]$ with $c_{t+1} \leq c_t$. Consider $\varphi(x,y)=u(c_1,c_2,\ldots,c_{t-1},x,y,c_{t+2},\ldots)$. Notice there exists $\alpha \in [c_{t+1},c_t]$ such that $\varphi(\alpha,\alpha)=u(C)=\varphi(c_t,c_{t+1})$. Introduce the indifference curve $y=\psi(x)$ such that $\varphi(x,\psi(x))=u(C)$. It is easy to show $-\psi'(x)=\varphi_x(x,y)/\varphi_y(x,y)\equiv\theta(x,y)$ for $y=\psi(x)$ and thus $\theta(x,\psi(x))$ is nonincreasing in x. For $\alpha=\psi(\alpha)$ and $c_t \geq \alpha$ we have $\theta(c_t,c_{t+1})=\theta(c_t,\psi(c_t))\leq\theta(\alpha,\psi(\alpha))\leq 1/\delta$. $\square$

Steady-state impatience may be defined by considering the marginal rate of substitution along constant programs. We defer discussion of steady-state impatience to Part V on optimal growth. There, we explore the connection between steady-state impatience and stability of optimal paths.

4. *Myopia and Support Prices*

Consider the commodity-price duality (ℓ^∞, ba) where ℓ^∞ has the sup norm topology. As noted in Part II, a linear functional on this space may take the form of a pure charge (e.g. a Banach limit). Countable additive elements of ba have ℓ^1 representations denoted by P. The value of a commodity X is (X,Π). A natural question in equilibrium analysis and welfare economics is when does a price system in ba have an ℓ^1 representation?[56] There are clear indications in the literature that some form of myopia and the possible representation of prices by elements of ℓ^1_+ are related properties. For example, Prescott and Lucas (1972) as well as Brown and Lewis (1981) introduced their myopia hypotheses in order to solve this problem.[57] We will illustrate the way in which this problem arises in an example of an exchange economy developed by Becker (1991b). The pasic model is originally due to Peleg and Yaari (1970). We exploit the β-myopia of the utility functions of consumers in the sample economy to derive ℓ^1_+ price supports for a weak Pareto optimal allocation.

The economy is defined by the triple $\{\langle A_\omega,A'_\omega\rangle,\succsim_i,\omega\}$, where $\succsim_i$ is the preference relation of consumer $i\,(i=1,\ldots,m)$ and ω is the social endowment vector. We assume $\omega=(\alpha,\alpha^2,\ldots,\alpha^t,\ldots)$ and $\alpha\geq 1$. The space A_ω is the pirncipal ideal generated by ω. The commodity-price duality is specified by the Riesz dual system (A_ω,A'_ω) where A'_ω is the α-norm dual of A_ω. The maintained assumptions on preference orders are (U1), (U2′), (U3) and (U4). An allocation is a nonnegative m-vector $(X_1,\ldots,X_m)$ in the commodity space satisfying $\sum_{i=1}^m X_i\leq\omega$. A Pareto optimal allocation has the usual meaning.

PROPOSITION 5. *Let $\{\langle A_\omega,A'_\omega\rangle,\succsim_i,\omega\}$ be an exchange economy where each agent has a β-myopic utility representation of $\succsim_i$. If $(X_1,\ldots,X_m)$ is a Pareto optimal allocation and $\beta>\alpha$, then there is a $P\in\ell^1_+$ such that for each i*

$$\sum_{t=1}^{\infty} p_t z_t/\beta^t \geq \sum_{t=1}^{\infty} p_t x_{i_t}/\beta^t \quad \text{for all} \quad Z\succsim_i X_i.$$

PROOF. The maintained assumptions on preferences imply each agents preference order

[56] Is there an economic interpretation of price systems which are pure charges? See Gilles (1989) and LeRoy (1989) for an affirmative answer.

[57] There is a voluminous literature on the price representation problem dating back to early work on intertemporal efficiency. Radner (1967) seems to have been the first to raise the question.

is uniformly α-norm proper (Proposition 2.1). Mas-Colell's Supporting Price Theorem[58] implies there is a price system P in ba such that $(Z,P) \geq (X_i,P)$ for all $Z \succsim_i X_i$. Since each agent has strictly monotone preferences, $P > 0$ and ω is extremely desirable for all i, it follows that P is β-myopic on $[0,\omega]$ and order continuous on A_ω [see Aliprantis, Brown and Burkinshaw (1989, p. 147]. Therefore, since the β-norm dual is isomorphic to ℓ^1 as noted in Part II, P has a weighted ℓ^1 representation. □

Many recursive utility functions are β-myopic, so Proposition 4 applies to exchange economies with those preferences. Characterizing support properties of Pareto optimal allocations in heterogenous agent economies with capital accumulation (both with a maximum sustainable stock and sustainable growth) for recursive utility maximizing agents would seem to be a natural follow-up problem for investigation.

IV. *The Aggregator Approach to Recursive Utility*

In Part II, we saw that recursive preferences give rise to an aggregator function that combines present consumption (or felicity from present consumption) and future utility to obtain present utility. This chapter takes that aggregator as a primitive.

In fact, there is a pre-Koopmans literature on recursive utility that uses the aggregator exclusively. An early example is Fisher (1930). Much of Fisher's analysis is carried out using a 2-good model. Utility depends on both current and future income. Early in the book, he explains that income is ideally thought of in utility terms, thus we should really think of current felicity and future utility combining to yield overall utility. This is precisely what the aggregator function does. Hayek (1941) also took the aggregator as a primitive, and even addressed stability issues in this framework.

The first modern paper to take the aggregator as primitive was Lucas and Stokey (1984). They started with an aggregator, and showed how a recursive utility function could be constructed from an aggregator function W, under the assumption that W was bounded. They then used this to characterize equilibria and examine stability when consumers have recursive preferences.

Taking the aggregator as fundamental provides detailed information about preferences in a compact form. First, it is a lot easier to specify an aggregator than a recursive utility function. Koopmans, Diamond and Williamson (1964) found an aggregator that had a specific property (increasing marginal impatience), but the corresponding utility function cannot be explicitly computed. It does not have a closed form expression. Second, the aggregator, with its sharp distinction between current and future consumption, often makes it easier to incorporate hypotheses about intertemporal behavior. It can be quite difficult to translate axioms into usable conditions on the utility function. The normality conditions used by Lucas and Stokey (1984), Benhabib, Jafarey and Nishimura (1988), Benhabib, Majumdar and Nishimura (1987) and Jafarey (1988) to study equilibrium dynamics are most eaisly imposed directly on the aggregator.[59] Finally, if we impose behavioral con-

[58] Mas-Colell (1986, p. 1048).

[59] Epstein (1987a) has discovered conditions on the utility function that imply a similar normality condition in models with continuous-time recursive utility.

ditions as axioms, there is the question of their consistency. With aggregators, this is never a problem. Once the utility function exists, consistency is automatic.

Of course, the use of the aggregator does partially obscure the actual utility function and its properties. Fortunately, the aggregator usually contains all the information required to construct the utility function. Lucas and Stokey (1984) made the aggregator approach feasible when they showed that the utility function could be reconstructed when the aggregator is bounded. Boyd (1990) introduced a refinement of the Contraction Mapping Theorem, the Weighted Contraction Theorem, which applies to a much broader class of utility functions that includes many standard examples. For many aggregators, this is enough to recover the utility function. Aggregators that allow $-\infty$ as a value require further treatment. Boyd combined the weighted contraction with a "partial sum" technique to construct the utility functions.

We will follow Boyd's (1990) treatment to find the utility function. In Section One, we examine the basic properties we require of the aggregator. Section Two gives a general existence and uniqueness theorem for the corresponding utility function when the aggregator is bounded below. Section Three illustrates the use of this theorem, and Section Four employs Boyd's "partial sum" technique to obtain existence for general aggregators.

1. Basic Properties of the Aggregator

As in the preceding parts, we assume that there is a single all-purpose good available in each time period for simplicity. The aggregator maps $\mathbf{X}\times\mathbf{Y}$ to $\mathbf{Y}$, where $\mathbf{X}$ is a subsee of $\mathbb{R}_+=\{x\in\mathbb{R}: x\geq 0\}$ and $\mathbf{Y}$ is a subset of $\mathbb{R}$. Aggregators will appear in the second argument, so W must take values in $\mathbf{Y}$. Recall the projection π and shift S are given by $\pi C=c_1$ and $SC=(c_2,c_3,\ldots)$ for $C\in\mathbb{R}^\infty$. The key property that makes a utility function U recursive is that $U(C)=W(\pi C,U(SC))$. Intuitively, we can find U by recursively substituting it in this equation. This substitution is performed by the recursion operator T_W defined by $(T_WU)(C)=W(\pi C,U(SC))$. Thus $(T_W^N0)(C)=W(c_1,W(c_2,\ldots,W(c_N,0))\ldots))$. The recursive utility function is the unique fixed point of T_W.

The most familiar aggregator is $W(c,y)=u(c)+\delta y$, which yields the additively separable utility function $U(C)=\sum_{t=1}^\infty \delta^{t-1}u(c_t)$. Obviously, $U(C)=u(c_1)+\delta U(SC)$. Other aggregators include the KDW (Koopmans, Diamond and Williamson, 1964) aggregator $W(c,y)=(1/\theta)\log(1+\beta c^r+\delta y)$, and modified Uzawa (1968) aggregator $W(c,y)=(-1+y)\exp[-u(c)]$ used by Epstein and Hynes (1983). This last aggregator yields the utility function[60]

$$U(C)=-\sum_{t=1}^{\infty}\exp\left[-\Sigma_{\tau=1}^{t}u(c_\tau)\right]$$

from Part II since

$$\left(-1-\sum_{t=2}^{\infty}\exp\left[-\Sigma_{\tau=2}^{t}u(c_\tau)\right]\right)\exp\left[-u(c_1)\right]=-\sum_{t=1}^{\infty}\exp\left[-\Sigma_{\tau=1}^{t}u(c_\tau)\right].$$

This form particularly intriguing since consumption only affects discounting, but

[60] Epstein and Hynes actually work in continuous time, but this is obviously a discrete version of their utility function.

does not seem to yield direct utility. Epstein (1983) considers a discrete-time formulation that permits uncertainty. His generalized Uzawa aggregator is $W(c,y)=(v(c)+y)e^{-u(c)}$. The Epstein-Hynes form is the special case $v(c)=-1$.

Without loss of generality, we may assume $0\in\mathbf{Y}$. In fact, if there is a $y\in\mathbf{Y}$ with $W(0,y)=y$, we may even assume $W(0,0)=0$. If $W(0,y)\neq y$, then $U(0)=y$. Now consider the utility function $\hat{U}(C)=U(C)-U(0)$. The adjusted aggregator, $\hat{W}(c,y)=W(c,y+U(0))-U(0)$ yields this utility function since $\hat{W}(c_1,\hat{U}(SC))=W(c_1,U(SC))-U(0)=U(C)-U(0)=\hat{U}(C)$. Both aggregators generate equivalent utility functions, and $\hat{W}(0,0)=0$.

When applied to the Epstein-Hynes (EH) aggregator, this yields

$$U(0)=W(0,\,U(0))=(-1+U(0))e^{-u(0)},$$

so $U(0)=1/(1-e^{u(0)})$. The adjusted aggregator is then $\hat{W}(c,y)=[y-e^{u(0)}/(1-e^{u(0)})]e^{-u(c)}-1/(1-e^{u(0)})$.

$$\hat{U}(C)=\sum_{t=1}^{\infty}e^{-tu(0)}\left[1-\exp\left(-\sum_{\tau=1}^{t}[u(c_\tau)-u(0)]\right)\right],$$

which is the discounted sum of functions depending on past consumption. Note the contrast with the original form of the utility function where consumption seemed to only affect discounting. This form also shows us that even though recursive utility is forward-looking, the functional form may superficially appear to be backward-looking.

AGGREGATOR. A function $W:\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ is an aggregator if:

(W1) W is continuous on $\mathbf{X}\times\mathbf{Y}$ and increasing in both c and y.

(W2) W obeys a Lipschitz condition of order one, i.e., there exists $\delta>0$ such that $|W(c,y)-W(c,y')|\leq\delta[y-y'|$ for all c in $\mathbf{X}$ and y,y' in $\mathbf{Y}$.

(W3) $(T_W^N y)(C)$ is concave in C for all N and all constants $y\in\mathbf{Y}$.

When W is differentiable the Lipschitz bound in (W2) is $\delta=\sup W_2(c,y)$. This uniformly bounded time perspective is similar to the time perspective studied axiomatically by Koopmans (1960) and Koopmans, Diamond and Williamson (1964). It insures that future utility is discounted by at least δ. In the additively separable case, W_2 is the discount factor. In the EH case, the fact that W is increasing in c implies $u'\geq 0$ since $W\leq 0$. The Lipschitz bound then becomes $e^{u(0)}$. We do not yet impose $\delta<1$ since we may want to consider undiscounted or even upcounted models.

The sole purpose of condition (W3) is to guarantee concavity of the utility function. It is not required for the existence results. Curiously, the aggregator need not be jointly concave in c and y for the associated utility function to be concave. Although the EH aggregator is not concave, the corresponding utility function $U(C)=-\sum_{t=1}^{\infty}\exp[-\sum_{\tau=1}^{t}u(c_\tau)]$ is concave. Epstein previously (1983) gave sufficient conditions for the concavity of generalized Uzawa utility functions. In the EH case, $u''<0$ is sufficient. More generally, when the utility function is the limit of the functions $T_W^N(0)(C)$, (W3) insures concavity is inherited by U. Thus,

LEMMA 3. *Suppose (W3) holds and $T_W^N S(0)(C)\to U(C)$. Then U is concave on its domain. If, in addition, W is strictly concave in c and strictly increasing in y, then U is strictly concave.*

Conversely, if U is concave, condition (W3) holds for all y in the range of U.

2. *The Existence of Recursive Utility*

When trying to construct the utility function, the first problem we confront is what domain to use. Obviously, the utility function will live on a subset of $\mathbb{R}_+^\infty$. The question is, which subset? Since one of the motivations for studying recursive utility is to admit non-degenerate equilibria, we must use subsets that are appropriate for equilibrium problems—linear spaces.[61]

Even in the additively separable case, it is unreasonable to expect the utility function to be defined on all of $\mathbb{R}^\infty$. Consider the additively separable aggregator $\sqrt{c}+\delta y$ where $\delta<1$. The utility function only makes sense when $\sum_{t=1}^{\infty}\delta^{t-1}\sqrt{c_t}$ converges. This will not happen for all vectors in $\mathbb{R}_+^\infty$. For example, the sum does not converge when $c_t=\delta^{-2t}$. This is where the weighted ℓ^∞ spaces come in. In this case, the utility function will only exist on $\ell_+^\infty(\beta)$ for $\beta<\delta^{-2}$. Our strategy will be to find a β so that the utility function exists and is β-continuous on $\ell_+^\infty(\beta)$.

Let $\mathbf{A}\subset\mathbb{R}^\infty$ with $\pi(\cup_{N=0}^{\infty}S^N\mathbf{A})\subset\mathbf{X}$.[62] Both the shift S and projection π are continuous in any topology on $\mathbf{A}$ that is stronger than the relative product topology, as are the β-too-ologies. Given a positive function φ, continuous on $\mathbf{A}$, let $\mathscr{C}$ be the space of continuous functions from $\mathbf{A}$ to $\mathbf{Y}$, and $\mathscr{C}_\varphi$ be the corresponding space of φ-bounded functions.[63] Since all the functions involved are continuous, $T_W:\mathscr{C}_\varphi\to\mathscr{C}$.

CONTINUOUS EXISTENCE THEOREM. *Suppose the topology on $\mathbf{A}$ is stronger than the relative product topology, $W:\mathbf{X}\times\mathbf{Y}\to\mathbf{Y}$ obeys (W1) and (W2), φ is continuous, $W(\pi C,0)$ is φ-bounded, and $\delta||\varphi\circ S||_\varphi<1$. Then there exists a unique $U\in\mathscr{C}_\varphi$ such that $W(\pi C,U(SC))=U(C)$. Moreover, $(T_W^N0)(C)\to U(C)$ in $\mathscr{C}_\varphi$.*

PROOF. Since W is increasing in y, the recursion operator T_W is increasing. Now

$$|T_W(0)|/\varphi(C)=|W(c_1,0)|/\varphi(C)<\infty$$

because $W(\pi C,0)$ is φ-bounded. Finally,

$$\begin{aligned}T_W(\xi+A\varphi)&=W(c_1,\xi(SC)+A\varphi(SC))\\&\leq W(c_1,\xi(SC))+A\delta\varphi(SC)\leq T_W\xi+A\delta||\varphi\circ S||_\varphi\varphi(C).\end{aligned}$$

The Weighted Contraction Theorem, with $\theta=\delta||\varphi\circ S||_\varphi<1$, shows that T_W is a contraction, and has a unique fixed point U.

Now consider $||U(C)-(T_W^N0)(C)||_\varphi\leq\delta^N||U(S^NC)||_\varphi\leq||U||_\varphi(\delta||\varphi\circ S||_\varphi)^N$. As the last term converges to zero, $(T_W^N0)(C)\to U(C)$. □

In fact, the full force of (W1) was not employed in the proof. The aggregator need not be increasing in c for the theorem to hold.

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[61] An alternative, as used by Streufert (1990), is to focus purely on capital accumulation problems. This allows him to further restrict the size of the subsets, and thus expand the range of aggregators he can use.

[62] This insures that the recursion operator always makes sense on $\mathscr{C}(\mathbf{A};\mathbf{Y})$.

[63] See the appendix for details.

3. *Examples with the Aggregator Bounded Below*

The easiest application of the Continuous Existence Theorem is to a bounded aggregator with $\delta<1$ and $\mathbf{A}=\ell_+^\infty(1)$. Take φ as the constant 1, and use the product topology. As in Lucas and Stokey (1984), this yields a recursive utility function that is not only β-myopic for all $\beta\geq 1$, but also continuous in the relative product topology on $\ell_+^\infty(1)$. In partiular, this applies to the EH aggregator with $u(0)>0$.

Another application is to W with $0\leq W(c,0)\leq A(1+c^\eta)$ as in the case where $W(c,0)$ has asymptotic exponent or asymptotic elasticity of marginal felicity [see Brock and Gale, (1969)] less than $\eta>0$ with $\delta\beta^\eta<1$. In this case, take $\mathbf{A}=\ell_+^\infty(\beta)$ and $\varphi(C)+1=||C||_\beta^\eta$. Then $||\varphi\circ S||_\varphi=\beta^\eta$, and the recursive utility function is β-myopic. This applies to the aggregator $W(c,y)=c^\eta+\delta y$. The utility function $\sum_{t=1}^\infty \delta^{t-1}c_t^\eta$ is continuous on each $\ell_+^\infty(\beta)$ for $\delta^\eta\beta<1$.

When $0\leq W(c,0)\leq A(1+\log(1+c))$, a similar argument shows that U is β-myopic for all $\beta<\infty$. Take $\gamma>0$ such take $\delta(\gamma+\log\beta)/\gamma<1$ and set $\varphi(c)=\gamma+\log(1+c)$. Then

$$\begin{aligned}\delta\varphi(||SC||_\beta)&=\delta\gamma+\delta\log(1+||SC||_\beta)\\&\leq\delta\gamma+\delta\log(1+\beta||C||_\beta)\\&\leq\delta\gamma+\delta\log\beta(1+||C||_\beta)\\&\leq\delta\gamma+\delta\log\beta+\delta\log(1+||C||_\beta)\\&\leq\gamma+\delta\log(1+||C||_\beta)\leq\varphi(||C||_\beta).\end{aligned}$$

In fact, when W is concave in $c, W(c,0)\leq W(1,0)+\alpha(c-1)$ for some supergradient α. (If differentiable, $\alpha=W_1(1,0)$.) Thus we may set $\varphi(C)=1+||C||_\beta$ for a β-myopic utility function when the aggregato is concave in c with $\delta\beta<1$.

Relaxing the condition $\beta^\eta\delta<1$ risks losing existence on $\ell_+^\infty(\beta)$. Again, the additively separable case makes this clear. Let $W(c,y)=c^\eta+\delta y$ and take $\beta=\delta^{-1/\eta}$. The utility function cannot be defined when C is given by $c_t=\beta^t$. No utility function can be constructed from the aggregator on $\ell_+^\infty(\beta)$. A smaller space must be used.[64]

4. *Unbounded Aeggrgators*

The Continuous Existence Theorem can also be used indirectly to deal with aggregators that are not bounded below, such as $W(c,y)=\log c+\delta y$. These obey:

(W1′) W is increasing in both c and y, upper semicontinuous on $\mathbf{X}\times\mathbf{Y}$, continuous when $c>0$ and $y>-\infty$ and obeys $W(c,-\infty)=W(0,y)=-\infty$ for all $c\in\mathbf{X}$ and $y\in\mathbf{Y}$.

For aggregators satisfying (W1′), paths that are near 0 can pose problems for the continuous existence theorem. When $W(c,y)=\log c+\delta y$ these problems result in a utility function that ts upper semicontinuous but not lower semncontinuous. However, they are not severe enough to preclude existence of the utility function.

[64] However, Streufert (1987) has discovered cases where $\beta^\eta\delta<1$ and U exists. These seem to require $W_{22}<0$, and may not be continuous on $\ell_+^\infty(\beta)$.

To circumvent the problems posed by paths that are too close to zero, Boyd (1990) considers a region that excludes them as the set **A**. More precisely, choose $\gamma \leq \beta < \infty$, and set ${}_\gamma||C|| = \inf|c_t/\gamma^{t-1}|$ if $0<\gamma$ and ${}_0||C|| = \infty$. Then take $\mathbf{A} = \ell_+^\infty(\beta,\gamma) = \{C \in \mathbb{R}_+^\infty : 0 < {}_\gamma||C||$ and $||C||_\beta < \infty\}$. This is the set of paths that have a growth rate between γ and β. Thus $\ell_+^\infty(\beta,0)$ is just our old friend $\ell_+^\infty(\beta)$.

To make this clear, consider the logarithmic case. For any such path, ${}_\gamma||C||\gamma^{t-1} \leq c_t \leq ||C||_\beta \beta^{t-1}$. Thus

$$\sum_{t=1}^\infty \delta^{t-1}[(t-1)\log\gamma + {}_\gamma||C||] \leq \sum_{t=1}^\infty \delta^{t-1}\log c_t \leq \sum_{t=1}^\infty (t-1)\delta^{t-1}[(t-1)\log\beta + ||C||]_\beta].$$

Since $\sum_{t=1}^\infty \delta^{t-1}\log c_t$ is squeezed between convergent series, it converges. However, the limit need not be continuous since the convergence is not uniform. We can get β-upper semicontinuity. It is enough to show this for each ball $\{C \in \ell_+^\infty(\beta) : ||C||_\beta < \kappa\}$. On this ball, $\sum_{t=1}^T \delta^{t-1}(\log c_t - (t-1)\log\beta - \log\kappa)$ has non-positive terms. Each of the partial sums is upper semicontinuous, and so is the limit. Since the limit differs from the original utility function by a constant, the utility function is upper semicontinuous too. In fact, we have escaped the lower bound on consumption by taking partial sums. Some sequences may even have utility $-\infty$. Nonetheless, the logarithmic case is well-behaved.

We are now forced to admit $-\infty$ as a possible value for utility. This causes some unpleasantness. Amazingly, $U(C) = -\infty$ satisfies the recursion too. The obvious solution is not the only one. Fortunately, we can restrict our attention to $\ell_+^\infty(\beta,\gamma)$ and see that this is not a reasonable solution.

The general case is similar. Intuitively, we expect to obtain the utility function by recursive substitution, as the limit of $(T_W^N u)(C) = W(c_1, W(c_2, \ldots, W(c_N, u)\ldots))$ with u constant. In fact, under appropriate conditions, the Continuous Existence Theorem applies on $\ell_+^\infty(\beta,\gamma)$, yielding a unique φ-bounded utility function Ψ. Of course, the iterates $T_W^N u$ converge to Ψ on $\ell_+^\infty(\beta,\gamma)$.

By using a process analogous to partial summation, Ψ can be extended to a utility function on all of $\ell_+^\infty(\beta)$. This extension is upper semicontinuous and recursive. Further, it is the *only* recursive upper semicontinuous extension of Ψ to $\ell_+^\infty(\beta)$.

UPPER SEMICONTINUOUS EXISTENCE THEOREM. *Suppose $W : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ obeys (W1′), the Lipschitz condition (W2) holds whenever W is finite, and there are increasing functions g and h with $g(c) \leq W(c,0) \leq h(c)$. Set $\varphi(C) = \max\{h(||C||_\beta), -g({}_\gamma||C||)\}$ and suppose $\varphi > 0$ with $\delta||\varphi \circ S||_\varphi < 1$ for some $\beta > \gamma > 0$ with $\beta \geq 1$. Then there exists a unique U that is φ-bounded on $\ell_+^\infty(\beta,\gamma)$, obeys $W(\pi C, U(SC)) = U(C)$ and is β-upper semicontinuous on $\ell_+^\infty(\beta)$.*

PROOF. First, temporarily give $\mathbf{A} = \ell_+^\infty(\beta,\gamma)$ the discrete topology. As all functions are continuous there, and $W(c,0)$ is clearly φ-bounded, the Continuous Existence Theorem applies, yielding a unique φ-bounded recursive utility function $\Psi : \ell_+^\infty(\beta,\gamma) \to \mathbb{R}$.

Second, let Z be an arbitrary element of $\ell_+^\infty(\beta,\gamma)$ and define the "partial sums" on all of $\ell_+^\infty(\beta)$ by

$$\Psi_N(C;Z) = [T_W^N \Psi(S^N Z)](C) = W(c_1, W(c_2, \cdots, W(c_N, \Psi(S^N Z))\ldots)).$$

Now for $Z, Z' \in \ell_+^\infty(\beta)$,

$$\begin{aligned}|\Psi_N(C;Z)-\Psi_N(C;Z')| &\le \delta^N|\Psi(S^N Z)-\Psi(S^N Z')|\\ &\le \delta^N M[\varphi(S^N Z)+\varphi(S^N Z')]\\ &\le M'(\delta\|\varphi\circ S\|_\varphi)^N\end{aligned}$$

for some M'. The first step uses the Lipschitz bound (W2). The second uses the φ-boundedness of Ψ on $\ell_+^\infty(\beta,\gamma)$, and the third uses the fact that $\varphi(S^N Z)\le(\|\varphi\circ S\|_\varphi)^N\varphi(Z)$ for any $Z\in\ell_+^\infty(\beta,\gamma)$. It follows that if $\lim_{N\to\infty}\Psi_N(C;Z)$ exists, it must be independent of Z. Note that for $C\in\ell_+^\infty(\beta,\gamma)$, $\Psi_N(C;C)=\Psi(C)$, so $\lim_{N\to\infty}\Psi^N(C;Z)$ exists on $\ell_+^\infty(\beta,\gamma)$ and is equal to Ψ there.

The third step is to show $U(C)=\lim_{N\to\infty}\Psi_N(C;Z)$ exists and is β-upper semicontinuous on all of $\ell_+^\infty(\beta)$. For κ arbitrary, take $C\in\ell_+^\infty(\beta)$ with $\|C\|_\beta<\kappa$ and set $z_t=\kappa\beta^{t-1}$. Since $c_t\le z_t, \Psi_N(C;Z)$ is a decreasing sequence. Its limit $U(C)$), which is also infimum, must exist. Further, each of the Ψ_N is the composition of non-decreasing β-upper semicontinuous functions, so their infimum $U(C)$ is also β-upper semicontinuous on $\{C: \|C\|_\beta\le\kappa\}$. Since upper semicontinuity is a local property, U is β-upper semicontinuous on all of $\ell_+^\infty(\beta)$.

The next step is to show that U is recursive. If $\pi C=0$ or if $U(SC)=-\infty$, (W1′) implies $W(\pi C,U(SC))=-\infty=U(C)$. Otherwise, we have

$$\begin{aligned}W(\pi C, U(SC)) &= W(\pi C, \lim_{N\to\infty}\Psi_N(SC;SZ))\\ &= \lim_{N\to\infty} W(\pi C, \Psi_N(SC;SZ))\\ &= \lim_{N\to\infty}\Psi_{N+1}(C;Z)=U(C).\end{aligned}$$

Therefore $W(\pi C,U(SC))=U(C)$ for all $C\in\ell_+^\infty(\beta)$.

The last step is uniqueness. Let Φ be a β-upper semicontinuous recursive utility function that is φ-bounded on $\ell_+^\infty(\beta,\gamma)$. Since Ψ is unique, Φ, Ψ and U agree on $\ell_+^\infty(\beta,\gamma)$. When $z_t=\|C\|_\beta\beta^{t-1}, C\le Z$ and so $\Phi(C)\le\Psi_N(C;Z)$. Thus $\Phi(C)\le\lim_{N\to\infty}\Psi_N(C;Z)=U(C)$. If $c_t=0$ for some $t, U(C)=-\infty=\Phi(C)$. If $c_t>0$ for all t, set $z_t=\max\{\gamma^{t-1},c_t\}$ and consider the sequence $C^n=(c_1,\ldots,c_n,z_{n+1},z_{n+2},\ldots)$. By construction, $\Psi(C^n)=\Psi_n(C;Z)$. Since $\gamma<\beta, C^n\to C$ in the β-topology. By upper semicontinuity of Φ, $\Phi(C)\ge\lim_{n+\infty}\Psi_n(C;Z)=U(C)$. It follows that $\Phi(C)=U(C)$, and thus U is the unique such function. □

Aggregators with $-1+\min\{0,\log c\}\le W(c,0)\le a+\log(1+c)$ fall into this framework. Given $\delta<1$ and $\beta\ge1$, the constant a may be assumed large enough that $\delta(a+\log\beta)/a<1$. Take $\gamma=1$ and let $\varphi(C)=\max\{a+\log(1+\|C\|_\beta), 1-\min\{0,\log_1\|C\|\}\}$. As $\delta\varphi(SC)/\varphi(C)\le\delta(a+\log\beta)/a<1$ since ${}_1\|C\|\le{}_1\|SC\|$ and $\|SC\|_\beta\le\beta\|C\|_\beta$, the utility function exists on $\ell_+^\infty(\beta,1)$ for any β. In other cases, upcounting ($\delta>1$) may be allowed. When $-c^\eta\le W(c,0)\le0$ with $\eta<0$, we set $\varphi(C)={}_\gamma\|C\|^\eta$ so $\delta\varphi(SC)/\varphi(C)\le\delta\beta^\eta\le\delta\gamma^\eta<1$. As $\eta<0$, $\beta^\eta<1$ and there are γ that permit $\delta>1$. The Upper Semicontinuous Existence Theorem applies to these examples.

As Boyd notes, the "partial sum" approach works on a wider range of aggregators than considered in the theorem. For example, if there is a function $v(c)$ with $v(c)=W(c,v(c))$, "partial sums" can be defined on $\{C: \|C\|_1<\kappa\}$ by $[T_W^N v(\kappa)](C)$. These form a decreasing sequence, so their limit is an upper semicontinuous function $U(C)$. As $W(c_1,U(SC))=$

$\lim[T_W^{N+1}v(\kappa)](C)=U(C)$, this yields a recursive utility function. This recursive utility function may fail to be lower semicontinuous. One such example is $W(c,y)=-1+e^{-c}y$ so that $v(c)=-1/(1-e^{-c})$ and utility is $U(C)=-\sum_{t=1}^{\infty}\exp(-\sum_{\tau=0}^{t}c_\tau)$. Consideration of the sequence $C^n=(c_1\ldots,c_n,0,\ldots)$, where $c_t=2\log(t+1)/t$, shows that this utility function is not lower semicontinuous since $U(C^n)=-\infty$ but $U(C)>-\infty$. Note that $\delta=1$ in this example.

V. *Properties of Optimal Paths*

Once we have a utility function, we can ask whether optimal paths exist. In the recursive case, the same conditions that guarantee existence of the utility function will also yield optimal paths, and a value function that satisfies Bellman's Equation. Our next task is to characterize these paths via Euler equations and a transversality condition, and then investigate their properties. Are optimal paths monotonic? Do they enjoy a turnpike property?

Section One shows that optimal paths exist, and are continuous in an appropriate topology. Section Two shows how dynamic programming may be used on recursive utility, and that the value function is the unique continuous solution to Bellman's equation. The transversality condition is taken up in Section Three. Monotonicity and the turnpike property are examined in Section Four. We conclude with a brief discussion of equilibrium models and the long-run distribution of income in Section Five.

1. *The Existence and Sensitivity of Optimal Paths*

Existence is quite straightforward. The existence of optimal paths is just one of the useful facts that follow from continuity of the utility function and compactness of the feasible set. When the aggregator defines a continuous utility function, a modern version of Weierstrass' theorem, the Maximum Theorem [see Berge (1963); Klein and Thompson (1984)] can be used to show continuity of optimal paths.[65] For example, when the budget set (and hence the optimal path) depends continuously on a parameter vector ω, the maximizer correspondence $\mathbf{m}(\omega)$ will be continuous.

MAXIMUM THEOREM. *Suppose $\mathbf{B}(\omega)$ is β-lower semicontinuous in ω and β-compact-valued.*

(1) *If U is β-upper semicontinuous, there exists a $C^*\in\mathbf{B}(\omega)$ such that $U(C^*)=\sup\{U(C): C\in\mathbf{B}(\omega)\}$.*

(2) *If U is β-continuous, the value function $J(\omega)=\sup U(\mathbf{B}(\omega))$ is continuous and the maximizer correspondence $\mathbf{m}(\omega)$ is upper semicontinuous. Further, if U is strictly concave, them $\mathbf{m}(\omega)$ is a continuous function of ω.*

This form of the maximum theorem will also demonstrate continuity of the optimal paths and value function. The remainder of this section shows how to employ the β-topologies in the one-sector model. The first step is to show that the feasible set is actually compact. This turns out to be fairly easy, since the β-topology often coincides with the

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[65] Magill and Nishimura (1984) also use the Maximum Theorem to obtain continuous policy functions with recursive utility.

product topology, which is quite easy to work with. In fact, Lemma 1 of section II.4 shows that the β-topology and product topology coincide on any α-bounded set whenever $\alpha<\beta$.

One application is to a one-sector model of optimal capital accumulation (*Ramsey model*). In the classical Ramsey model, the technology is described by a (gross) *production function*. The production function f is a continuous, non-decreasing function $f:\mathbb{R}_+\to\mathbb{R}_+$. Note that $f(0)\geq 0$. In the time-varying Ramsey model, the technology is described by a sequence, $\{f_t\}_{t=1}^{\infty}$, of such production functions. Given this production technology, the set of feasible paths of accumulation from initial stock k (the *production correspondence*) is $\mathbf{F}(k)=\{C\in\mathbb{R}_+^{\infty}:0\leq k_t\leq f_t(k_{t-1}),\ k_0=k\}$. The set of feasible consumption paths (the *consumption correspondence*), $\mathbf{B}(k)$ is $\{C\in\mathbb{R}_+^{\infty}: 0\leq c_t\leq f_t(k_{t-1})-k_t$ for some $K\in\mathbf{F}(k)\}$. Define f^t inductively by $f^1=f_1$ and $f^t=f_t\circ f^{t-1}$. The *path of pure accumulation* is $\{f^t(k)\}_{t=1}^{\infty}$. Both $\mathbf{B}(k)$ and $\mathbf{F}(k)$ are closed in the product topology and $\mathbf{B}(k)\subset\mathbf{F}(k)\subset\prod_{t=1}^{\infty}[0,f^t(k)]$. As this last set is compact by Tychonoff's Theorem, $\mathbf{B}(k)$ is also compact in the product topology.

When $\lim[f^t(k)/\alpha^t]<\infty$, both $\mathbf{F}(k)$ and $\mathbf{B}(k)$ are α-bounded subsets of $\ell_+^{\infty}(\beta)$. More generally, we call the technology α-bounded if $\mathbf{F}(k)$ is α-bounded. This happens in the case of exogeneous technical progress where $f_t(x)=e^{nt}x^{\rho}$. The path of pure accumulation grows at asymptotic rate $\exp\{n/(1-\rho)\}$, so the technology is α-bounded for $\alpha>\exp\{n/(1-\rho)\}$. As any concave production function obeys $f(x)\leq f(a)+\xi(x-a)$ whenever ξ is a supergradient at a (e.g. $\xi=f'(a)$), it is α-bounded for any $\alpha>\xi$. Thus, any stationary, concave, production technology is α-bounded for all $\alpha>f'(\infty)$. Provided U is upper semicontinuous on $\ell_+^{\infty}(\beta)$ for some $\beta>\alpha$, Lemma II.4.1 and the Maximum Theorem combine to show existence of at least one optimal path.[66]

Let's temporarily confine our attention to the case where there is a unique optimal capital-consumption path $\{k_t(k),c_t(k)\}_{t=1}^{\infty}$. This will occur if the production function is concave and the utility function strictly concave. Define the *consumption policy function* $g(k)=c_1(k)$. The policy function gives the optimal consumption level as a function of the previous period's capital stock. The maximum theorem guarantees that g exists and is continuous. There is an associated *capital policy function* $h(k)=f(k)-g(k)$. The optimal paths are then $c_t(k)=g(k_{t-1}(k))$ with $c_1=g(k)$ and $k_t(k)=h(k_{t-1}(k))=f(k_{t-1}(k))-g(k_{t-1}(k))$.

To obtain continuity of the value function and policy functions, it is enough to show that the production correspondence $\mathbf{F}(k)$ is product lower semicontinuous since the set of feasible paths is the continuous image of the production correspondence. For k' near k, $\mathbf{F}(k')\subset\mathbf{F}(k+1)$. Locally, everything takes place in an α-bounded set, and we may use the product topology.

For lower semicontinuity, it is enough to show lower semicontinuity for the basic open sets $\mathscr{S}(Y,\epsilon,N)=\{X\in\mathbb{R}_+^{\infty}: |x_t-y_t|<\epsilon$ for all $t<N\}$. Let ϵ, $N>0$ be given. Take $Y\in\mathbf{F}(k)$. By continuity of the f_t, we can choose δ with $|f^t(k')-f^t(k)|<\epsilon$ for all $t\leq N$ when $|k-k'|<\delta$. For any such k', take the path $x_t=\min\{y_t,f^t(k')\}$. Note that $f^t(k')+\epsilon>f^t(k)\geq$

[66] In fact, we could use the weaker condition that U be continuous on $\ell_+^{\infty}(\Theta)$ with Θ the path of pure accumulation, i.e., $\theta_t=f^t(k)$. Continuity of U can be obtained for a general class of aggregators and production functions by using upper and lower approximations like those used by Streufert (1990). Alternatively, a brute force calculation will also often show $W(c_1,W(c_2,\ldots)\ldots,)$ converges uniformly to a continuous utility function in this case.

y_t for $t<N$, so $y_t \geq x_t > y_t - \epsilon$ for all $t \leq N$. Hence $X \in \mathscr{G}(Y,\epsilon,N)$. Further, $f_{t+1}(x_t) = \min\{f_{t+1}(y_t), f^{t+1}(k')\} \geq x_{t+1}$ and $x_1 \leq f_1(k')$, so $X \in \mathbf{F}(k')$. It follows $\mathbf{F}(k') \cap \mathscr{G}(Y,\epsilon,N) \neq \phi$ whenever $|k-k'|<\delta$, establishing lower semicontinuity.

An immediate application is to demonstrate β-continuity of optimal paths as a function of initial capital stock. One consequence is that $c_t(k)$ is continuous in k for each t. In general, this only holds for $\beta > \alpha$. For $\beta = \alpha$, it can fail even in models with additively separable utility. Amir, Mirman and Perkins (1991) and Dechert and Nishimura (1983), using a non-convex stationary technology, find that optimal paths converge to zero if the initial capital stock is below some critical value. Optimal paths starting above the critical value converge to a steady state that lies above the critical value. They assume a maximum sustainable stock, so $\alpha=1$ will do. The optimal path is not norm ($\alpha=1$) continuous because of the long-run jump as you cross the critical value.

Variations on this are possible. Stronger forms of the maximum theorem allow the utility function to depend on the parameter ω. If the bounds of Part IV hold uniformly in ω, the optimal paths will be continuous in ω.[67] A simple example is an optimal growth model with additively separable utility $W(c,y) = u(c) + \delta y$. Take $(k,\delta) = \omega \in \Omega = \mathbb{R}_+ \times [0,\delta]$ with a strictly concave, bounded u and $\delta < 1$. With a stationary concave production function f, a unique optimal path $\{c_t(k,\delta)\}$ exists. Further, $\{c_t(k,\delta)\}$ is β-continuous, hence $c_t(k,\delta)$ is a continuous function of (k,δ) for all $(k,\delta) \in \Omega$. A non-separable example in a similar vein is the EH form $W(c,y) = [-1 + e^{\delta - u(c)} y]$. When $\delta < u(0)$, this yields a β-continuous utility function for any $\beta > \alpha$.

When the turnpike property holds, β-continuity of optimal paths will imply α-continuity. In fact, if optimal paths starting in some interval of initial stocks converge to the same steady state, α-continuity follows on that interval.

2. *Recursive Dynamic Programming*

The limited separability in recursive utility is sufficient to do dynamic programming. Not surprisingly, the weighted contraction theorem is also useful here. The usual Principle of Optimality applies, yielding Bellman equation $J(k) = \sup\{W(c, J(f(k)-c) : 0 \leq c \leq f(k)\}$.

Define the Bellman operator by

$$(T\xi)(k) = \sup\{W(c, \xi(f(k)-c)) :\ 0 \leq c \leq f(k)\}.$$

When W is continuous on $[0,\infty)$, the maximum theorem shows that the Bellman operator maps continuous functions into continuous functions. Further, the supremum is actually attained for each continuous function ξ. A function solves the Bellman equation if and only if it is a fixed point of the Bellman operator. A contraction mapping argument will now show that the Bellman operator has a unique fixed point, which must be the value function.

Suppose u is continuous on $\mathbb{R}_+$, and let $\varphi > 0$ be increasing and continuous with $W(f(k), 0)/\varphi(k)$ bounded. The Bellman operator is clearly monotone. Further,

$$(T0)(k) = \sup\{W(c,0) :\ 0 \leq c \leq f(k)\} = W(f(k), 0) \leq \varphi(k).$$

[67] Details may be found in Boyd (1986).

Finally,

$$T(\xi+A\varphi)(k)=\sup\{W(c,\xi(f(k)-c))+A\varphi(f(k)-c)\}\leq(T\xi)(k)+A\delta\varphi(f(k))$$

since φ is increasing. Provided that $\delta\sup_x\varphi(f(x))/\varphi(x)<1$, the conditions of the weighted contraction theorem hold. In sum, we have the following proposition:

PROPOSITION 6. *Suppose $W(\cdot,0)$ is continuous on $\mathbb{R}_+$ and there is an increasing continuous $\varphi>0$ with $\theta=\delta\sup_x[\varphi(f(x))/\varphi(x)]<1$ and $W(f(x),0)/\varphi(x)$ bounded. Then the Bellman equation has a unique continuous solution.*

The fact that T is a contraction actually gives more information. Consider $\xi_n(k)=T^n(0)(k)$. Then $\|\xi_n-TJ\|_\varphi\leq\theta\|\xi_{n-1}-J\|_\varphi$. By induction, we obtain $\|\xi_n-J\|_\varphi\leq\theta^n\|\xi_0-J\|_\varphi=\theta^n\|J\|_\varphi$ since $\xi_0=0$. Thus $\xi_n\to J$ in $\mathscr{C}_\varphi$. This fact allows us to numerically approximate the value function to any desired degree of accuracy.

A class of models covered by proposition 1 are those where $f(x)\leq\alpha+\beta x$ with $\beta>1$ and $u(c)=c^\eta$ for $0<\eta\leq1$. Set $\varphi(x)=\lambda+x^\eta$ where λ obeys $1+\alpha^\eta/\lambda\leq\beta^\eta$. Then $\varphi(f(x))\leq\lambda+(\alpha+\beta x)^\eta\leq\lambda+\alpha^\eta+\beta^\eta x^\eta\leq\beta^\eta(\lambda+x^\eta)=\beta^\eta\varphi(x)$. The Bellman equation has a unique solution provided $\beta^\eta\delta<1$.

One example is the case where $u(c)=c^\eta$ and $f(k)=\beta k$ with $0<\eta\leq1$ and $\beta^\eta\delta<1$. This satisfies the hypothesis of Proposition 1. The value function has the form Ak^η. The constant A can be determined by substituting this functional form in the Bellman equation, and solving for A. The fact that Ak^η solves the Bellman equation verifies that it is the value function since Proposition 1 guarantees that solutions to the Bellman equation are unique.

Streufert (1990) provides an alternative to contraction mapping methods. He considers the case where there are best and worst paths. These yield upper and lower partial sums. He considers the case where they both converge to the recursive utility function (biconvergence). He shows that the value function is the unique admissible solution to the Bellman equation, where admissibility rules out certain obviously absurd functions.

3. *Characterization of Optimal Paths*

We call an optimal path (C^*,K^*) *regular* if $c_t^*>0,k_t^*>0$ for all t. For simplicity, this section focuses on regular optimal paths. The analogous results for non-regular paths may be found in Boyd (1990).

Optimal paths for the Ramsey model are characterized in this section. A useful envelope theorem and the Euler equations are developed first. We then proceed to the main result that the Euler equations, together with the transversality condition, completely characterize optimal paths for a large class of aggregators.

The following assumptions will be maintained throughout this section. The utility function U obeys $U(0)=0$ and is concave and φ-bounded on $\ell_+^\infty(\beta)$ for some φ with $\|\varphi\circ S\|_\varphi<1/\delta$. In addition, the feasible set $\mathbf{B}$ is generated by an α-bounded technology for some $\alpha<\beta$ given by a sequence continuous, concave, increasing production functions $\{f_t\}$ with $f_t(0)=0$. As a consequence, the theorems of the previous sections apply. The value function $J(y)$ is defined and continuous in initial income $y=f_1(k)$. When U is differentiable with respect to consumption at time t, denote $\partial U/\partial c_t$ by U_t. Except as noted, assume U is differentiable at each time.

ENVELOPE THEOREM. *The value function J is non-decreasing and concave. If U is differentiable with respect to consumption in period 1, and optimal paths are regular, then J is differentiable and obeys $dJ(y)/dy=U_1(C)$ where C is any optimal path from y.*

PROOF. The value function is increasing since the feasible set grows when the initial stock increases. Concavity follows since U is concave and $\alpha\mathbf{B}(k)+(1-\alpha)\mathbf{B}(k')\subset\mathbf{B}(\alpha k+(1-\alpha)k')$ for $0\leq\alpha\leq 1$.

Differentiability is established as follows.[68] Let $h>0$, $H=(h,0,\dots)$, and let C be an optimal path with initial income y so that $J(y)=U(C)$. Clearly, $J(y+h)\geq U(C+H)$ and thus $J(y+h)-J(y)\geq U(C+H)-U(C)$. Dividing by h and taking the limit shows that the right-hand derivative $J'(y+)$ satisfies $J'(y+)\geq U_1(C)$. Since C is regular, c_1 is non-zero. We may then repeat this with $-c_1<h<0$, to show $J'(y-)\leq U_1(C)\leq J'(y+)$. As J is concave, $J'(y+)\leq J'(y-)$, thus $J'(y)=U_1(C)$. □

COROLLARY. *Suppose U is recursive, the aggregator is differentiable, and optimal paths are regular. Then $dJ(y)/dy=W_1(c_1,U(SC))$ where C is any optimal path from y.*

Henceforth, assume that U is differentiable at each time t, and that the optimal path is regular. Now let C^* be optimal and let K^* be the associated sequence of capital stocks. Set $\mathbf{B}_N=\{K\in\mathbf{B}: k_N\geq k_N^*\}$. Let $V_N(K)=U(f_1(k_0)-k_1,\dots,f_N(k_{N-1})-k_N,f_{N+1}(k_N)-k_{N+1}^*, f_{N+2}(k_{N+1}^*)-k_{N+2}^*,\dots)$. By the Principle of Optimality, K^* solves the problem of maximizing V_N over $\mathbf{B}_N$. Setting the derivative with respect to k_t equal to zero for $t=1,\dots,N-1$, we obtain the necessary conditions

$$U_{t+1}(C^*)f'_{t+1}(k_t^*)-U_t(C^*)=0.$$

These are referred to as the *Euler Equations.*

Since $U_t(C)/U_{t+1}(C)=1+R_{t,t+1}(C)$, we can rewrite the Euler equations as:

$$f'_{t+1}(k_t^*)=1+R_{t,t+1}(C^*)=1+R(S^{t-1}C^*).$$

I.e., the net marginal product of capital is equal to the marginal rate of impatience. In the additively separable case $1+R_{t,t+1}(C)=u'(c_t)/\delta u'(c_{t+1})$, so these reduce to the usual Euler equations.

The Euler equations, are instrumental in proving the Transversality Theorem.

TRANSVERSALITY THEOREM. *Suppose U is recursive and differentiable at each time. A regular path C^* is optimal if and only if the Euler equations hold and $k_t^*U_t(C^*)\to 0$ as $t\to\infty$ (the Transversality Condition).*

PROOF. Suppose C^* is optimal. As above, the optimal path must satisfy the Euler equations. Note that $k_t^*>0$ for all t by regularity. Let $y_t^*=f_t(k_{t-1}^*)$ denote the income stream associated with the optimal path C^* and J_t denote the value function at time t with $c_t^*>0$. Since $J_t(0)=0$, and J_t is concave, $J_t(y)\geq yJ_t'(y)$ for all $y\geq 0$. Setting $y=y_t^*$ yields

$$k_t^*W_1(c_t^*,U(S^tC^*))\leq y_t^*W_1(c_t^*,U(S^tC^*))\leq J_t(y_t^*). \tag{1}$$

[68] This method is adapted from Mirman and Zilcha (1975).

Now $J_t(y_t)=U(S^{t-1}C^*)$. Multiplying through by δ^{t-1} and using the Euler equations yields

$$0\leq k_t^* U_t(C^*)\leq \delta^{t-1}U(S^{-1}C^*).$$

Combining the φ-boundedness of U with $\delta\|\varphi\circ S\|_\varphi<1$ shows $k_t^* U_t(C^*)\to 0$ along any subsequence with $c_t>0$. The sufficiency of the transversality condition is implied by Lemma 2 since U is continuous. □

LEMMA 4. *Suppose U is concave and product lower semicontinuous on the feasible set. Then a path K^* is optimal if it satisfies the Euler equations and the transversality condition is satisfied.*

PROOF. Consider an arbitrary feasible path K with associated C. Define an approximate utility function $\mathscr{U}_N$ by $\mathscr{U}_N(C)=U(c_1,\ldots,c_N,c_{N+1}^*,c_{N+2}^*,\ldots)$ where C^* is the consumption path corresponding to K^*. Since U and f are concave, we have

$$\mathscr{U}_N(C)-U(C^*)\leq \sum_{t=1}^{N-1}(\partial U/\partial k_t)[k_t-k_t^*]-U_N(k_N-k_N^*)$$

Now $\partial U/\partial k_t=0$ by the Euler equations. Thus $\mathscr{U}_N(C)-U(C^*)\leq -U_N(k_N-k_N^*)\leq U_N k_N^*$. Letting $N\to\infty$ and using the transversality condition shows lim sup $\mathscr{U}_N(C)\leq U(C^*)$. By lower semicontinuity of U, $U(C)\leq$ lim sup $\mathscr{U}_N(C)\leq U(C^*)$ for all feasible C. Therefore C^* is optimal. □

If U is not differentiable, a similar result could be obtained by using supergradients instead of derivatives. In fact, Malinvaud's (1953) sufficiency proof doesn't even need recursivity.

4. *Monotonicity, Stability and Turnpikes*

Beals and Koopmans (1969) have given conditions where a convex technology would yield monotonic optimal paths in the one-sector model. A necessary and sufficient condition for monotonicity is not known with more general technologies, although progress has been made by Benhabib, Majumdar and Nishimura (1987). More is known about the additively separable case, where Dechert and Nishimura (1983) carried out an analysis of monotonicity in a reduced form model.

In this section we assume the feasible set $\mathbf{B}(k)$ is α-bounded and convex, that U is strictly concave, satisfies the Inada conditions, and is β-continuous for some $\beta>\alpha$. Optimal paths are then unique, and the policy correspondencestare functions. The Inada conditions imply that the optimal path is strictly positive for $k>0$. Recall that the Euler equations are $1+R(S^{t-1}C)=f'(k_t)$ where $R=R_{1,2}$.

MONOTONICITY THEOREM. *Suppose $\partial R/\partial c_1<0$. For any initial stock k, $k_t(k)$ is a strictly increasing function of k and the optimal path is strictly monotonic.*

PROOF. Let $k<k'$, and let K and K be optimal from k and k', respectively. Suppose $k_1=k_1'$. Consider the path k'' defined by $k_0''=k_0$ and $k_t''=k_t'$ for $t=1,2,\ldots$. This path is optimal

by the Principle of Optimality. Further, $c_1''=f(k)-k_1''<f(k')-k_1'=c_1'$ and $c_2''=f(k_1'')-k_2''=c_2'$. Thus $c_t'=c_t''$ for $t=2,3,\ldots$. The Euler equations yield $1+R(c_1'',c_2'',\ldots)=f'(k_1'')=f'(k_1')=1+R(c_1',c_2',\ldots)$. Since $c_t'=c_t''$ for $t=2,3,\ldots,R(c_1'',c_2'',\ldots)=R(c_1',c_2'',\ldots)$. But this is impossible since R is decreasing in c_1 and $c_1''<c_1'$. Thus $k_1\neq k_1'$.

Now suppose $k_1>k_1'$. Since $k_1(0)=0<k_1'<k_1(k)$, and $k_1(k)$ is continuous, there is a k'' with $0<k''<k$ and $k_1(k'')=k_1'$. This is impossible by the preceding argument. Therefore k_1 is strictly increasing. Since $k_t(k)$ is the t^{th} iterate of k_1, it too is strictly increasing. Now if $k_1<k$, $k_2=k_1(k_1)<k_1(k)=k_1$. Iteration shows $k>k_1>k_2>\ldots$. The case $k_1>k$ is similar. □

The condition on the rate of impatience says that, all other things equal, we become more patient (the rate of impatience decreases) when current consumption rises. This seems quite intuitive, and holds in the addivitely separable case where $1+R(C)=u'(c_1)/\delta u'(c_2)$ and $u''<0$. In fact it holds whenever $W_{12}\geq 0$. This condition on the aggregator is not necessary for a decreasing rate of impatience, since the EH aggregator has $W_{12}=-u'(c)e^{-u(c)}<0$. Yet $1+R(C)=u'(c_1)(-1+U(SC)/[u'(c_2)(-1+U(S^2C))]$ is decreasing in c_1 since $u''<0$. We obtain a turnpike result for these cases. Optimal paths either converge to a steady state, or to ∞.

The next question of interest is stability of the steady states. Define the *steady-state rate of impatience*, ρ by $\rho(c)=R(C_{con})$ where $C_{con}=(c,c,\ldots)$. This is the marginal rate of impatience, evaluated along the constant path C_{con}. Of course, $\rho>0$.[69] For convenience, define $\Phi(c)$ to be the utility of the constant path C_{con}. The rate of impatience is then $\rho(c)=1/W_2(c,\Phi(c))-1$. With additively separable preferences ($W(c,U)=u(c)+\delta U$) this reduces to the usual rate of impatience $\rho=\delta^{-1}-1$. Epstein's generalized Uzawa aggregator $W(c,U)=(v(c)+U)e^{-u(c)}$ has $\Phi(c)=v(c)/(e^{u(c)}-1)$ and $\rho(c)=e^{u(c)}-1$. This exhibits *increasing steady-state impatience* ($\rho'>0$), as does the KDW aggregator where $\rho(c)=\theta e^{\theta\Phi(c)}/\delta-1$.

Initial stocks can be divided into three disjoint sets. Let $\mathscr{T}^0=\{k: k=0 \text{ or } f'(k)=1+\rho(f(k)-k)\}$, $\mathscr{T}^+=\{k: f'(k)>1+\rho(f(k)-k)\}$ and $\mathscr{T}^-=\{k: f'(k)<1+\rho(f(k)-k)\}$. For $k\in\mathscr{T}^0$, the Euler equations and transversality condition are clearly satisfied by the stationary path $k_t=k$. Thus every element of $\mathscr{T}^0$ is a steady state. The Euler equations also show that all steady states are in $\mathscr{T}^0$. Accumulation is definitely possible in $\mathscr{T}^+$ since $f'(k)>1+\rho(f(k)-k)>1$.

One way to think about $1+\rho(f(k)-k)$ is as the long-run (steady-state) supply price of capital. The marginal product f' gives the long-run demand price. Thus long-run demand lies above supply in $\mathscr{T}^+$ and below supply in $\mathscr{T}^-$. Intuitively, the quantity should rise in the long-run in $\mathscr{T}^+$, and fall in $\mathscr{T}^-$. That this intuition is correct is the content of the turnpike theorem below.

Since both $\mathscr{T}^+$ and $\mathscr{T}^-$ are open, they are the countable union of open intervals. The end points of these intervals must be in $\mathscr{T}^0$. Now label the endpoints $\bar{k}_i$ such that $\bar{k}_i<\bar{k}_{i+1}$. If $k\in(\bar{k}_t,\bar{k}_{t+1})$, the optimal path cannot cross the steady states at the endpoints, so $k_t\in(k_i,k_{i+1})$. Further, since k_t is monotonic, it must converge to some $\bar{k}$. Taking

[69] Buckholtz and Hartwick (1989) consider a generalized Uzawa aggregator with $v(c)=a[e^{u(c)}-1]$, which is not increasing in c. The constant function $U(C)=a$ is the only utility function satisfying the recursion. Obviously $\rho=0/0$ is then undefined. Buckholtz and Hartwick reject this interpretation, and argue for the use of an overtaking criterion, which has a zero rate of impatience.

the limit in the Euler equations shows $f'(\bar{k})=1+\rho(f(\bar{k})-\bar{k})$. The optimal path converges to one of the endpoints. Similarly, if k is greater than all of the steady states it either converges to the largest steady state, or to ∞. The next theorem shows that $k_t \to \bar{k}_{i+1}$ when $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathscr{T}^+$ and $k_t \to \bar{k}_i$ when $k \in (\bar{k}_i, \bar{k}_{i+1}) \subset \mathscr{T}^-$. However, we need a preliminary lemma before proceeding to the turnpike result.

NON-OPTIMALITY LEMMA. *Suppose $k \in \mathscr{T}^+$ ($k \in \mathscr{T}^-$) and $k_t \le k$ ($k_t \ge k$) for $t<n$ with $k_t = k$ for $t \ge n$. Then $U(C) \le \Phi(f(k)-k)$ and K is not optimal.*

First suppose $k \in \mathscr{T}^+$ and let $\Psi(k)=\Phi(f(k)-k)$. That $U(C) \le \Psi(k)$ is trivial for $n=1$. We proceed by induction. Suppose $U(C) \le \Psi(k)$ when $n=m \ge 1$ and consider a path K with $k_t \le k$ and $k_t = k$ for $t \ge m+1$. If $k_m = k, U(C) < \Psi(k)$ by the induction hypothesis, so we may suppose $k_m \le k$.

Consider the path K' defined by $k'_t = k$ for $t \ne m$ and $k'_m = k+\delta$. Obviously $f'(k)>1$, so this path will be feasible from k for $\delta>0$ small enough. Taking a Taylor expansion shows $U(C')-\Psi(k)=W_1(W_2)^{m-1}[W_2 f'-1]\delta+o(\delta)\delta$ where all derivatives are evaluated at k. Since $1+\rho(f(k)-k)=1/W_2(k,\Psi(k))<f'(k)$, δ may be chosen small enough that $U(C')>\Psi(k)$. Note that remaining at k cannot be optimal.

Now take λ. $0<\lambda<1$ with $\lambda(k+\delta)+(1-\lambda)k_m=k$. Then $K''=\lambda K'+(1-\lambda)K$ satisfies the hypotheses of the lemma for $n=m$, so $U(C'') \le \Psi(k)$ by the induction hypothesis. Now $\Psi(k) \ge U(C'') \ge \lambda U(C')+(1-\lambda)U(C) > \lambda\Psi(k)+(1-\delta)U(C)$. Thus $\Psi(k)>U(C)$. The inequality holds for all n by induction. Further, since the stationary path $k_t=k$ is feasible and not optimal, K cannot optimal.

The case of $k \in \mathscr{T}^-$ is similar. □

TURNPIKE THEOREM. *Suppose $\partial R/\partial c_1<0$. The optimal path from k is stationary if $k \in \mathscr{T}^0$, increasing if $k \in \mathscr{T}^+$ and decreasing if $k \in \mathscr{T}^-$.*

PROOF. Consider the case where $k \in \mathscr{T}^+$. We know that k_t is strictly monotonic. Suppose $k_t \downarrow k'$. Take a sequence of feasible paths K^ν such that $K^\nu \to K$ in the product topology with $k_t^\nu \le k$ for all t and $k_t^\nu = k$ for large t. (This is possible since $f'>1$ on $[k_t^\nu, k]$.) Then $U(K^\nu) \le \Phi(f(k)-k)$ by the Non-Optimality Lemma. Since U is product continuous on the feasible set, $U(K) \le \Phi(f(k)-k)$, contracting the fact that K is optimal.

The case $k \in \mathscr{T}^-$ is similar, except that the optimal path may simply be truncated to obtain the desired K^ν. □

Benhabib, Majumdar and Nishimura (1987) examine long-run dynamics in two-sector models. They find monotonic convergence to a steady state under a normality condition and a condition on factor intensity, and oscillation if either condition is reversed.

5. *Long-Run Income Distribution in Dynamic Economies*

One big contrast between general recursive utility and additively separable models comes when we examine dynamic equilibrium models. In the additively separable case, the most patient agent(s) end up with all of the capital, while relatively impatient agents use all of their labor income to service their debt. This is obviously absurd since the impatient agents would not be able to survive, much less pay the interest on their debt.

This occurs since the long run capital supply is perfectly elastic at the (fixed) steady-state rate of impatience. If the interest rate is above the rate of impatience, agents will lend as much as they can. If the interest rate is below the rate of impatience, agents will borrow as much as possible. The long-run equilibrium thus has the interest rate set at the most patient agent's rate of impatience, and all others borrow as much as possible.

With recursive utility, the steadt-state rate of impatience varies depending on long-run consumption. All agents can have the same rate of impatience in the steady state. The consumption levels are non-zero, but vary across agents depending on their respective rates of impatience. If the steady-state rate of impatience is increasing in steady state consumption, the more patient individuals consume more (and have higher wealth) is the steady state.

To see this, consider the case where each individual earns wages w and faces a fixed interest rate r. Steady-state consumption is $w+rk$ is net savings. Consider the case of two agents, with $\rho^1(c)<\rho^2(c)$ for all c (agent one is more patient). In steady-state equilibrium, $\rho^1(c_1)=\rho^2(c_2)>\rho^1(c_2)$. With ρ^1 increasing in c, $w+rk_1=c_1>c_2=w+rk_2$ thus $k_1>k_2$. Curiously, if the rate of impatience were decreasing in consumption, the more patient individual would own *less* capital.[70]

For these considerations to be relevant, we also need a stability result. The equilibrium must converge to the steady state. The stability of recursive dynamic equilibrium has been investigated in a number of papers. The simplest case is a representative agent economy. In that case, the equivalence principle holds [Becker and Majumdar (1989)]. The equilibrium problem is equivalent to a planner's problem with the same preferences as the representative agent. Since the solution to the planner's problem is stable, so is the equilibrium.

The heterogeneous agent recursive case was first rigorously examined by Lucas and Stokey (1984), who considered a two-agent, one-good exchange economy. They assumed that both current consumption and future utility were normal in the sense that $W_1(c,y)/W_2(c,y)$ is decreasing in c and increasing in y. They also required an increasing steady-state rate of impatience.[71] Jafarey (1988) has found that decreasing impatience insures instability, and that more generally, stability depends on the relative rates of impatience at zero and at the endowment. In the increasing impatience case, this last condition merely insures there is an interior steady-state equilibrium.

Both set up a dynamic programming problem that generates the equilibrium. The idea is that any equilibrium is Pareto optimal, and so solves a social planner's problem for some set of weights. This is then recast in a dynamic programming framework. This results in some complication, but can be done. To see the complication, consider the additively separable case where $U_i(C_i)=\sum_{1=t}^{\infty}\delta_i^{t-1}u_i(c_{it})$ for $i=1,2$. The planner's objective is $\lambda_1U(C_1)+\lambda_2U(C_2)$. This objective can be rewritten $\lambda_1u_1(c_1)+\lambda_2u_2(c_2)+\lambda_1\delta_1U_1(SC_1)+\lambda_2\delta_2U_2(SC_2)$. The weights on future utility are different from the weights on current felicity. This problem can be circumvented by explicitly including future utility in the planner's objective, as detailed in Lucas and Stokey. The planner's problem then contains the needed

[70] Boyd (1986) examines such cases in a Ramsey equilibrium framework.

[71] Hayek (1941) was the first to point out the importance of increasing impatience for stability, although his argument does not meet modern-day standards of rigor.

information on stability.

Benhabib, Jafarey and Nishimura (1988) study the long-run behavior of production economies with heterogeneous agents. They again set up the planner's dynamic programming problem, but then use a linearization to study stability. As in the single agent case, increasing marginal impatience combines with a normality condition to yield stability.

A comprehensive analysis of the planner's problem has been carried out in a series of papers by Dana and Le Van (1989, 1990, 1991). They find (1990a, b) that a similar programming problem can be set up in a general model with many agents and many goods. They then obtain the Euler equations, and examine the uniqueness and stability of steady states of the planner's problem. The other paper (1989) examines equilibria corresponding to initial endowments. This amounts to picking the correct social welfare function. They obtain detailed information about this mapping of endowments into weights.

VI. *Conclusion*

The upshot of all this is that many of the results and techniques we take for granted in the additively separable model carry over to recursive utility. Although we concentrated on one-sector models, many of these methods have applications to multi-sector models. Koopmans' original results on representation were shown in a multi-sector framework. The weighted contraction technique also applies to multi-sector models. Just replace absolute values by $\mathbb{R}^n$ norms, and work in subsets of $(\mathbb{R}^n)^\infty$. Existence of optimal paths and continuity of policy functions easily follows by the Maximum Theorem. Similarly, the characterization via Euler equations and transversality condition is easily extended. Of course, the stability results of Benhabib et al. and Dana and Le Van are already in a multi-sector framework, although the statement of necessary conditions for stability may get quite complex.

Throughout the paper, we have focused on the theoretical aspects of recursive utility. In case the reader is wondering about empirical work on the subject, we close by mentioning the paper by Zin (1987). He finds empirical support for recursive preferences exhibiting increasing impatience in United States macroeconomic data.

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Appendix: The Weighted Contraction Theorem

Let $f\in\mathscr{C}(\mathbf{A},\mathbf{B})$, the space of continuous functions from $\mathbf{A}$ to $\mathbf{B}$. Suppose $\varphi\in\mathscr{C}(\mathbf{A};\mathbf{B})$ with $\mathbf{B}\subset\mathbb{R}$ and $\varphi>0$. A function f is *φ-bounded* if the *φ-norm* of f, $\|f\|_\varphi=\sup\{|f(x)|/\varphi(x)\}$ is finite. The same isometry trick we used on $\ell^\infty(\beta)$ shows that $\mathscr{C}_\varphi(\mathbf{A};\mathbf{B})=\{f\in\mathscr{C}(\mathbf{A};\mathbf{B}):f$ is φ-bounded$\}$ is a Banach space under the φ-norm. Just set $(Vf)(x)=f(x)/\varphi(x)$. In particular, $\mathscr{C}_\varphi(\mathbf{A};\mathbf{B})$ is a complete metric space. Recall that a transformation $T:\mathscr{C}_\varphi\to\mathscr{C}_\varphi$ is a *strict contraction* if $\|Tx-Ty\|_\varphi<\theta\|x-y\|_\varphi$ with $\theta<1$. For such T, we have:

CONTRACTION MAPPING THEOREM. *A strict contraction on a complete metric space has a unique fixed point.*

The proof is well-known, and can be found in various standard references [e.g., Reed and Simon (1972), Smart (1974)].

In applications, the main problem is to show that T is a strict contraction. An easy way to do this is by using monotonicity properties, as is common in dynamic programming. In the weighted contraction context, this yields the following form of the theorem.

WEIGHTED CONTRACTION MAPPING THEOREM (MONOTONE FORM). *Let* $T: \mathscr{C}_\varphi \to \mathscr{C}$ *such that*

(1) *T is non-decreasing ($\xi \leq \psi$ implies $T\xi \leq T\psi$).*

(2) $T(0) \in \mathscr{C}_\varphi$.

(3) *$T(\xi + A\varphi) \leq T\xi + A\theta\varphi$ for some constant $\theta < 1$ and all $A > 0$.*

Then T has a unique fixed point.

PROOF. For all $\xi, \psi \in \mathscr{C}_\varphi, |\xi - \psi| \leq ||\xi - \psi||_\varphi \varphi$. So, $\xi \leq \psi + ||\xi - \psi||_\varphi \varphi$ and $\psi \leq \xi + ||\xi - \psi||_\varphi \varphi$. Properties (1) and (3) yield $T\xi \leq T\psi + \theta||\xi - \psi||_\varphi \varphi$ and $T\psi \leq T\xi + \theta||\xi - \psi||_\varphi \varphi$. Thus $||T\xi - T\psi||_\varphi \leq \theta||\xi - \psi||_\varphi$.

Setting $\psi = 0$, we have $||T\xi - T(0)||_\varphi \leq \theta||\xi||_\varphi$, and so $||T\xi||_\varphi \leq \theta||\xi||_\varphi + ||T(0)||_\varphi < \infty$ by property (2). Hence $T: \mathscr{C}_\varphi \to \mathscr{C}_\varphi$. As $\theta < 1$, T is a strict contraction on $\mathscr{C}_\varphi$. By the contraction mapping theorem, it has a unique fixed point. □

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