I. Growth Theory and Related Topics

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ACHIEVING A GENERAL CONSUMPTION SET IN AN INFINITE MODEL OF COMPETITIVE EQUILIBRIUM*

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I. *The Problem*

The original papers, which are now classic, on equilibrium in competitive models with an infinity of goods are Peleg and Yaari (1970) and Bewley (1972). In the Peleg-Yaari paper, where an exchange economy is considered, the consumption set is assumed to equal the positive orthant. Bewley avoids this assumption in his Theorem 1 on the exchange economy, but he introduces it later so that all prices may be reduced to prices of individual goods and in order to deal with production. It should be noted that the consumption set wherever it is located can be moved into the positive orthant by the stratagem used in Arrow and Hahn (1971). That is, for each negative component $-b_i$ in the greatest lower bound $-b$ of the consumption set introduce a stock of the corresponding good equal to b_i. For instance if the ith good is a labor service, b_i might be thought of as the quantity of leisure available to be transfromed into this labor service. Since the net trading set is arrived at by subtracting endowments from the consumption set, the net trading set is not changed by this move, but the consumption set is moved into the positive orthant. The crucial question is whether the greatest lower bound of the consumption set lies in the consumption set. From the viewpoint of economic realism it must not be required to lie there.

The illegitimacy of the requirement is quite clear in the case where labor services are traded. Each labor service must have its corresponding type of leisure, but the sum of the quantities of leisure not consumed, which is equal to the sum of the quantities of labor services supplied, cannot exceed 24 hours per day, while the bound on the quantity of each type of labor service that can be supplied might be, day, 8 hours. Then the lower bound of the consumption set represents a quantity of leisure not consumed equal to 8 hours multiplied by the number of kinds of labor service, a quantity likely to be much in excess of 24 hours. On the other hand, even if labor services are not traded, it is not realistic to suppose that economic activity requires that a minimum quantity of each good is consumed whether that quantity be positive or zero for particular goods. The minimum level of subsistence should allow for substitution between goods.

If this problem is viewed in an intertemporal context, assuming the lower bound of the consumption set to lie in the consumption set means that the level of consumption which can be sustained in any given period is independent of the levels of consumption that have

* I thank John Boyd for his assistance.

occurred in earlier periods. This is as unrealistic as the neglect of substitution in the subsistence bundles of a single period.

There is no objection to the consumption set that Bewley uses in his Theorem 1 for production economies, given that the commodity space is ℓ_∞. This is a convex closed subset of the positive orthant of ℓ_∞. However the theorem provides prices which are allowed to lie anywhere in the dual space ba of ℓ_∞, a space which includes linear functionals which are purely finitely additive measures such as limits at infinity on convergent sequences, extended to the whole space by the Hahn-Banach theorem. If only a finite number of trades are possible, the presence of prices which are not decomposable into sequences of periodwise prices might be acceptable. However such prices are excluded by arbitrage in a market where goods can be traded for any future date and the number of independent trades can be infinite. If the linear functional in ba gave to a stream of goods a price different from the price arrived at by trading for each date separately, a profit of any magnitude could be obtained through trading both ways. This eliminates the budget constraint and excludes the possibility of competitive equilibrium when complete satiation is not feasible. Therefore, if the consumption sets lie in ℓ_∞, linear functionals which lie in ba but not in ℓ_1 cannot be realized in a competitive equilibrium. It is for this reason that Bewley's Theorem 1 cannot be used unless trading is constrained. In terms used on the finanical markets one might say that bubbles are not possible in equilibrium unless markets are incomplete.

There is a second issue for describing the set over which the consumption choice ranges. Peleg and Yaari use the space s, or $\mathbb{R}^\infty$, the space of all sequences of real numbers, for the goods space. Goods are arranged in temporal order with some arbitrary order for the finite number of goods consumed in any given period. They put the product topology on $\mathbb{R}^\infty$ and assume that consumer preferences are continuous in this topology. Continuity implies that if x is preferred to y then every z in a small neighborhood of x is preferred to y. In the product topology this implies for a particular z and y that there is N such that for $i > N$ whether z is preferred to y is independent of the components z_i of z so long as z lies in the consumption set (in their case the positive orthant, but it might be a closed convex subset of the positive orthant as for Bewley in his Theorem 1). If the preference order has a utility representation which is periodwise separable, this is equivalent to requiring the utility function to be a well defined sum of periodwise felicity functions. This is the assumption used by Weitzman (1973) for his theorem on support prices for infinite optimal programs of capital accumulation. The assumption is implied if the utility function is a discounted sum of bounded felicity functions where the discount factors over single periods are bounded below 1.

On the other hand, Bewley uses for the period model, to which we will confine our attention, the space ℓ_∞ of bounded sequences of real numbers and he finds price functionals which lie in ℓ_1, the space of sequences of real numbers which are absolutely summable. This places a constraint on the choices of the consumer which is additional to the constraint arising from the budget condition interpreted in the usual way. That is, when faced with a price vector p, also a sequence of real numbers, he is not free to chose any sequence x of consumption goods which satisfies $\sum_{i=1}^{\infty} p_i x_i \leq m$, where m is his income. He must select a sequence which is uniformly bounded as well. Of course it may turn out that sequences which are unbounded over time are not feasible, but the consumer is represented as making his choice independently of feasibility. Indeed he is not assumed to know what consumption sequences are feasible. His exclusion of an option is considered to be a consequence of the fact that it is too expensive or does not lie in his consumption set. On the other hand, in models of economic growth it is often assumed that unbounded paths of growth are feasible and choices over them are allowed.

II. *Earlier Proofs of Existence*

Bewley listed in his paper of 1972 three ways of proving an existence theorem for models with an infinite number of goods. One is the method of Peleg-Yaari which uses the Debreu-Scarf theorem (1963) that an allocation which remains in the core as the economy is increased by duplication for an indefinite number of times can be supported as a competitive equilibrium. A sketch of the extension of this theorem to production economies was included in the original paper by Debreu amd Scarf. The Peleg-Yaari approach has been used by Aliprantis, Brown, and Burkinshaw (1987) and by Boyd and McKenzie (1992).

A second method is that of Bewley who proves existence for finite dimensional subspaces (defined by a finite number of infinite dimensional vectors) and extends the result to the whole space by a limiting argument depending on the theorem of Alaoglu [Berge (1963)] that a weakly closed and bounded set in a dual space is weakly compact. Stigum (1973) also uses a limiting argument but he first proves the theorem for an economy lasting for a finite number of periods and increases the number of periods without limit. In Stigum's approach the number of goods is infinite only in the limit. Zame (1987) also uses a limiting argument similar to Bewley's but based on subeconomies defined by a family of principal order ideals in a normed lattice. He proves an existence theorem for a production economy.

A third method addresses the problem of existence in the space of utilities which is finite dimensional when the number of consumers is assumed to be finite. This method had been introduced by Negishi (1960) for the finite dimensional case and also used by Arrow and Hahn (1971) for that case. It has been applied to the case of an infinite number of goods by Mas-Collel (1986a) who introduces the notion of uniform properness. Uniform properness in effect uniformly bounds rates of substitution between goods. This assumption is inconsistent with the Inada condition which is often used in the capital accumulation models to bound paths interior to the feasible set. The method was extended to the production case by Mas-Collel (1986b) and Richard (1989). This is also the method used by Chichilnisky and Heal (1991).

The first person to move in the direction which I have indicated was Stigum (1973). He used the space of real sequences for his goods space and his price space, as did Peleg-Yaari. Moreover he explicitly allowed for unbounded sequences in his consumption sets and faced the problem of defining a profit condition in the production sector which would allow for sequences whose values did not converge. However he used conditions on his consumption sets which are very difficult to interpret but which suggest that he intended these sets to include their lower bounds. In any case his proofs are very tedious and as a result his papers have been neglected. Stigum's paper is distinguished by the fact that he avoided all advanced theorems of topology in his proofs, using only elementary limiting processes.

An important extension of the early arguments to more general consumption sets was made by Back (1984, 1988) who based his arguments for the production case (1984) on Bewley's Theorem 1 where a competitive equilibrium was shown to exist with prices in the space ℓ_a, which is dual to ℓ_∞ with the topology of the sup norm. This dual space contains purely finitely additive measures which are subject to the objections we have outlined. Thus the problem is to show that the ℓ_1 part of ℓ_a can supply the equilibrium prices. It was in order to establish this result that Bewley resorted to consumption sets equal to the positive orthant. Back proved that the result can be accomplished with consumption sets which are closed convex subsets of the positive orthant of ℓ_∞ given some assumptions on these sets and on the possible production sets. For the discrete case these assumptions involve, loosely speaking, the possibility of making substitutions, in the production possibility set or in the preferred sets, of the tail of one vector for the tail of another vector while remaining in the set. Such a replacement for the tails of production vectors goes back to the early paper on efficiency prices by Radner (1967) whose objective was to generalize the results of Malinvaud (1953).

A third attack on the problem of a general consumption set was made by Chichilnisky and Heal (1991). Their argument requires that the preference order, which is defined on a consumption set that is closed and convex and lies in the positive orthant of a Sobolev space, which might be, for example, a weighted ℓ_2 space, should be continuously extendible into an open neighborhood of the positive orthant. This seems to require that the uniform properness assumption of Mas-Colell be satisfied [Richard and Zame (1986), Back (1988)]. In addition there are special assumptions to exclude a zero price vector from the price set which are difficult to interpret.

III. *Existence in the Malinvaud Model*

The primary purpose of this paper is to describe the McKenzie-Boyd theorem applied to the Malinvaud model of production and to a generalized Neo-Austrian model of production and to compare their results with those of Bewley and Back. The discussion will be concentrated on the production economy over a sequence of periods with a finite number of goods in each period. In this way we give the assumptions a concrete interpretation in economic terms so that we may see to what extent they have economic interest in a context that is well understood. I will expound a somewhat simplified version of the paper of Boyd and McKenzie (1993) in which proofs are sometimes sketched and sometimes omitted. Finally I will indicate the important respects in which the papers of Bewley and Back differ from Boyd-McKenzie.

The commodity space used in Boyd-McKenzie is the same space used by Peleg-Yaari which they call s and others have called $\mathbb{R}^\infty$. However we will suppose that the set of goods is the same in each period and describe the space as $s^n = \prod_{t=0}^{\infty} \mathbb{R}^n(t)$, that is, the space of sequences of n-vectors. We give this space the product topology where each $\mathbb{R}^n(t)$ has the ℓ_1 norm topology. The ℓ_1 norm is the sum of absolute values, that is, $|z_t|_1 = \sum_{i=1}^{n} |z_{ti}|$. In the product topology an open set $U = \prod_{t=0}^{\infty} U_t$ where U_t is an open set for all t and $U_t =$

$\mathbb{R}^n(t)$ for all save a finite number of values of t.

There are a finite number H of traders each of whom owns a trading set $C^h \subset s^n$. Since the trading sets are primitives in the model, there is no need to list endowments separately. It may be considered that the presence of endowments allows negative components to appear in the vectors of the trading sets. Capital stocks do not appear in the consumer trading sets except at $t=0$ where they are endowments. In an economy with certainty the ownership of capital stocks after the first period is inessential. Only the value of investment is significant for the consumer, and the sequence of investment values is implicit in the pattern of consumption over time. Preference orders P^h are defined on the trading sets along with preference correspondences P^h where $P^h(x^h)=\{z^h \in C^h | z^h P^h x^h\}$. $R^h(x^h)$ is defined by $z^h \in R^h(x^h)$ if $z^h \in C^h$ and $x^h \notin P^h(z^h)$.

The production sector is given by a convex cone Y with vertex at the origin. As is well known, this does not exclude economies with production sectors composed of a finite set of firms who own convex production sets (McKenzie (1959)]. It is assumed that $Y=\sum_{t=1}^{\infty} \bar{Y}_t$ where $\bar{Y}_t$ contains vectors of the form $y_t=(0, \dots, 0, u_{t-1}, v_t, 0, \dots)$. In other words, the production sector is that introduced by Malinvaud (1953) in which the output appearing at time t is the result of inputs committed at time $t-1$. This is a Markov type assumption on the nature of production processes. Everything relevant to production in a period is included in the goods committed to production at the beginning of the period. The inputs and outputs of the production sector include the capital stocks. Let Y_t be the projection of $\bar{Y}_t$ into the coordinate subspace $\mathbb{R}^n(t-1) \times \mathbb{R}^n(t)$. Then $(u_{t-1}, v_t) \in Y_t$ implies $u_{t-1} \leq 0$ and $v_t \geq 0$.

In stating the assumptions on the net trading sets we will need the notion of irreducibility. First we say that the goods bundle x^h belonging to the net trading set of the h*th* trader is strongly individual rational if $x^h \in P^h(y^h)$ for all $y^h \in C^h \cap Y$. Then the economy is strongly irreducible if, whenever I_1, I_2 is a nontrivial partition of the traders and $x_{I_1}+x_{I_2} \in Y$ with $x^h \in C^h$ for all h, there are $z_{I_1}+z_{I_2} \in Y$ with $z^h P^h x^h$ for $h \in I_1$, and, for $h \in I_2$, $z^h \in C^h$ when x^h is strongly individually rational and not an extreme point of C^h, and $z^h \in \alpha C^h$ for some $\alpha>0$ otherwise. This strong form of irreducibility is needed so that allocations to replicas of a trader in the core of the replicated economy will be indifferent. If net trading sets include their greatest lower bounds, strong irreducibility is implied by strict monotonicity.

The assumptions on the production sector are simple realizations of the assumptions used in Boyd and McKenzie. Otherwise the assumptions here are the same as there. The Malinvaud economy $\mathscr{E}$ is given by the list $(Y, C^1, \dots, C^H, P^1, \dots, P^H)$ where $Y=\sum_{t=1}^{\infty} \bar{Y}_t$. Let $e_0=(1, \dots, 1)$.

1. Y_t is a closed, convex cone with vertex at the origin.
2. $Y_t \subset \mathbb{R}^n_-(t-1) \times \mathbb{R}^n_+(t)$ with $Y_t \cap (\{0\} \times \mathbb{R}^n_+(t))=\{(0,0)\}$.
3. C^h is convex, closed, and bounded below by $z \in \ell_\infty$.
4. For all h the correspondence P^h is convex valued and, relative to C^h, open valued with open lower sections. The preference relation P^h is irreflexive and transitive. $R^h(x^h)$ is the closure of $P^h(x^h)$ for all $x^h \in C^h$ when $P^h(x^h)$ is not empty.
5. $x^h \in C^h$ and $z_t^h \geq x_t^h$, with strict inequality for some t, implies $z \in P^h(x^h)$.
6. The economy $\mathscr{E}$ is strongly irreducible.
7. There is $\bar{x}^h \in C^h - Y$ with $\bar{x}^h \leq 0$. Moreover, $\bar{x}=\sum_{h=1}^{H} \bar{x}^h < 0$ and, for some $\xi>0$,

$\bar{x}_t < -\xi_{e_0}$ for all t. For any x^h, let $z^h \in R^h(x^h) - Y$ and $\delta > 0$, then there is a τ_0 such that for each $\tau > \tau_0$, there is an $\alpha > 0$ with $(z_0^h + \delta e_0, z_1^h, \ldots, z_\tau^h, \alpha \bar{x}_{\tau+1}^h, \ldots) \in R^h(x^h) - Y$.

Assumption 1 implies constant returns to scale. However, diminishing returns may be accommodated tnrough the presence of artificial entrepreneurial factors [McKenzie (1959)]. The essential feature of Assumption 2 is the implication that inputs precede outputs and that elementary production processes converge to 0. This allows the profit condition to be given by the profit condition for elementary processes. Moreover, if $y_t = 0$, $y_{t+1} = 0$ since outputs require inputs. This, together with Assumption 3, allows us to prove that the feasible set is bounded. It is also implied that $Y - s_+^n$ can contain no straight lines, which allows us to prove that the feasible set is closed.

Assumption 3 that C^h is bounded below also enters the proof that a price vector in ℓ_1 gives a valuation to all preferred trades and in the proof that the convex hull of preferred sets over all traders is closed. The last part of Assumption 4 implies local nonsatiation. Open lower sections are needed in the proof that the cores are closed sets of allocations. Assumption 5 is a weak form of periodwise monotonicity. Assumption 6 is needed at two points in the proof of existence, to show that in the replicated economy replicas of a given trader receive bundles in the core which are indifferent, and to show that a quasi-equilibrium is a competitive equilibrium. The last part of Assumption 7 is what remains of the assumption made by Back which would be implemented in a period model by supposing that tails can be substituted between vectors in the consumption sets or in the production set. Our assumption is weaker for two reasons, a particular vector is selected to provide the tail, after multiplication by a constant, for the substitutions, and the substitutions are made in feasible net trading vectors with production. The first part of assumption 7 is an adequacy assumption which is also used by Back and in stronger form by others. Iy may be regarded as the Slater condition [see Uzawa (1958, p. 34)] that a feasible point exists with slack constraints.

The set of *possible trades with production* for the hth consumer is $C^h - Y$. The set of *admissible price vectors* is s^n. Like Peleg-Yaari but unlike Back, the admissible price vectors are not all contained in the dual of the commodity space. The lower bound on C^h insures that pw is either finite or $+\infty$ for all $w \in C^h$. For $p \in S$ the *budget set* of the hth trader is $B^h(p) = \{x | x \in C^h \text{ and } px \leq 0\}$. A *competitive equilibrium* for the economy $\mathscr{E}$ is a list $(p, y, x^1, \ldots, x^H)$ such that p is admissible and the following conditions are met.

I. $px^h \leq 0$ and $z \in P^h(x^h)$ implies $pz > 0$.

II. $y \in Y$ where $y_0 = u_0$ and $y_t = u_t + v_t$ for $t \geq 1$. Also $p_{t-1}u_{t-1} + p_t v_t = 0$ for $t \geq 1$, and $z \in Y$ implies $p_{t-1}u'_{t-1} + p_t v'_t \leq 0$ for all $t \geq 1$, where $z_t = u'_t + v'_t$.

III. $\sum_{h=1}^{H} x^h = y$.

The first condition is the demand condition. The second condition is the profit condition. The third condition is the balance condition. The proof that an equilibrium exists for the economy $\mathscr{E}$ under the assumptions we have made proceeds in several steps. It is first proved that $\mathscr{E}$ has a nonempty core. An allocation of net trades $\{x^h\}$ admits an *improving coalition* if there is an allocation $\{w^h\}_{h \in B}$ over the members of a subset B of traders such that $\sum_{h \in B} w^h \in Y$ and $w^h \in P^h(x^h)$ for all $h^h \in B$. The *core* of the economy $\mathscr{E}$ is the set of feasible allocations which do not admit an improving coalition. Using the lower bound on consumption sets from Assumption 3 and the impossibility of production without inputs from Assumption 2 the set of feasible allocations is shown to be compact by the classical argument from finite models, extended to the infinite model by induction and Tychonoff's theorem. This proof is available because of the separability of the production set in the Malinvaud model. The feasible set is also seen to be nonempty from Assumption 5 and 7 and to be closed from the closedness of the sets Y and C^h. Then a classical argument gives a continuous utility function defined over the feasible set. Applying the Scarf theorem (1967) proves that the core of $\mathscr{E}$ is not empty.

The next step of the proof follows the line of argument of Peleg-Yaari and Aliprantis and Brown (1983) to prove that there is an allocation which remains in the core after indefinitely many replications. This is called an *Edgeworth equilibrium*. The notion is defined of an *equal treatment core* in which the replicas of a particular trader receive the same allocation. Strong irreducibility implies that the replicas of a particular trader receive allocations of the same utility in the core. Then convexity of preferences implies that the equal treatment core K_r of the economy replicated r times is not empty. Open lower sections of the preferred sets $R^h(x^h)$ allow us to prove that the cores of replicated economies are closed and thus compact since they lie in a compact feasible set. Since the sequence of cores K_1 is nested, we finally have that $K=\cap_{r=1}^{\infty} K_r$ is not empty. Any allocation $(x^1, \ldots, x^H)$ in K is an Edgeworth equilibrium.

Let $y=\sum_{h=1}^{H} x^h$. The final problem is to show that an admissible price vector p exists such that $(p,y,x^1, \ldots ,x^H)$ is a competitive equilibrium. Let G be the convex hull of $\cup_{h=1}^{H} R^h(x^h)$. The key is to separate $G-Y$ from the origin with a hyperplane. An admissible price vector that defines this hyperplane is an equilibrium price vector.

The proof proceeds by a series of lemmas.

Lemma 1. G is closed in s^n.

Proof. By Assumption 3 each C^h is bounded below by z, and therefore G is bounded below by $\bar{z}$. Suppose $z^s \in G$ and $z^s \to z$. We must show that $z \in G$. Let $z^s=\sum_{h=1}^{H} w^{hs}=\sum_{h=1}^{H} \alpha_{hs} z^{hs}$ where $\alpha_{hs} \geq 0$, $\sum_{h=1}^{H} \alpha_{hs}=1$ and $z^{hs} \in C^h$.

The α_{hs} are contained in the unit interval, so we may assume that they converge to α_h by passing to a subsequence if necessary. If the w^{hs} were unbounded for some h, the fact that each w^{hs} is bounded below would imply that z^s is also unbounded, contradicting the convergence of z^s. Therefore each of the w^{hs} is bounded. Then by a Cantor diagonal process [Dunford and Schwartz (1958, p. 23)] we can choose a further subsequence where each of the w^{hs} converges to w^h.

Let $I=\{h|\alpha_h>0\}$. For $h \in I$, $w^{hs}/\alpha_{hs}=z^{hs} \to w^h/\alpha_h \in C^h$. For $h \notin I$, $\alpha_{hs}\bar{z} \leq w^{hs}$ where $\bar{z}$ is the lower bound on C^h from Assumption 3. Taking the limit gives $0 \leq w^h$. Now consider $w^h/\alpha_h+\sum_{i \notin I} w^i$, which is in $R(x^h)$ by periodwise monotonicity. Moreover, $\sum_{h \in I} \alpha_h(w^h/\alpha_h+\sum_{i \notin I} w^i)=\sum_{h=1}^{H} w^h=z$. Therefore $z \in G$. □

The following theorem is adapted from Choquet (1962).

Theorem [Choquet]. If Z is a product closed convex set in s^n which contains no straight lines, then for any two product closed subsets X, Y of Z, the sum $X+Y$ is closed.

Lemma 2. $G-Y$ is closed in s^n.

Proof. Recall $G-Y\subset \bar{z}+s_+^n-Y$. Since Y and s_+^n contain 0, both $G-\bar{z}$ and $-Y$ are closed and contained in s_+^n-Y. Also s_+^n-Y contains no straight lines because Assumption 2 implies that $-y_0=-u_0>0$ and if $y_0=0$, $y=0$. Thus we need only show s_+^n-Y is closed and apply Choquet's theorem.

Let $z^n\to z$ with $z^n\in Y-s_+^n$. Then there are $y^n\in Y$ with $z^n<y^n$. Since z^n converges, the y_0^n are bounded below. But $y_0^n=u_0^n\leq 0$, so the y_0^n are bounded. By the same classical argument that proves the feasible set to be compact when Assumptions 1, 2, and 3 hold y_t^n is bounded for each t. By a Cantor diagonal process we may find a convergent subsequence of y^n with limit $y\in Y$. Since $z^n\leq y^n$, $z\leq y$ and $z\in Y-s_+^n$. So $Y-s_+^n$ is closed. □

Lemma 3. If $K\neq\phi$ there is no $z\in G$ and $y\in Y$ such that $z-y\leq 0$ and $z_t-y_t\leq 0$ for some t.

Proof. Let G^0 be the convex hull of the $P^h(x^h)$. In the light of the periodwise monotonicity assumption for preferences, it is sufficient to prove that there is no $z\in G^0$ and $y\in Y$ such that $z-y=0$. However this may be done by the argument used by Debreu and Scarf (1963) for the finite case. □

We must define a vector $\bar{c}$ which may be used in normalizing the price vectors p by puting $p\bar{c}=1$. This involves using Assumption 7 to substitute for the tails of the allocations x^h in the Edgeworth equilibrium that has been shown to exist. Choose α and τ in accordance with Assumption 7 so that $d^h=(x_0^h+e_0,x_1^h,\ldots x_\tau^h,\alpha\bar{x}_{\tau+1}^h,\ldots)\in R^h(x^h)-Y$. Let $\bar{d}^h=d^h+(2e_0,\ldots)$. By monotonicity $\bar{d}^h\in(G-Y)\cap\ell_\infty$. Since $\bar{x}^h\leq 0$, it also follows by monotonicity that $\bar{c}^h=\bar{d}^h-\alpha\bar{x}^h\in(G-Y)\cap\ell_\infty$. Note that $\bar{c}_t^h=0$ for $t=\tau+1,\ldots$. Let $\bar{d}=(1/H)\sum_{h=1}^H\bar{d}^h$. We now define $\bar{c}=(1/H)\sum_{h=1}^H\bar{c}^h=\bar{d}-\alpha\bar{x}/H$.

The dual space of ℓ_∞ is ℓ_{ba}, the space of bounded additive measures on the integers. By a theorem of Yosida and Hewitt the space ℓ_{ba} contains the sums of members of ℓ_∞ and purely finitely additive measures which are 0 on all elements of ℓ_∞ that converge to 0. Although the purely finitely additive part of ℓ_{ba} does not provide useful prices for competitive theory it is helpful to allow them provisionally and then show that they are not needed for the support of the competitive equilibrium. Therefore for $0<\varepsilon<1$ we define a provisional *price set*

$$S(\varepsilon)=\{p\in\ell_{ba+}|p\bar{c}=1 \text{ and } pz\geq-\varepsilon \text{ for all } z\in(G-Y)\cap\ell_\infty\}.$$

In order to show that $S(\varepsilon)$ is not empty we will find a price vector $p\geq 0$, $p\in s^{n*}$, the dual space of s^n, which supports $G-Y$. Since the elements of s^{n*} are vectors with a finite number of nonzero components, they are also elements of ℓ_1 and thus of ℓ_{ba}. We then note that $S(\varepsilon)$ is closed in the weak* topology of ℓ_{ba} and thus compact by Alaoglu's theorem. This will imply that the intersection S of the $S(\varepsilon)$ sets is nonempty as $\varepsilon\to 0$. An element of this intersection will provide the equilibrium price vector $p^*\in S$. Note that because of periodwise monotonicity the equilibrium price vector must have an infinite number of nonzero components, so it cannot lie in s^{n*}.

Lemma 4. For any $\varepsilon>0$ there is $p\in s^{n*}$ such that $p\in S(\varepsilon)$ with $p\geq 0$ and $|p_0|>0$. Also, for any $p\in S(\varepsilon)$, we have $p_{t-1}u_{t-1}+p_tv_t\leq 0$ for all $(u_{t-1},v_t)\in Y_t$ for all $t\geq 1$.

Proof. For $\varepsilon>0$ let $a(\varepsilon)=(-\varepsilon e_0,0,0,...)$ where $e_0=(1, \ldots ,1) \in\mathbb{R}_0^n$. By Lemma 3, $a(\varepsilon)\notin G-Y$. By Lemma 2, $G-Y$ is closed. Also the single point set $\{a(\varepsilon)\}$ is compact. By a separation theorem [Berge (1963, p. 251)] there is a continuous linear functional $f\in s^{n*}$ with $f\neq 0$ such that $f(z)>f(a(\varepsilon))+\delta$ for any $z\in G-Y$ and some $\delta>0$. Any such f may be represented by a vector $p\in s^n$ with $p\neq 0$ but $p_t=0$ for all but finitely many t, where

$$f(z)=pz=\sum_{t=0}^{\infty} p_t z_t \geq -\varepsilon|p_0|+\delta,$$

for any $z\in G-Y$ and some $\delta>0$. Periodwise monotonicity implies that $p\geq 0$. Thus we have for some $p\geq 0$, $p\neq 0$,

$$pz>-\varepsilon|p_0| \text{ for all } z\in G-Y. \tag{1}$$

On the other hand $x^h\in R^h(x^h)$ for all h and $\sum_{h=1}^{H} x^h=y$ for some $y\in Y$ implies that $0\in G-Y$. Since $Y=Y+Y$ and $0\in G-Y$ it follows that $-Y\subset G-Y$. Therefore the definition of $S(\varepsilon)$ implies $py\leq\varepsilon|p_0|$ for all $y\in Y$ and $|p_0|\neq 0$. Since $\alpha y\in Y$ for any $\alpha>0$ it follows that $py\leq 0$ for all $y\in Y$. However $(0, \ldots, 0, u_{t-1}, v_t, 0, \ldots)\in Y$ for all $t\geq 1$ and $(u_{t-1}, v_t)\in Y_t$. Therefore $p_{t-1}u_{t-1}+p_t v_t\leq 0$ for all $(u_{t-1}, v_t)\in Y_t$. □

Lemma 5. The intersection S of the price sets $S(\varepsilon)$ for $0<\varepsilon<1$, is not empty. Moreover, $\bar{p}\in S$ implies $\bar{p}z>0$ for all $z\in(G-Y)\cap\ell_\infty$.

Proof. Lemma 4 implies that $S(\varepsilon)$ is not empty. Furthermore $S(\varepsilon)$ is closed in the weak* topology of ℓ_∞ since the inner product is weak* continuous and the inequalities that define it are weak. Let p be an arbitrary element of $S(\varepsilon)$. Consider the point $\bar{d}=\bar{c}+\alpha\bar{x}/H$, which is in $(G-Y)\cap\ell_\infty$ by construction. Therefore $p\bar{c}+\alpha p\bar{x}/H\geq-\varepsilon$ by definition of $S(\varepsilon)$. In other words,

$$-p\bar{x}\leq H(1+\varepsilon)/\alpha. \tag{2}$$

Since $p\geq 0$, and $\bar{x}$ is bounded below 0 by Assumption 7, $S(\varepsilon)$ is bounded as a consequence of the relation (2). Then Alaoglu's theorem [Berge (1963, p. 262)] implies that $S(\varepsilon)$ is weak* compact since it is closed and bounded. Finally the $S(\varepsilon)$ are nested sets, so their intersection S is not empty by the finite intersection property [Berge (1963, p. 69)]. The last statement of the Lemma is immediate. □

Let $\bar{p}$ be an element of S. The Yosida-Hewitt theorem [Dunford and Schwartz (1958, p. 163)] gives a decomposition of $\bar{p}$ into the sum of a vector $p^*\in\ell_1$ and a purely finitely additive measure which is equal to 0 on all elements of ℓ_∞ which converge to 0. We may note that this result for ℓ_∞ is given a completely elementary proof by Prescott and Lucas (1972). Since $\bar{c}$ has only a finite number of nonzero components, $p^*\bar{c}=\bar{p}\bar{c}=1$, so p^* is not 0. Moreover, using the extended real line, the inner product with p^* may be applied over all of $G-Y$.

Lemma 6. $p^*z=\sum_{t=1}^{\infty} p_t^* z_t$ is well defined for all $z\in G-Y$.

Proof. Let $z=w-y$ where $w\in G$ and $y\in Y$. Define $w_{it}^-=0$ for $w_{it}\geq 0$ and $w_{it}^-=w_{it}$ for $w_{it}<0$. For $w\in C^h$, $0\geq w^-\geq \bar{z}$, so $w^-\in\ell_\infty$. Also $p^*w=p^*(w-w^-)+p^*w^-$. The first term is either finite or $+\infty$ and the second term is finite. Thus p^*w is either finite or $+\infty$. On the other hand consider $y\in Y$. We may write $y=\sum_{t=1}^{\infty} y(t)$ with $y(t)\in \bar{Y}_t\subset Y$. Each $y(t)\in \ell_\infty$ and $p^*y(t)\leq 0$ by Lemma 4. Moreover $\sum_{t=1}^{\tau} y(t)=\sum_{t=1}^{\tau-1} y_t+v_\tau$. Since $p^*\sum_{t=1}^{\tau-1} y(t)\leq 0$ and $v_t\geq 0$, $p^*\cdot\sum_{t=1}^{\tau-1} y_t\leq 0$, all τ. Thus p^*y is either finite or $-\infty$. Combining these results shows that p^*z is either finite or $+\infty$ for all $z\in(G-Y)$. □

We must now show that p^* separates 0 from $G-Y$.

Lemma 7. The vector p^* satisfies $p^*z\geq 0$ for all $z\in G-Y$.

Proof. Take $z\in G-Y$. If $p^*z=+\infty$, we are done, so we may assume p^*z is finite. Let $\varepsilon>0$. Write $z=\sum_{h=1}^{H}\alpha_h z^h$ with $z^h\in R^h(x^h)-Y$. For τ large, $\hat{z}^h=(\varepsilon e_0+z_0^h, z_1^h, \ldots, z_\tau^h, 0,\ldots)\in R^h(x^h)-Y$ by Assumption 7 and monotonicity. Thus $\hat{z}=\sum_{h=1}^{H}\alpha_h\hat{z}^h\in G-Y$. Apply $\bar{p}\in S$ to obtain $\varepsilon|p_0^*|+\sum_{h=1}^{H}\alpha_h\sum_{t=0}^{H} p_t^*z_t^h=\bar{p}\hat{z}\geq 0$ for τ large by the definition of S. Letting $\tau\to\infty$ we find $\varepsilon|p_0^*|+p^*z\geq 0$. Since ε is arbitrary, $p^*z\geq 0$. □

I claim that $(p^*,y,x^1,\ldots,x^H)$ where $y=\sum_{h=1}^{H} x^h$ is a competitive equilibrium for $\mathscr{E}$. By Lemma 7 we have

$$p^*z\geq 0 \tag{3}$$

for $z\in G-Y$. On the other hand, $G\subset G-Y$ and $x=(x^1,\ldots,x^H)\in G$ implies that $x^h\in G$, so $p^*x^h\geq 0$ for all h. Also $(x^1,\ldots,x^H)$ feasible implies $\sum_{h=1}^{H} x^h\in Y$. Therefore $p^*\cdot\sum_{h=1}^{H} x^h\leq 0$ by Lemma 4. This implies, for all h,

$$p^*x^h=0, \text{ so } p^*y=0. \tag{4}$$

must hold for all h.

(4) establishes the first part of Condition 1 and the first part of Condition II for competitive equilibrium. The second part of Condition II is given by Lemma 4.

To complete the proof that condition I holds we must show that $w^h\in P^h(x^h)$ implies $w^h\notin B^h(p^*)=\{z^h|p^*z^h\leq 0\}$. A final lemma is

Lemma 8. If there is $w^h\in C^h$ such that $pw^h<0$ and $pz^h\geq 0$ for all $z^h\in P^h(x^h)$, then $pz^h>0$ for all $z^h\in P^h(x^h)$.

Proof. Suppose $z^h\in P^h(x^h)$ and $pz^h=0$. Since $P^h(x^h)$ is open in the product topology relative to C^h by Assumption 4, there is a point $y^h\in\alpha w^h+(1-\alpha)z^h$ such that $y^h\in P^h(x^h)$ and $py^h<0$. This contradicts the hypothesis. Therefore no such z^h can exist. □

From Lemma 8 we see that Condition I will be completed if it can be proved that every consumer has a net trade w^h such that $p^*w^h<0$. However $\bar{x}\in C$ and $\bar{x}<0$ implies that $p^*\bar{x}^h<0$ for some h.

Let I_1 be the set of indices h such that there is $w^h\in C^h$ with $p^*w^h<0$. Let I_2 be the complementary subset of indices. We have just shown that I_1 is not empty. Suppose that $I_2\neq\phi$. By Assumption 5 there are z_{I_1} and z_{I_2} with $z^hP^hx^h$ for all $h\in I_1$ and $z^h\in\alpha C^h$, $\alpha>0$, for $h\in I_2$, where $z_{I_1}+z_{I_2}=y'\in Y$. By Lemma 8 we have $p^*z_{I_1}^h>0$. However $p^*z_{I_2}^h\geq 0$ by assumption. Thus $p^*(z_{I_1}+z_{I_2})=p^*y'>0$. However, since $y'\in Y$ we have $p^*z\leq 0$ for all $z\in Y$. Thus I_2 must be empty. Then by Lemma 8, $w^h\in P^h(x^h)$ implies $p^*w^h>0$ for all h. This establishes the second part of Condition I for competitive equilibrium. Condition

III follows from the definition of a feasible trade. Therefore $(p^*, y, x^1, \ldots, x^h)$, where $y=\sum_{h=1}^{H} x^h$, is a competitive equilibrium of $\mathscr{E}$.

We have proved

Theorem 1. Under Assumptions 1–7 the economy $\mathscr{E}$ has a competitive equilibrium with prices in ℓ_1.

We have also shown that any Edgeworth equilibrium is a competitive equilibrium. Classical arguments show that any competitive equilibrium is a Edgeworth equilibrium.

Corollary. Under Assumptions 1–7 an allocation is an Edgeworth equilibrium of the economy $\mathscr{E}$ if and only if there is a price vector $p^* \in \ell_1$ for which it is a competitive equilibrium.

IV. *Existence in a Generalized Neo-Austrian Model*

The Malinvaud model assumes that all the factors relevant to output at the end of the any period are summarized in the list of inputs which are traded on markets and made available at the start of that period. This is not, however, the production of Hicks in *Value and Capital* (1939, pp. 194–5). He describes a *production plan* for the firm. "An input is merely something which is bought for the enterprise, output something which is sold. Thus, if the whole concern were to be wound up, and all its equipment sold, this equipment could be regarded as an 'output' of the date at which the sale took place—all subsequent outputs being zero. This idea allows us to think of the entrepreneur as planning ahead for a limited period (n weeks): for we regard the plant he plans to have left over at the end of that time as particular kind of output (say Z_n), a kind which is only produced in the last week."

Later in his book *Capital and Time: a Neo-Austrian Theory* (1973) Hicks describes production in terms of processes of finite duration in which a net input occurs at the beginning of the process and net outputs occur at later times. Atsumi (1991) introduces a generalization of the Hicks model in which the processes begin with an net input but net inputs may also occur later in a process. We will use a generalization of the Atsumi model.

In the spirit of Atsumi's model we will consider production processes which last for an indefinite length of time but in which the initial components are net inputs and later components may be either net inputs or net outputs. Also net outputs of certain size in any period require net inputs of at least a certain size in an earlier period. Each process continues for all subsequent periods. However the process may end with a string of zeros. In the generalized Neo-Austrian economy Assumption 1 and Assumptions 3 to 7 are unchanged. Assumption 2 is replaced by

Assumption 2′. Y is a closed convex cone with vertex at the origin. Moreover given any t and $\varepsilon>0$ there is $\delta>0$ such that $|y_t^+|>\varepsilon$ implies $|y_s^-|>\delta$ for some $s<t$.

It is an immediate consequence of Assumption 2′ that $Y \cap s_+^n=(0, \ldots)$. In other words, net provisions cannot be made available to traders from the production sector unless at some time traders supply quantities of goods to the production sector. The economy with

Assumption 2′ for the production sector will be referred to as $\mathscr{E}'$.

It is also appropriate to change the definition of competitive equilibrium for the economy $\mathscr{E}'$. In place of Condition II we introduce

II′. $py=0$, and for all $z\in Y$, $\limsup \sum_{t=0}^{\infty} p_t z_t \leq 0$.

Since p is an admissible price functional, $p\in s^n$ is understood. The question whether the profit condition should be stated on finite processes or on the whole set of infinite paths is of some interest. If the profit condition is put on finite processes, the condition may be satisfied by inefficient paths as Malinvaud discovered, and Atsumi found in his analysis of the Neo-Austrian model. On the other hand Debreu showed that all equilibria will be Pareto Optimal when the profit condition is satisfied on the set of infinite paths of net outputs $y\in Y$ with well defined linear functionals, that is, in our case lim sup may be replaced by limit, and the functionals also support the preferred sets. However, it has been proved by Michel (1990) that the existence of a supporting price satisfying II′ is a necessary and sufficient condition for the optimality of an infinite path of capital accumulation in the neoclassical model under the overtaking criterion. Therefore its appearance in the context of infinite competitive paths is not surprising.

It has usually been the custom in the Neo-Classical models to place the profit condition on the one period production processes. In effect, producers are treated as having one period horizons. Foresight is left to traders. However, it is not clear that traders should be prohibited from exercising their foresight to combine processes into infinite paths of output which may have positive value. Such possibilities of profit may be avoided by requiring Condition II′ as the profit condition.

Much of the proof remains as it was detailed in Section 1. On the other hand, where the constitution of the production sector is involved new proofs are needed.

It is obvious that the production set Y cannot contain any straight lines since if $y\in Y$ it is necessary that the first component y_t be negative. Then $-y$ begins with y_t positive which is excluded, so $-y\notin Y$. Then the proof that $G-Y$ is closed is the same as before. Since the New-Austrian model does not derive its net output stream from periodwise production processes unlike the Malinvaud model, we will give a proof of the compactness and non-emptiness of the feasible set.

Lemma 1. The set F of feasible allocations is non-empty, compact, and convex.

Proof. By Assumption 7 there is $\bar{x}^h\leq 0$ with $\bar{x}^h\in C^h-Y$. Then by Assumption 5, $0\in C^h-Y$, so $C^h\cap Y$ is not empty. Let x^h lie in $C^h\cap Y$ for all h. Then $(x^1,\ldots,x^H)\in F$, so F is not empty. The set F is convex since Y and the C^h are convex.

The C^h are closed for all h, and, since they are bounded below by $\bar{z}$, they are contained in $\bar{z}+s^+$. Therefore, Choquet's theorem implies that C is closed by the argument used for Lemma 3.2. Then $F=C\cap Y$ is closed as an intersection of closed sets. It only remains to prove that $C\cap Y$ is compact.

We first show that bounded net inputs yield bounded outputs at time τ. Suppose not. Then there is a sequence y^s, $s=1,2,\ldots$, contained in $C\cap Y$ such that y^s_τ is unbounded above. Consider $y^s_t/|y^s_\tau|\to 0$ for all $t<\tau$ and $\bar{y}_\tau=y^s_\tau/|y^s_\tau|>0$ where $|\bar{y}_\tau|=1$. This implies that for large s Assumption 2′ is violated. Therefore no such sequence can exist. Since τ is arbitrary, this implies F is bounded in every component. Since F is closed, F is compact in the product topology by Tychonoff's theorem. $\square$

The compactness of F allows the existence of a utility function and the nonemptiness of the core to be proved in the same way as for the Malinvaud model. Then the existence of the Edgeworth equilibrium proceeds as before. Lemmas 3.1, 3.2, and 3.3 are proved as before. However, Lemma 3.4 must be reformulated to take account of Assumption 2′. We replace Lemma 3.4 with

Lemma 2. For any ε, $0<\varepsilon<1$, there is $p\in(s^n)^*$ such that $p\in S(\varepsilon)$ with $|p_0|>0$. Moreover, whenever $p\in S(\varepsilon)$, $py\leq 0$ for all $y\in Y\cap \ell_\infty$.

Proof. The proof of the first proposition of Lemma 2 remains the same. For the second proposition let $y\in Y\cap \ell_\infty$ and $p\in S(\varepsilon)$. Since $Y=Y+Y$ and $0\in G-Y$, it follows that $-Y\subset G-Y$. Therefore, $pz>-\varepsilon|p_0|$ for all $z\in(G-Y)\cap\ell_\infty$ implies that $pz>-\varepsilon|p_0|$ for all $z\in Y\cap\ell_\infty$ by the first proposition of the Lemma. Since $\alpha z\in Y\cap \ell_\infty$ for any $\alpha>0$, it follows that $pz\leq 0$ for all $z\in Y\cap\ell_\infty$. $\square$

Lemma 3.5 is not changed. Lemmas 3.6 and 3.7, however, are replaced by

Lemma 3. The vector p^* satisfies $p^*w=\lim_{\tau\to\infty}\sum_{t=0}^{\tau}p_t^*w_t\geq 0$ for all $w\in G$ and $\limsup_{\tau\to\infty}\sum_{t=0}^{\tau}p_t^*y_t\leq 0$ for all $y\in Y$.

Proof. Write $z=\sum_{h=1}^{H}\alpha_h z^h$ with $z^h\in R^h(x^h)-Y$. Let $z(\tau)=(z_0,\ldots,z_\tau,0,\ldots)$. For $\varepsilon>0$ and τ large, $\hat{z}^h(\tau)=(\varepsilon e_0+z_0^h, z_1^h,\ldots,z_\tau^h,0,\ldots)\in R^h(x^h)-Y$ by Assumption 7 and monotonicity. Since it is a finite vector, $\hat{z}(\tau)=\sum_{h=1}^{H}\alpha_h\hat{z}^h(\tau)\in(G-Y)\cap\ell_\infty$. Apply $\bar{p}\in S$ to obtain $\varepsilon|p^*|+\sum_{h=1}^{H}\alpha_h p^*z^h(\tau)=\bar{p}\hat{z}(\tau)\geq 0$ for τ large. Letting $\tau\to\infty$ we find $\varepsilon|p_0|+\liminf_{\tau\to\infty}p^*\hat{z}(\tau)\geq 0$. Since ε was arbitrary, $\liminf_{\tau\to\infty}p^*z(\tau)\geq 0$. The proof that $p^*w\geq 0$ follows as before from the fact that C^h is bounded below by $\bar{z}\in\ell_\infty$. As we have seen, $-Y\subset G-Y$. Therefore, $y\in Y$ implies $\liminf_{\tau\to\infty}p^*(-y(\tau))\geq 0$ or $\limsup_{\tau\to\infty}\sum_{t=0}^{\tau}p_t^*y_t\leq 0$, for all $y\in Y$. $\square$

We may now prove

Theorem 1. Under Assumptions 1, 2′, and 3–7, the economy $\mathscr{E}'$ has a competitive equilibrium (in the new definition).

Proof. Consider $(p^*, y, x^1,\ldots,x^H)$ where $(x^1,\ldots,x^H)\in K$, $y=\sum_{h=1}^{H}x^h$, and p^* is the ℓ_1 part of $\bar{p}\in S$. Since $x^h\in G$, Lemma 3 implies $p^*x^h\geq 0$. Lemma 3 also implies $\limsup_{\tau\to\infty}\sum_{t=0}^{T}p^*z\leq 0$ for all $z\in Y$. Then it follows that $p^*y=0$. Therefore, $p^*x^h=0$ for all h. The proof that $z^h\in P^h(x^h)$ implies $p^*z^h>0$ goes just as before. This establishes Condition I of competitive equilibrium. Condition II is provided by Lemma 4. Condition III follows from the definition of y. $\square$

V. *The Theorem of Back-Bewley*

As we have mentioned the theorem of Back is based on Theorem 1 of Bewley which establishes the existence of competitive equilibrium in the space $\mathscr{L}_\infty$ under assumptions fairly close to ours except in the space and topology used. However, the equilibrium price vector for the discrete model found by Bewley in his Theorem 1 is only known to lie in the space ℓ_{ba} which is dual to ℓ_∞ endowed with the sup norm topology. In this space the price functionals are sums of vectors in ℓ_1 and purely finitely additive measures, which are zero on all vectors having only a finite number of components. Back shows that with some additional assumptions the purely finitely additive measures may be excluded from the price functionals, so that valuations of goods vectors become inner products with price vectors in ℓ_1.

Besides using ℓ_∞ for the goods space and the Mackey topology rather than the product topology, the Bewley-Back assumptions are very close to ours except for Assumption 7. Their assumption corresponding to the first part of Assumption 7 is the so-called *Adequacy Assumption*. For a convex closed production set $Y \subset \ell_\infty$, which contains 0, let $A(Y)$ be the asymptotic cone of Y, that is, the largest convex cone at 0 contained in Y. The assumption in Back (1984) is

Adequacy Assumption. For each trader h, there exist $\bar{x}^h \in C^h$ and $\bar{y}^h \in A(Y)$ such that $\bar{x}^h - \bar{y}^h < -\delta < 0$.

This is stronger than our assumption which only requires the strict inequality hold for the whole set of traders. Of course there is also a difference in the production set assumed. Bewley in effect uses our production set. However, Back (1984) assumes that firms own convex closed production sets containing the origin. Note that in the absence of entrepreneurial factors the asymptotic cone may equal the origin.

Corresponding to the second part of Assumption 7 Bewley used two assumptions for his production model (Theorem 3), one on the consumption side and one on the production side. For consumption he assumed that the possible consumption set C^h is the positive orthant. This allowed him to truncate any possible consumption stream and remain in the possible consumption set. That is, if $w^h = (w_0^h, \ldots)$ is a possible consumption for the hth trader, so is $(w_0, \ldots, w_\tau, 0, \ldots)$. In terms of our trading sets this amounts to assuming that the greatest lower bound of the trading set is in the trading set.

On the production side he made an *Exclusion Assumption*. In the discrete model this assumption may be stated in the form.

Exclusion Assumption. If $y \in Y$ then $(y_0, \ldots, y_\tau, 0, \ldots) \in Y$ for all $\tau > 0$.

This assumption is valid in the Malinvaud model provided there is free disposal. Free disposal is needed since $y_\tau = v_\tau - u_\tau$ while $y(\tau+1) = (u_\tau, v_{\tau+1}) \in Y_{\tau+1}$. But $y_{\tau+1} = 0$, so either $u_t = v_t$ for all $t \geq \tau + 1$, or $v_{\tau+1} = 0$. Either alternative amounts to free disposal. The Exclusion Assumption is needed to establish that Y is supported at the equilibrium net output vector y by a price functional in ℓ_1.

Rather than making separate assumptions on C^h and Y we make an assumption on vectors in $G - Y$. Thus our version of the adequacy assumption is joint on the trading sets and the production set. Given $w^h \in C^h$, let z be an arbitrary vector in $R^h(w^h) - Y$. Suppose additional amounts of all goods are available at $t = 0$. We assume in Assumption 7 that some multiple of a sufficiently late tail of $\bar{x}^h$ may be substituted for the corresponding tail of z and the resulting vector will remain in $R^h(w^h) - Y$. This assumption is weaker than the assumption $C^h = b + s^n$ plus the exclusion assumption. We show that this assumption is in fact a theorem in the classical model of capital accumulation.

Back accepts Bewley's Adequacy Assumption implicitly since he assumes Bewley's Theorem 1. Also he states the implication of the assumption for the whole set of traders explicitly. However he replaces the assumption that $C^h = \ell_\infty^+$ and the assumption that any $y \in Y$ may be truncated and remain in Y by a property which he assumes is possessed by the sets C^h and Y separately. This property is referred to as Property M (for mixture) in Back (1984). He attributes the idea to Majumdar (1972), but it goes back at least to Radner (1967), in the context of supporting efficient points in a production set over an infinite horizon. For any $\pi \in \ell_\infty$ write $\pi = \pi_c + \pi_p$ where $\pi_c \in \ell_1$ and π_p is a purely finitely additive measure. This is possible by the theorem of Yosida-Hewitt. We will state Back's assumption for the period model.

Mixture Property. Let z, z' be arbitrary vectors in $Z \subset \ell_\infty$. Let $z(\tau) = (z_0, \ldots, z_\tau, z'_{\tau+1}, \ldots)$. Consider a sequence $\{z^1, z^2, \ldots\}$ where $z^s \in \ell_\infty$ such that $\pi_p z^s \to 0$ and $z^s \to 0$ pointwise. Then the z^s may be chosen so that $z(\tau) + z^s \in Z$ for all $\tau \geq 0$.

The Mixture Property is assumed for each of the sets C^h and Y. The relation to Assumption 7 is easy to see. Make the assumption of the Mixture Property for the sets $Z = R^h(w) - Y$ but only for $z' = \alpha \bar{x}^h$ for some $\alpha \geq 0$ and for large τ. Choose $z^n = (\varepsilon^n e_0, 0, \ldots)$. Then Assumption 7 follows.

Other significant differences between our paper and Back's paper are that we use s^n while he uses ℓ_∞ and he uses the Mackey topology while we use the product topology (the topology of pointwise convergence). As explained earlier we prefer the space s^n to the space ℓ_∞ since then all the constraints on the consumer's choice are derived from the budget. However, the Mackey topology is not available for this space. On the other hand, if the model is confined to the space ℓ_∞ the Mackey and the product topology are the same on sets which are bounded in the topology of the sup norm. As Stigum pointed out early, the fact that the feasible set is bounded allows the proof to be begin with net trading sets which are truncated so that they are compact. This is the way the proof is sometimes given in the case of finitely many goods. Then it can be shown that an equilibrium in the truncated economy is still an equilibrium in the original economy using the same argument as in the finite case.

Finally it is a major difference between our papers that we prove an equivalence theorem for Edgeworth and competitive equilibria. This is a result in the tradition of Peleg-Yaari and Aliprantis, Brown, and Burkinshaw. However, they assume $C^h = -b + s^n_+$.

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