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ON INCENTIVES COMPATIBILITY AND CONSTRAINED OPTIMALITY OF INCOMPLETE MARKET EQUILIBRIA*

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I. *Introduction*

Competitive equilibria in models of sequential trade under uncertainty will not typically be Pareto optimal when markets are incomplete due to insufficient asset structure. This is not very surprising since the test of Pareto optimality allows an imaginary planner to reallocate goods for which there is no market. A more interesting question is to ask if there is a room for Pareto improvement when the planner is allowed to intervene only in the incomplete asset markets, leaving complete spot market prices free to adjust. Surprisingly, even when the planner's ability is constrained in this way, Stiglitz (1982) observed that Pareto improvement is still possible with some non-generic exceptions. Later, Geanakoplos Polemarchakis (1986) formalized Stiglitz' idea and showed that, generically in preferences and endowments, every rational expectation equilibrium can be Pareto improved upon by interventions confined to the asset markets.

In this paper, I restrict the planner's ability even further to see if competitive equilibria are optimal or efficient in some weaker sense. I assume that the planner cannot directly observe the characteristics of consumers, so the intervention must be self-selecting; that is to say, I require the intervention to be anonymous, or incentive compatible as the reallocation of asset must be acceptable for each consumer, which will be an additional constraint for the planner.

More precisely, there are a continuum of consumers divided into finitely many types in the model. The planner knows the statistical information of these types, but he cannot observe the types directly. The consumers trade assets in the beginning of the first period, and then the planner intervenes by showing a list of asset portfolios, which I will call a proposal, from which each consumer chooses a portfolio freely. That is, the allocation of assets will be incentive compatible, or perhaps more correctly, anonymous. In the next period, the state of economy is resolved and spot markets for consumption goods open, where a consumer can spend the return of his final portfolio, i.e., the portfolio acquired in the initial trading plus the portfolio selected from the planner's list.

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A proposal is called *anonymous constrained feasible* if the chosen portfolios sum up to zero. An equilibrium allocation is *anonymous constrained optimal* if there is no anonymous feasible proposal that leads to a Pareto superior allocation of consumption goods. This paper asks the question of whether one should expect an incomplete market equilibrium to be anonymous constrained optimal.

It turns out that there are robust examples of economies whose equilibria are anonymous constrained optimal and also there are robust examples of economies with no anonymous constrained optimal equilibrium. So, constrained suboptimality is not a generic property of competitive equilibria if the extra constraint of anonymity is imposed. Although the assumption of no learning of the planner is somewhat against the spirit of rational expectation, this paper suggests that the optimality issue of incomplete market equilibria is subtle if incentive problem is taken into account.

The assumption of unobservable characteristics is compatible with the basic idea behind optimal taxation theory, or second-best theory.[1] One can interpret the planner in the model of this paper as a government which can intervene in asset markets by using a general nonlinear tax schedule. So, an interpretation of the result in this context is that the incomplete competitive market sometimes outperforms the government.

Of course, one can consider different anonymity constratints by changing the way consumers anticipate the intervention and the timing of the intervention. I refer reader to Kajii (1991) and Younés (1992) which study some of the possibilities.

The plan of this paper is as follows. The model is in section 2. The concept of anonymous constrained optimality is discussed in section 3 and section 4 contains examples.

II. *The Model*

I will consider a two-period sequential exchange economy. There are two periods, 0 and 1. The state of economy is resolved in the beginning of period 1. There are L goods and S states, $L,S<\infty$. There is no production.

Every consumer has a consumption set $X=\mathbb{R}_+^{LS}$. Consumption takes place in period 1 *only* and therefore the state of economy is resolved before consumption takes place. There are a continuum of consumers but there are T types of them, $1<T<\infty$. Each type t, $t=1,\ldots,T$, of concumers is characterized by its utility function $u_t:X\to\mathbb{R}$ and initial endowments $\omega_t\in\mathbb{R}^{LS}$. Abusing notation, the set of types is denoted by T. The utility of consumption plan $x\in\mathbb{R}^{LS}$ is $u_t(x)$.

For simplicity, each type of consumers is assumed to have the same population. I interpret this economy as an economy with a continuum of consumers and finitely many types.

I am interested in allocations with equal treatment property; that is, allocations in which the same type of consumers receive the same consumption. So, I will identify, for example, a consumption plan of a consumer of type t with the common consumption plan which all the type t consumers have. The common consumption bundle of type t con-

[1] The relation of anonymous allocations and optimal taxation in general equilibrium context has been studied by, among others, Hammond (1979), Guesnerie (1981), and Dierker-Haller (1990).

sumers is denoted by $x_t \in X$, and the allocation of goods is denoted by $\boldsymbol{x}=(x_1,\ldots,x_t,\ldots,x_T)\in \boldsymbol{X}^T$. An allocation $\boldsymbol{x}$ is said to be *feasible* if $\sum_{t=1}^T(x_t-\omega_t)=0$.

In period 0, J real assets are traded. Asset j, $j=1,\ldots,J$, yields a commodity bundle $r_j^s \in \mathbb{R}^L$ in state s. Let $\boldsymbol{R}^s$ be the $L\times J$ matrix whose jth column is $\boldsymbol{r}^s$. The return (vector) of a portfolio $y=(y_1,\ldots,y_j,\ldots,y_J)'\in\mathbb{R}^J$ in the state s is given by $\boldsymbol{R}^s y=\sum_j r_j^s y_j \in \mathbb{R}^L$. Again, I am interested in asset allocations with equal treatment property. I denote the common portfolio of type t consumers by a vector $y_t=(y_{t1},\ldots,y_{tj},\ldots,y_{tJ})'\in\mathbb{R}^J$, and the asset allocation by $\boldsymbol{y}=(y_1,\ldots,y_t,\ldots,y_T)'\in\mathbb{R}^{JT}$.

Let $\boldsymbol{P}=\mathbb{R}_+^{LS}$ and $\boldsymbol{Q}=\mathbb{R}^J$. A generic element of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are denoted by $p=(p^1,\ldots,p^S)$ and $q=(q_1,q_2,\ldots,q_J)$, respectively.

Definition 1. An allocation $(\boldsymbol{x},\boldsymbol{y})=(x_t,y_t)_{t=1}^T\in X^T\times\mathbb{R}^{JT}$ is an *equilibrium allocation* if there exist spot market prices $p\in\boldsymbol{P}$, and asset market prices $q\in\boldsymbol{Q}$, such that
(a) for every t, (x_t,y_t) maximizes $u_t(x_t)$
subject to $p^s(x_t^s-\omega_t^s)\leq p^s\boldsymbol{R}^s y_t$ for all s and $qy_t\leq 0$;
(b) $\sum_{t=1}^T(x_t-\omega_t)=0$ and $\sum_{t=1}^T y_t=0$.

It is often convenient to use the indiiect utility function instead of the direct utility function u_t. For any $(p,q)\in\boldsymbol{P}\times\boldsymbol{Q}$ and $y\in\mathbb{R}^J$, define $v_t(p,y)$ by the rule:

$$v_t(p,y)=\mathrm{Max}\{u_t(x): p^s(x^s-\omega_t^s)\leq p^s\boldsymbol{R}^s y \text{ for all } s\}$$

In words, $v_t(p,y)$ is the maximum utility level of type t consumer that can be obtained through spot market trade given prices p if he has a portfolio y in the beginning of the second period. Note that if $(\boldsymbol{x},\boldsymbol{y},p,q)$ is an equilibrium, then for each t, y_t maximuzes $v_t(p,y)$ in y subject to $qy=0$. So, if v_t is differentiable, assuming interior solution, $(\partial/\partial y)v_t$ is proportional to q in equilibrium for all t.

III. *Anonymous Constrained Optimality*

As I said in section 1, I assume that the planner cannot observe the type of consumers and he cannot intervene in the spot markets in period 1, although he has the statistical information on consumers' types. The planner can intervene in the period 0 asset markets with a *proposal* z, which is a function $t\rightarrow z_t$ from T to $\mathbb{R}^J$ (or, z can be seen as an element of $\mathbb{R}^{JT}$), and each consumer is free to announce his type. If a consumer reports his type to be t, he receives an asset portfolio z_t. I shall focus on direct revelation proposals in which each consumer chooses to announce their true types.[2]

Imagine a situation where the planner can intervene in the asset markets with some proposal after asset trade is completed among the consumers and before spot market transactions take place. There is a possibility that the planner can obtain some information about the consumers' types by observing the asset transactions and the market clearing asset prices. Anticipating such planners' learning scheme, some types of consumers may be

[2] This is standard since there are a continuum of consumers. See Hammond (1979), Dasgupta-Hammond-Maskin (1979), Champsaur-Laroque (1981), and Dubey-Mas-Colell-Shubik (1980).

better off by pretending other types. That is, some sort of strategic "pooling" equilibria is possible, as is often the case in contract theory. But in this paper, I will simply do away with this complexity by assuming that the planner cannot observe the transactions nor prices and that the consumers do not anticipate the planner's intervention at all. These assumptions are by no means "realistic," but the exercise is still delicate and I firmly believe that it is very instructive to understand a subtle nature of incentive problem in sequential trade models with incomplete markets.

If the intervention is not anticipated, then the type t consumer with rational expectation will choose an equilibrium portfolio y_t in period 0 asset markets corresponding to the given equilibrium price expectations $\bar{p}$ and allocation $\bar{x}_t$. Having observed asset transactions, the planner may find a room for Pareto improvement and wish to intervene in the asset markets using a proposal.

When consumers report their types, they expect some prices for period 1 markets to decide which portfolio is the best. If the consumers continue to expect the original $\bar{p}$ to prevail, then it is clear that no proposal z can Pareto improve upon x even if types are observable. Indeed, by definition, each type t consumer's indirect utility $v_t(\bar{p},y)$ is maximized at $y=\bar{y}_t$ subject to $q\cdot y\leq 0$, hence $q\cdot z_t>0$ must hold for all t if z induces a Pareto improvement. But this is impossible since $\sum_t z_t=0$.[3]

Although I assume that the intervention is not anticipated when assets are traded in the markets, it does not follow that the consumers should continue to believe that the initially expected prices at the time when consumers are asked to report their types. Indeed, since at this point consumers understand the planner intends to implement some outcomes which may be different from competitive market equilibria, it is rather inconsistent to assume that the consumers ignores the possibility of some aggregate effect in the end. Therefore, I will require consumers' expectations to be updated at the moment the planner lays out the proposal, in such a way that the new price expectations are self-fulfilling, i.e., if all the consumers truthfully report their types, then the expected prices will be the market clearing prices in every state. Under this assumption, the planner may find a room for Pareto improvement through the change of price expectations unlike in the previous case of no expectation adjustment.

Formally, the concept of anonymous proposals the planner can use is defined as follows:

Definition 2. Let $\boldsymbol{y}=(y_t)$ be a given asset allocation. A proposal z_t is *anonymous with respect to* $\boldsymbol{y}$ provided there exists a price system $p\in \boldsymbol{P}$ such that
(a) for all t and k in T,

$$v_t(p, z_t)\geq v_t(p, z_k),$$

(b) there exist $\boldsymbol{x}^*=(x_t^*)$ such that $\sum_t x_t^*=\sum_t \omega_t$, and for all t, $x=x_t^*$ maximizes $u_t(x)$ subject to $p^s x^s\leq p^s[\omega_t^s+\boldsymbol{R}^s(y_t+z_t)]$ for all s given p, y, z; that is, p are equilibrium prices for spot markets where type t consumers are endowed with $\omega_t^s+\boldsymbol{R}^s(y_t+x_t)$ in state s,
(c) $\sum_{t\in T} z_t=0$.
The prices p above will be called price expectations associated with anonymous proposal z.

[3] This is slightly different from Grossman (1977)'s Social Nash Optimality where it is assumed consumers continue to choose $\bar{x}_t$ when the asset markets are intervened.

(a) is the anonymity, or incentive compatibility constraint, given price expectations p. Note that it is assumed that price expectations do not change as a consumer deviates by not reporting his true type since there is a continuum of consumers. (b) means that price expectations are updated in a self-fulfilling way and (c) is the feasibility. Note that the original portfolio allocation y_t need not be an equilibrium allocation for the definition to make sense.

Definition 3. An allocation $(\bar{x},\bar{y})$ is *anonymous constrained optimal* if there is no anonymous proposal z with respect to $\bar{y}$ associated with price expectations $p \in \boldsymbol{P}$ such that $v_t(p,\bar{y}_t+z_t)>u_t(\bar{x}_t)$ for all $t \in T$. $(\bar{x},\bar{y})$ is *anonymous constrained local optimal* provided there is an $\varepsilon>0$ such that there is no anonymous proposal z with respect to $\bar{y}$ associated with price expectations $p \in \boldsymbol{P}$ such that $v_t(p,\bar{y}_t+z_t)>u_t(\bar{x}_t)$ for all $t \in T$ and $\|z\|<\varepsilon$.[4]

Should one expect a competitive equilibrium to be anonymous constrained efficient?

Proposition. There are robust examples of economies in which there is no anonymous constrained optimal competitive equilibrium. Also, there are robust examples of economies in which every equilibirum is anonymous constrained local optimal.

Proof is by constructing examples. I will give some robust examples of economies with two types of consumers that exhibit the properties asserted above in the next section. I will spend the rest of this section to illustrate why it is possible to construct such examples since it is certainly more important for the reader to get some intuition behind the examples.

Consider an economy where there are two types of consumers of equal population. Suppose the planner intends to transfer an asset portfolio z from type 2 consumers to type 1 consumers. Let $W_t(z,p(z))$ be the utility of type t consumers, $t=1,2$, where $p(z)$ is the resulting commodity price vector after the transfer z is made. If both types of consumers reveal their types, then each type 1 consumer receives z and each type 2 consumer receives $-z$. Suppose the economy is in equilibrium before the transfer takes place; that is, the vectors of marginal utilities $\boldsymbol{D}_z W_t(0,p(0))$, $t=1,2$, are proportional to some vector q.

A small transfer z is anonymous, or incentive compatible, if $F_1(z) \equiv W_1(z,p(z)) - W_1(-z,p(z)) \geq 0$ and $F_2(z) \equiv W_2(-z,p(z)) - W_2(z,p(z)) \geq 0$. Note that $p(z)$ is unaffected since each individual consumer is negligible in size. Up to first order, $F_1(z) \simeq F_1(0) + \boldsymbol{D}F_1(0) \cdot z = 2\boldsymbol{D}_z W_1(0,p(0)) \cdot z$, hence type 1 consumer would not accept z if $q \cdot z<0$. Note that the terms containing the derivative with respect to p, $\boldsymbol{D}_p W$, cancel out. Similarly, if $q \cdot z>0$, type 2 consumers would not accept $-z$. Therefore, if transfer z is anonymous, then it must be the case that $q \cdot z=0$. Hence the second derivative of F_t matters.

If $\boldsymbol{D}^2 F_1(0)$ is negative definite, then the equilibrium is anonymous constrained local optimal. Indeed, for all small z such that $q \cdot z=0$, I have $F_1(z) \simeq F_1(0) + \boldsymbol{D}F_1(0) \cdot z + (1/2) z' \boldsymbol{D}^2 F_1(0) z<0$, so there is no way to make a transfer z without giving type 1 consumers an incentive to lie. On the other hand, if $\boldsymbol{D}^2 F_1(0)$ is positive definite, then any small z such that $q \cdot z=0$ will satisfy the anonymity constraint. That is, anonymity has no bit, locally. So, if this is the case, I will be able to conclude that the allocation in question is not anonymous constrained optimal since the result by Geanakoplos-Polemarchakis implies that there is an arbitrary small reallocation that leads to a Pareto improvement.

[4] Here, z is regarded as a point in $\mathbb{R}^{JT}$.

I will elaborate on the last part of the argument below. But before doing it, I want to argue D^2F_1 could be negative definite and so the argument is not vacuous. A simple calculation shows that $D^2F_1=4(\partial^2/\partial z\partial p)W_1\cdot Dp$ at equilibrium. In principle, there is no restriction on the cross second derivative of the indirect utility function $(\partial^2/\partial z\partial p)W_1$ and the derivative Dp, thus it is not surprising that there are economies where D^2F_1 is negative definite. Such economies will be robust since the negative definiteness of D^2F_1 is a robust property.

Now I come back to the point I left. Let me first recall the relevant part of the Geanakoplos-Polemarchakis result. Let $(x,y;p,q)$ be an equilibrium. A proposal z_t is said to be *strongly SGP constrained feasible* if the conditions (b) and (c) of Definition 1 are satisfied and that $qz_t=0$ for all t. Then the equilibrium $(x,y;p,q)$ is called *weak SGP constrained local optimal* if there is no arbitrary small strongly SGP constrained feasible proposal z that makes every consumer better off as in Definition 3.[5]

Geanakoplos and Polemarchakis (1986) show that generically in utility functions and endowments, every equilibrium is *not* SGP weak constrained local optimal under some regularity conditions and restrictions on the number of consumers relative to the number of commodities. Notably, it is assumed that the number of consumers is less than the number of (contingent) commodities.[6]

Notice that in the illustration a reallocation of assets must respect the original asset market budget constraint if it is anonymous. This is in fact a general and instructive property to see the relation between the Geanakoplos-Polemarchakis result and the anonymous constrained optimality.

Lemma. Let $(\bar{x},\bar{y},\bar{p},\bar{q})$ be an equilibrium. Assume the indirect utility function $v_t(p,y)$ is continuously differentiable at $(\bar{p},\bar{y}_t)$ for each t. Suppose z_t^n, $t\in T$, $n=1,\ldots$, is a sequence of anonymous proposals such that $z_t^n\to 0$ as $n\to\infty$ for all $t\in T$. Then for each t there exists an accumulation point $\bar{z}_t$ of the sequence $z_t^n/||z_t^n||$ such that $\bar{q}\bar{z}_t=0$ for all t.

proof

I can choose a subsequence of z_t for every t such that $\sum_t z_t^n=0$ for all n and $z_t^n/||z_t^n||$ converges. Let $\bar{z}_t$ be the limit of the subsequence of $z_t^n/||z_t^n||$. I claim $\bar{q}\bar{z}_t=0$ for all t. Suppose not. Then $\bar{q}\bar{z}_t<0$ for some t and $\bar{q}\bar{z}_k>0$ for some k since $\sum_t z_t^n=0$. Let $F_{tk}(z_t,z_k)=v_t(\bar{p},\bar{y}_t+z_t)-v_t(\bar{p},\bar{y}_t+z_k)$. Since $\bar{y}$ is an equilibrium asset allocation, marginal utilities from assets must be equated to the prices; that is, $\frac{\partial}{\partial z_t}F_{tk}(0,0)=\xi_t\bar{q}$ and $\frac{\partial}{\partial z_k}F_{tk}(0,0)=\xi_k\bar{q}$ for some ξ_t, $\xi_k>0$. Since as n goes to infinity z_t^n gets arbitrary close to $||z_t^n||\cdot\bar{z}_t$ and z_k^n gets arbitrary close to $||z_k^n||\cdot\bar{z}_k$. Hence $F_{tk}(z_t^n,z_k^n)$ becomes arbitrary close to $\xi_t\bar{q}(||z_t^n||\cdot\bar{z}_t)-\xi_k\bar{q}(||z_k^n||\cdot\bar{z}_k)$, which is always negative since $\bar{q}\bar{z}_t<0$ and $\bar{q}\bar{z}_k>0$. Therefore, for any p close to $\bar{p}$, $v_t(p,\bar{y}_t+z_t^n)-v_t(p,\bar{y}_t+z_k^n)<0$ if n is large enough. But this is a contradiction because type t consumers will be better off by choosing z_k^n. □

In the light of Lemma, I have:

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[5] "Strong" and "weak" are my creation.

[6] Geanakoplos-Magill-Quinzii-Dréze (1990) deal with the case with production and they perturb production set and leave utility functions intact. See also Kajii (1992).

Corollary. If an equilibrium is weak SGP constrained local optimal, then it is anonymous constrained local optimal.

The concept of weak SGP constrained optimality has not been appreciated in the literature since it has not been clear whether the weak SGP optimality is an interesting economic property at all. Indeed, the requirement that the net transfer must respect the original equilibrium asset prices is hardly natural in the context of the central planner's intervention in asset markets. It cannot be interpreted that the planner achieves the reallocation through open matket operation, since there is no reason to expect the original prices q to clear the markets when the operation takes place. The result above, however, seems to indicate that the weak SGP optimality is worth further investigation. In particular, it is interesting to study if it is possible to construct robust class of economies with a large but finite number of consumers that have weak constrained optimal equilibria.[7]

IV. *Examples*

I shall provide some examples of an economy with two types of consumers. In every example there is a unique equilibrium in which asset allocation is zero, i.e., no asset trade is made in equilibrium.

Example 1 shows that there is an economy with two assets whose equilibrium is not anonymous constrained optimal. In example 2 I will see on the contrary that there is an economy with an equilibrium which is not weak SGP constrained optimal but anonymous constrained local optimal. Examples 3 and 4 deal with economies with three assets. Example 3 domenstrates that there is an equilibrium that is not anonymous constrained optimal whereas I shall show in Example 4 that there is an equilibrium which is not weak SGP constrained optimal but anonymous constrained local optimal.

In every example, there are four states and two goods in each state. The price of the first good is normalized to be one. Assets pay only in the first good, so they are real numéraire assets. Preferences of both consumers are homothetic in each state, and they are represented by utility functions of the form:

$$u_t(x_t^1, x_t^2, x_t^3, x_t^4) = \sum_{s=1}^4 \pi_t^s[\alpha_t^s \log(x_t^s) + (1-\alpha_t^s)\log(x_t^s)], \quad t=1,2,\ \alpha_t^s \in (0,1),\ \pi_t^s > 0.$$

The corresponding indirect utility functions are:

$$\begin{aligned} v_t(p^1, p^2, p^3, p^4; m_t^1, m_t^2, m_t^3, m_t^4) &= \sum_{s=1}^4 \pi_t^s v_t^s(p^s, m_t^s) \\ &= \sum_{s=1}^4 \pi_t^s[\log(m_t^s) + (1-\alpha_t^s)\log(p^s)], \end{aligned}$$

$t=1, 2.$

Initial endowments of type t consumers in state s is denoted by:

$$\omega_t^s = (\omega_t^{s1}, \omega_t^{s2}) \in \mathbb{R}_{++}^2.$$

[7] Mas-Colell (1987) shows if there is a continuum of consumers, such economies can be constructed.

Let me do some primary calculations. Let $p^s(z)$ be the equilibrium spot price of good 2 when consumer 1 receives z units of good 1 from consumer 2 in state s. Because of the separability of utility function transfer of goods in state s does not affect equilibrium spot prices in the other states. It is straightforward to compute $p^s(z)$, which turns out to be:

$$p^s(z)=\beta^s+\gamma^s z,$$

where
$$\beta^s=[(1-\alpha_1^s)\omega_1^{s1}+(1-\alpha_2^s)]/(\alpha_1^s\omega_1^{s2}+\alpha_2^s\omega_2^{s2}),$$

$$\gamma^s=(\alpha_2^s-\alpha_1^s)/(\alpha_1^s\omega_1^{s2}+\alpha_2^s\omega_2^{s2}).$$

Example 1. [Two assets, an equilibrium is not anonymous constrained optimal]
$(\alpha_1^1, \alpha_1^2, \alpha_1^3, \alpha_1^4)=(1/3, 2/3, 1/3, 2/3)$
$(\alpha_2^1, \alpha_2^2, \alpha_2^3, \alpha_2^4)=(2/3, 1/3, 2/3, 1/3)$
$(\omega_1^1, \omega_1^2, \omega_1^3, \omega_1^4)=((1, 2), (1, 2), (2, 1), (2, 1))$
$(\omega_2^1, \omega_2^2, \omega_2^3, \omega_2^4)=((2, 1), (2, 1), (1, 2), (1, 2))$
Asset structure is given by (with abuse of notation):
$\boldsymbol{r}_1=(1, 1, 1, 1)$
$\boldsymbol{r}_2=(0, 2, 0, 2)$;
that is, asset 1 pays one unit of good one in every state, and asset 2 pays 2 units of good 1 in state 2 and 4, nothing otherwise.
Equilibrium prices corresponding transfer z are given by:
$p^1(z)=1+4z$
$p^2(z)=1-5z$
$p^3(z)=1+5z$
$p^4(z)=1-4z$

Claim. If both π_1 and π_2 satisfy
$\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4=0,\ t=1,2,$
then $p^s=1$, $s=1,\dots,4$, $q=(1,1)$, $y_1=y_2=0$ constitute an equilibrium.

proof. From the computation above, $p^s=1$ clear spot markets. So, all I have to show is that $y_t=0$ is utility maximizing given that $p^s=1$ for all s and $q=(1,1)$, which is equivalent to show that $\sum_s \pi_t^s \partial_m v_t^s \boldsymbol{r}^s$ is proportional to (1,1) when evaluated at $y_t=0$.[8] Note that if $p^s=1$ for all s and $y_t=0$, both consumers receive three units of income in every state, so $\partial_m v_t^s=1/m_t^s=1/3$. So, at $y_t=0$, I have $\sum_s \pi_t^s \partial_m v_t^s r_1^s=(1/3)(\pi_t^1+\pi_t^2+\pi_t^3+\pi_t^4)$ and $\sum_s \pi_t^s \partial_m v_t^s r_2^s=(1/3)(2\pi_t^2+2\pi_t^4)$ for each t. Therefore, if $\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4=0$, $\sum_s \pi_t^s \partial_m v_t^s \boldsymbol{r}^s$ is proportional to (1,1). □

Fix π_t such that $\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4$, $t=1,2$, arbitrary. Then equilibrium asset prices are (1,1) and because of Lemma I focus on transfers of the form $(a,-a), a\in\mathbb{R}$; that is, type 1 consumers receive a units of the numéraire in state 1 and 3, and $-a$ units in state 2 and 4. The corresponding spot prices are, abusing notations, given by: $p^1(a)=1+4a$, $p^2(a)=1+5a$, $p^3(a)=1+5a$, $p^4(a)=1+4a$.

[8] The first order condition for utility maximization is sufficient since this is a concave problem.

I can explicitly compute the level of utility $V_t(a)$ of consumer t associated with the transfer $(a, -a)$.

$$V_1(a)=\pi_1^1[\log(3+9a)-(2/3)\log(1+4a)]+\pi_1^2[\log(3+9a)-(1/3)\log(1+5a)] +\pi_1^3[\log(3+6a)-(2/3)\log(1+5a)]+\pi_1^4[\log(3+3a)-(1/3)\log(1+4a)]$$

$$V_2(a)=\pi_2^1[\log(3+3a)-(1/3)\log(1+4a)]+\pi_2^2[\log(3+6a)-(2/3)\log(1+5a)] +\pi_2^3[\log(3+9a)-(1/3)\log(1+5a)]+\pi_2^4[\log(3+9a)-(2/3)\log(1+4a)]$$

The derivatives evaluated at $a=0$ are given by:
$V_1'(0)=(1/3)[\pi_1^1+4\pi_1^2-4\pi_1^3-\pi_1^4]=(5/3)[\pi_1^2-\pi_1^3]$
$V_2'(0)=(1/3)[-\pi_2^1-4\pi_2^2+4\pi_2^3+\pi_2^4]=(5/3)[\pi_2^3-2\pi_2^2]$.
Therefore, the equilibrium is not weak SGP constrained optimal whenever $[\pi_1^2-\pi_1^3]$ and $[\pi_2^3-\pi_2^2]$ have the same sign, or equivalently,
$[\pi_1^2-\pi_1^3]\cdot[\pi_2^3-\pi_2^2]>0$.

Now I shall examine if the transfer is anonymous. Recall that each consumer believes that his report does not affect prices. Therefore, every consumer expects $p^1(a)=1+4a$, $p^2(a)=1+5a$, $p^3(a)=1+5a$, $p^4(a)=1+4a$, regardless of his report. Income of consumers of type 1 is given by the following table:

	reveal its type	tell a lie
state 1	$3+9a$	$3+7a$
state 2	$3+9a$	$3+11a$
state 3	$3+6a$	$3+4a$
state 4	$3+3a$	$3+5a$

Let $F_t(a)$=(utility of type t consumer if reveal its type)$-$(utility of type t consumer if tell a lie). Then $(a, -a)$ is anonymous iff $F_t(a)>0$ for $t=1,2$. The explicit form of F_1 is as follows.

$$F_1(a)=\pi_1^1[\log(3+9a)-\log(3+7a)]+\pi_1^2[\log(3+9a)-\log(3+11a)] +\pi_1^3[\log(3+6a)-\log(3+4a)]+\pi_1^4[\log(3+3a)-\log(3+5a)].$$

Note that because of the separability of indirect utility function the price terms have been canceled out. F_2 can be derived similarly:

$$F_2(a)=\pi_2^1[\log(3+3a)-\log(3+5a)]+\pi_2^2[\log(3+6a)-\log(3+4a)] +\pi_2^3[\log(3+9a)-\log(3+11a)]+\pi_2^4[\log(3+9a)-\log(3+7a)].$$

The first derivatives vabish at $a=0$. This is not coincidental since up to first order incentive constraints are degenerate at equilibrium as is seen in the previous sections. The second derivatives evaluated at equilibrium is given by:

$F_1''(0)=(1/9)[-32\pi_1^1+40\pi_1^2-20\pi_1^3+16\pi_1^4]=(4/9)[-4\pi_1^1+6\pi_1^2-\pi_1^3]$
$F_2''(0)=(1/9)[16\pi_2^1-20\pi_2^2+40\pi_2^3-32\pi_2^4]=(4/9)[-\pi_2^2+6\pi_2^3-4\pi_2^4]$.

where I used the relation $\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4=0$ to obtain the final expression. Therefore, if both $F_1''(0)$ and $F_2''(0)$ are positive, or equivalently, $[-4\pi_1^1+6\pi_1^2-\pi_1^3]>0$ and $[-\pi_2^2+6\pi_2^3-4\pi_2^4]>0$, then both $F_1(a)$ and $F_2(a)$ are positive for sufficiently small a. In conclusion, if π_1 and π_2 satisfy $[-4\pi_1^1+6\pi_1^2-\pi_1^3]>0$ and $[-\pi_2^2+6\pi_2^3-4\pi_2^4]>0$ as well as $[\pi_1^2-\pi_1^3]\cdot[\pi_2^3-\pi_2^2]>0$ and $\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4=0$, then the equilibrium is not anonymous constrained optimal. For instance, $\pi_1=(1/4,1/3,1/4,1/6)$ and $\pi_2=(1/6,1/4,1/3,1/4)$ meet this requirement. This completes Example 1.

Now I shall show that Example 1 is robust against perturbations of utility functions and endowments. In fact, I shall argue for a general case since the same technique can be used in the subsequent examples. Fix an eocnomy $(\bar{u}_t,\bar{\omega}_t)$ arbitrary. Let $(\bar{x}_t,\bar{y}_t,\bar{p},\bar{q})$ be an equilibrium and consider the following functions F_{tk}, $t,k\in T$, $t\neq k$:

$$F_{tk}(z_t,z_k)=\sum_s v_t^s(p^s, p^s\omega_t^s+r^s(\bar{y}_t+z_t))-\sum_s v_t^s(p^s, p^s\omega_t^s+r^s(\bar{y}_t+z_k)),$$

where p^s is a spot market clearing price system in state s when consumers of type t receives a portfolio y_t+z_t such that $\sum_t z_t=0$. Suppose the equilibrium is generically (strongly) regular,[9] i.e., p^s is a smooth function of $(z_1,\ldots,z_{T-1})\in\mathbb{R}^{J(T-1)}$, locally. For instance, the equilibrium in Example 1 is regular. Then by direct calculation one gets $\frac{\partial}{\partial z_t}F_{tk}(0,0)=-\frac{\partial}{\partial z_k}F_{tk}(0,0)=\sum_s\partial_m v_t^s(\bar{p},\bar{p}\omega_t^s+r^s\bar{y})r^s=\xi_t\bar{q}$, for some $\xi_t>0$. What I have shown is that $\frac{\partial^2}{\partial z_t\partial z_k}F_{tk}(0,0)$ is positive definite on the set $\{z:[\sum_s\partial_m v_t^s(\bar{p},\bar{p}\omega_t^s+r^s\bar{y})r^s]z=0\}$. Since F and v are continuous functions of utility functions and endowments, $\frac{\partial^2}{\partial z_t\partial z_k}F_{tk}(0,0)$ is still positive definite on the set $\{z:[\sum_s\partial_m v_t^s(\bar{p},\bar{p}\omega_t^s+r^s\bar{y})r^s]z=0\}$ for any economy (u_t,ω_t) close enough to $(\bar{u}_t,\bar{\omega}_t)$.

Example 2. [Two assets, an equilibrium is not weak SGP constrained optimal but anonymous constrained local optimal]

I shall consider the same setting as Example 1, and I will examine the situation $F_1''(0)<0$. That is, I shall consider the case where π_1 and π_2 satisfy $[-4\pi_1^1+6\pi_1^2-\pi_1^3]<0$, $[\pi_1^2-\pi_1^3]\cdot[\pi_2^3-\pi_2^2]>0$ and $\pi_t^1-\pi_t^2+\pi_t^3-\pi_t^4=0$ for both t. For instance, take $\pi_1=(1/4,1/6,1/4,1/3)$ and $\pi_2=(1/3,1/4,1/6,1/4)$. Suppose the planner wants to use a transfer of assets of the form $(a,-a)$. But then $F_1(a)=F_1(0)+F_1'(0)a+F_1''(0)a_2/2+o(a_2)=F_1''(0)a^2/2+o(a_2)<0$ if a is small, which implies any small a will give consumers of type 1 an incentive to lie. On the other hand, I know from Lemma in the last section that if an asset reallocation does not satisfy the original budget constraint, then some consumers have incentive to lie.

To sum up, there is no small reallocation of assets that makes both type of consumers better off and that induces truth-telling. Interestingly, for $\pi_1=(1/4,1/6,1/4,1/3)$ and $\pi_2=(1/3,1/4,1/6,1/4)$, both types of consumers have an incentive to lie for any small transfer.

One might argue that the two asset case is special because of the fact the projection

[9] See Geanakoplos-Polemarchakis (1986) for a more precise definition.

of vectors of marginal utilities from assets on the budget line are degenerate as I explained in the last section. So, I shall supply examples of the three asset case.

Example 3. [Three assets, an equilibrium is not anonymous constrained optimal]

I shall consider the economy with the same α and ω as Example 1. There are three assets. The returns of assets are given by:

$\boldsymbol{r}_1=(1,1,1,1)$,
$\boldsymbol{r}_2=(0,2,0,0)$,
$\boldsymbol{r}_3=(0,0,0,2)$.
π is given by:
$\pi_1=(5/12,1/3,1/4,1/3)$
$\pi_2=(1/6,1/4,1/3,1/4)$.

It is easy to check that $p^s=1$, $s=1,\ldots,4$, $q=(1,1/2,1/2)$, $y_1=y_2=0$ is an equilibrium. See Example 1.

Now consider a transfer of the form $(a,-a,-a)$. Note that this transfer satisfies the budget constraint for assets. Since one unit of asset 2 and one unit of asset 3 yield (0,2,0,2) which is identical with the return of the asset 2 in Example 1, the return of a transfer $(a,-a,-a)$ coincides with that of the transfer $(a,-a)$ considered in Example 1. Therefore, this transfer generates exactly the same utility allocation as in Example 1. I have shown that the transfer $(a,-a)$ in Example 1 improves upon both consumers welfare anonymously if a is small enough and if $[-4\pi_1^1+6\pi_1^2-\pi_1^3]>0$, $[-\pi_2^2+6\pi_2^3-4\pi_2^4]>0$, $[\pi_1^2-\pi_1^3]\cdot[\pi_2^3-\pi_2^2]>0$ hold, so does $(a,-a,-a)$ in this case if a is small. This implies that the equilibrium is neither anonumous constrained optimal nor weak SGP constrained optimal.

Example 4. [Three assets, an equilibrium is not weak SGP constrained optimal but anonymous constrained local optimal]

I shall consider the following economy with three assets:

$(\pi_1^1,\pi_1^2,\pi_1^3,\pi_1^4)=(12,2,6,21)$
$(\pi_2^1,\pi_2^2,\pi_2^3,\pi_2^4)=(16,6,3,18)$
$(\alpha_1^1,\alpha_1^2,\alpha_1^3,\alpha_1^4)=(1/3,1/3,1/3,1/3)$
$(\alpha_2^1,\alpha_2^2,\alpha_2^3,\alpha_2^4)=(2/3,2/3,2/3,2/3)$
$(\omega_1^1,\omega_1^2,\omega_1^3,\omega_1^4)=((1,1),(1,1),(2,1),(2.1))$
$(\omega_2^1,\omega_2^2,\omega_2^3,\omega_2^4)=((1,1),(1,1),(2,1),(1,2))$
$\boldsymbol{r}_1=(1,1,1,1)$
$\boldsymbol{r}_2=(1,2,1,2)$
$\boldsymbol{r}_3=(0,2,3,0)$

Equilibrium prices corresponding transfer z are given by:

$p^1(z)=1+3z$
$p^2(z)=1+3z$
$p^3(z)=1+4z$
$p^4(z)=1+4z$

One can check $p^s=1$, $s=1,\ldots,4$, $q=(1,3/2,1/2)$, $(y_1^1,y_1^2,y_1^3)=(y_2^1,y_2^2,y_2^3)=(0,0,0)$ constitute an equilibrium as in Example 1. So, I can concentrate on transfer of the form $(a,b,-2a-3b)$ because of Lemma. I shall show that each type 1 consumer has an incentive to lie no matter how small a and b are. Then this implies that it is impossible to improve upon welfare of both consumers anonymously.

Income of cunsumers of type 1 is given by the following table:

	reveal its type	tell a lie
state 1	$2+4(a+b))$	$2+2(a+b)$
state 2	$2-4(3a+4b)$	$2-2(3a+4b)$
state 3	$3-9(5a+8b)$	$3-7(5a+8b)$
state 4	$3+9(a+2b)$	$3+7(a+2b)$

Let $F_1(a)$=(utility of type 1 consumer if reveal its type)$-$(utility of type 1 consumer is tell a lie). Then

$$\begin{aligned} F_1(a,b)=&12[\log(2+4(a+b)-\log(2+2(a+b))] \\ &+2[\log(2-4(3a+4b))-\log(2-2(3a+4b))] \\ &+6[\log(3-9(5a+8b))-\log(3-9(5a+8b))] \\ &+21[\log(3+9(a+2b))-\log(3+7(a+2b))]. \end{aligned}$$

By construction, the gradient of F_1 vanishes if it is evaluated at $(a,b)=(0,0)$. One can directly compute the second derivative of F_1 at (0,0):

$$D_2F_1(0,0)=-(1/3)\begin{bmatrix}2094 & 3332\\ 3332 & 5388\end{bmatrix},$$

which is a negative definite matrix. Therefore, $F_1(a,b)<0$ for all small (a,b).

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REFERENCES

Champsaur, P. and G. Laroque (1981), "Fair Allocations in Large Economies," *Journal of Economic Theory* 25, pp. 269–282.

Dasgupta, P., P. Hammond, and E. Maskin (1979), "The Implementation of Social Choice Rules," *Review of Economic Studies* 46, pp. 185–211.

Dierker, E. and H. Haller (1990), "Tax System and Direct Mechanisms in Large Finite Economies," *Journal of Economics* 52, pp. 99–116.

Dubey, P., A. Mas-Colell, and M. Shubik (1980), "Efficiency Properties of Strategic Market Games: An Axiomatic Approach," *Journal of Economic Theory* 22, pp. 339–362.

Geanakoplos, J., M. Magill, M. Quinzii, and J. Dreze (1990), "Generic Inefficiency of Stock Market Equilibrium When Markets are Incomplete," *Journal of Mathematical Economics* 19, pp. 113–152.

Geanakoplos, J. and H. Polemarchakis (1986), "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete," in: W. Heller, R. Starr and D. Starrett, eds., *Uncertainty, Information and Communication*, Cambridge, U.K., Cambridge University Press.

Guesnerie, R. (1975), "Production of the Public Sector and Taxation in a Simple Second Best Model," *Journal of Economic Theory* 10, pp. 127–156.

Guesnerie, R. (1981), "On Taxation and Incentives: Further Reflections on the Limits of Redistribution," Discussion Paper No. 89, Sonderforschungsbereich 21, University of Bonn.

Grossman, S. (1977), "A Characterization of the Optimality of Equilibrium with Incomplete Markets," *Journal of Economic Theory* 15, pp. 1–15.

Hammond, P. (1979), "Straightforward Individual Incentive Compatibility in Large Economies," *Review of Economic Studies* 46, pp. 263–282.

Kajii, A. (1991), "Anonymity and Optimality of Competitive Equilibria When Markets are Incomplete," CARESS Working Paper #91-25R, University of Pennsylvania. Forthcoming in *Journal of Economic Theory.*

Kajii, A. (1992), "Constrained Suboptimality with Many Agents," CARESS Working Paper #92-29, University of Pennsylvania.

Mas-Colell, A. (1987), "On the Theorem of Geanakoplos Polemarchakis," mimeo, Harvard University.

Mirrlees, J. (1986), "The Theory of Optimal Taxation," in: K.J. Arrow and M.D. Intriligator, eds., *Handbook of Mathematical Economics* Vol. 3, New York, North-Holland.

Newbery, D. and J. Stiglitz (1982), "The Choice of Technique and the Optimality of Equilibrium with Rational Expectations," *Journal of Political Economy* 90, pp. 223–246.

Stiglitz, J. (1982), "The Inefficiency of Stock Market Equilibrium," *Review of Economic Studies* 49, pp. 241–261.

Stiglitz, J. (1987), "Pareto Efficient and Optimal Taxation and the New New Welfare Economics," in: A. Auerbach and M. Feldstein, eds., *Handbook of Public Economics*, Vol. 2, New York, North-Holland.

Thomson, W. and H. Varian (1985), "Theories of Justice Based on Symmetry," in: L. Hurwicz, D. Schmeidler, and H. Sonnenschein, eds., *Social Goals and Social Organization: Essays in Honor of Elista Pazner*, Cambridge, U.K., Cambridge University Press.

Younés, Y. (1992), "Towards a Theory of Incentive Compatibility for Economies with Incomplete Market Structures," CARESS Working Paper #92–10, University of Pennsylvania.