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OPTIMAL GROWTH UNDER UNCERTAINTY: A COMPLETE CHARACTERIZATION OF WEAKLY MAXIMAL PROGRAMS*

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Abstract

In this paper a general reduced model of capital accumulation under uncertainty is presented, and the existence of a price system supporting the weakly maximal program of capital accumulation is proved. Also, the conditions under which such a price system is a sequence of integrable functions are shown. In addition, the weakly maximal program is characterized by using a price system.

I. *Introduction*

The purpose of this paper is to prove the existence of price systems for weakly maximal programs of capital accumulation under uncertainty and to characterize the weak maximality of programs. The result established in this paper is a generalization of those of Radner (1973) and Zilcha (1976). Also it includes the results in deterministic cases by McKenzie (1986).

In the economy considered in this paper, there is uncertainty in production technologies and utility functions. In each period in time, the current production technology and utility function are certain, but future technologies and utilities thereafter are uncertain. The economic model presented in this paper is a general reduced form which includes many cases of economic application.

In proving the existence of price systems supporting weakly maximal programs, there are two key arguments. One is an induction argument for the proof of existence of prices, which has been developed by McKenzie (1986) in deterministic models and first applied by Zilcha (1976) to models with uncertainty. The other is a decomposition theorem on finitely additive measures by Yosida & Hewitt (1952), which has been used in proving the integrability of prices, first by Bewley (1972) in general equilibrium models, and also by Radner (1973) and Zilcha (1976) in growth models with uncertainty. In their growth models, uncertainty exists only in production technologies and is assumed to be stationary.

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In this paper we shall consider a general non-stationary model with uncertainty in both production technology and utility, and prove a general support price theorem for weakly maximal programs of capital accumulation.

This paper is formulated in the following fashion. In section 2 we shall construct a general reduced model of capital accumulation under uncertainty. In section 3 an existence theorem of a price system for the weakly maximal program will be proved. In section 4 the conditions under which price systems are integrable functions will be shown. In section 5 the weakly maximal program will be characterized by a price system.

II. *A General Reduced Model*

First we shall present a general reduced model of capital accumulation in which future utilities and production technologies are uncertain.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Each element in Ω denotes a possible state of nature, which may be interpreted as a stream of environments in all past, present, and future periods. Family $\mathscr{F}$ is the set of all possible events and P denotes the probability distribution of states.

Let $T=\{0, 1, 2, \dots\}$ be the space of time. The uncertainty of states is described by a filtration $\{\mathscr{F}_t \mid t\in T\}$, i.e., $\mathscr{F}_t$ is a σ-sub-algebra of $\mathscr{F}$ such that $\mathscr{F}_t\subset\mathscr{F}_{t+1}$ for all $t\in T$. Each family $\mathscr{F}_t$ is interpreted as the information about states that will become known up to period t.

The production technology available at each period $t>0$ is described by a relation D_t: $\Omega\to R^m\times R^m$, that is,

$$\omega\in\Omega\longrightarrow D_t(\omega)\subset R^m\times R^m,$$

where R^m denotes an m-dimensional Euclidean space. We assume that the graph of D_t defined by

$$G(D_t)=\{(x, y, \omega)\mid(x, y)\in D_t(\omega)\}$$

is $\mathscr{B}(R^m)\times\mathscr{B}(R^m)\times\mathscr{F}_t$-measurable, where $\mathscr{B}(R^m)$ is the family of all Borel subsets of R^m. By $D_t(\omega)$ we represent the possibility of transformation of capital stocks. That is, $(x, y)\in D_t(\omega)$ means that under state ω it is possible to transform capital stock x at time t-1 into capital stock y at time t.

The satisfaction in the economy at each period $t>0$ is described by a utility function u_t: $G(D_t)\to R\cup\{-\infty\}$. that is,

$$(x, y, \omega)\in G(D_t)\longrightarrow u_t(x, y, \omega)\in R\cup\{-\infty\},$$

where R denotes the real line. We assume that u_t is a $\mathscr{B}(R^m)\times\mathscr{B}(R^m)\times\mathscr{F}_t$-measurable function, which may take value $-\infty$. Value $u_t(x, y, \omega)$ is interpreted as the maximum level of social welfare under state ω attained in period between time t-1 to time t if capital stocks at time t-1 and time t are x and y respectively.

In order to show a program of capital accumulation, we will use a stochastic process, i.e., a function $K: T \times \Omega \rightarrow R^m$ such that $K(t, \cdot)$ is $\mathscr{F}_t$-measurable for each $t \in T$. To denote a stochastic process $K: T \times \Omega \rightarrow R^m$, we also write as $K = \{k_t | t \in T\}$, where k_t is a function defined by $k_t(\omega) = K(t, \omega)$. In addition, we shall restrict ourselves only to essentially bounded processes, and namely assume that $k_t \in \mathscr{L}_\infty(\mathscr{F}_t)$ for all $t \in T$, where $\mathscr{L}_t(\mathscr{F}_t)$ is the set of all essentially bounded $\mathscr{F}_t$-measurable functions on Ω to R^m with respect to P.

The set of all possible ways to transform capital stocks from time t-1 to time t is defined by

$$\mathscr{D}_t = \{(f, g) \in \mathscr{L}_\infty(\mathscr{F}_{t-1}) \times \mathscr{L}_\infty(\mathscr{F}_t) \,|\, (f(\omega), g(\omega)) \in D_t(\omega) \text{ a.s.}\}.$$

Definition 2.1: A stochastic process $K = \{k_t | t \in T\}$ is called a *program* if $k_t \in \mathscr{L}_\infty(\mathscr{F}_t)$ and $(k_{t-1}(\omega), k_t(\omega)) \in D_t(\omega)$ a.s. for each $t > 0$, i.e., $(k_{t-1}, k_t) \in \mathscr{D}_t$ for all $t > 0$.

In a program $K = \{k_t | t \in T\}$, for each $t \in T$, k_t is a random variable and $k_t(\omega)$ denotes quantities of capital stock planned to accumulate at time t in state ω. Since D_t is $\mathscr{B}(R^m) \times \mathscr{B}(R^m) \times \mathscr{F}_t$-measurable, production technology D_t is perfectly known in determining capital stock k_t at time t. However, in determining k_{t-1} at time t-1, production technology D_t is unknown. In this sense, uncertainty exists in production technologies. Similarly, utility function u_t is perfectly known in determining capital stock k_t at time t, but unknown in determining k_{t-1} at time t-1. Thus, uncertainty also exists in utility functions.

For a program $K = \{k_t | t \in T\}$, we denore, by $U_t(K)$, the sum of expected utilities that will be obtained up to time t by program K. Namely, under some appropriate conditions which will be shown later, we can define

$$U_t(K) = \sum_{s=1}^{t} \int_\Omega u_s(k_{s-1}(\omega), k_s(\omega), \omega) dP(\omega).$$

Since value $U_t(K)$ may become infinity as t goes to $+\infty$, the so-called overtaking criterion should be used to evaluate programs.

Definition 2.2: A program $K = \{k_t | t \in T\}$ is said to be *weakly maximal* if any other program starting from the same initial condition can not overtake program K, i.e., there is no other program $K' = \{k_t' | t \in T\}$ with $k_0 = k_0'$ such that

$$\liminf_{t \to +\infty} [U_t(K') - U_t(K)] > 0.$$

The above definition is a generalization of maximization in usual problems where utility sums are bounded.

III. *Price Systems Supporting Weakly Maximal Programs*

In this section we shall establish a general version of the support price theorem. The following are basic assumptions for the model.

(A.1) (convexity) For each t and ω, $D_t(\omega)$ is convex and $u_t(x, y, \omega)$ is concave in (x, y).

(A.2) (boundedness) For each t, for any $\alpha>0$ there exists a number β such that $(x, y)\in D_t(\omega)$ and $||x||\leqq\alpha$ imply $||y||\leqq\beta$ and $u_t(x, y, \omega)\leqq\beta$.

Assumption (A.1) is the convexity of the model, and means that production sets are convex and utility functions are concave. Assumption (A.2) is the boundedness of the model in each period, and means that, if capital stock at time $t-1$ is bounded, then capital stock and utility at time t are also bounded.

Remark 3.1: Assumption (A.2) implies that given $f\in\mathscr{L}_\infty(\mathscr{F}_{t-1})$, there exists a number b_0 such that $||g||_\infty\leqq b_0$ and $\int u_t(f, g, \cdot)dP\leqq b_0$ for all g with $(f, g)\in\mathscr{D}_t$.

We do not have to take into consideration programs which are obviously bad. Let us consider a program $K^*=\{k^*_t\,|\,t\in T\}$ satisfying the following condition.

(C.1) $\int u_t(k^*_{t-1}, k^*_t, \cdot)dP>-\infty$ for each $t>0$.

Remark 3.2: Under assumption (A.2), condition (C.1) implies that in program K^*, $\int u_t(k^*_{t-1}, k^*_t, \cdot)dP$ is finite for each $t>0$. Thus, value $U_t(K^*)$ is well-defined, and the definition of weak maximality can be applied to program K^*.

Given program K^*, for each $t\in T$ let us define a normalized utility function, v_t: $G(D_t)\rightarrow R\cup\{-\infty\}$, by

$$v_t(x, y, \omega)=u_t(x, y, \omega)-u_t(k^*_{t-1}(\omega), k^*_t(\omega), \omega).$$

For each $t\in T$ and $f\in\mathscr{L}_\infty(\mathscr{F}_t)$, we denote by $\mathscr{K}_t(f)$ the set of all feasible programs beginning with capital stock f at time t, i.e.,

$$\mathscr{K}_t(f)=\{K=\{k_t\,|\,t\in T\}\,|\,K \text{ is a stochastic process such that } k_t=f \text{ and } (k_s, k_{s+1})\in\mathscr{D}_{s+1} \text{ for each } s\geqq t\}.$$

Now, by virtue of Remark 3.2, we can define a normalized value function. For each $t\in T$, let us define a function V_t: $\mathscr{L}_\infty(\mathscr{F}_t)\rightarrow R$ by

$$V_t(f)=\sup_{K\in\mathscr{K}_t(f)}\left\{\liminf_{r\to+\infty}\sum_{s=t+1}^{r}\int_\Omega v_s(k_{s-1}, k_s, \cdot)dP\right\} \qquad \text{for } f\in\mathscr{L}_\infty(\mathscr{F}_t).$$

Here, we should note that functions V_t are defined for a particular program K^*, and they depend on the program.

Remark 3.3: It can be shown by definition of V_t that for each $t\in T$,

$$V_t(f) \geqq \int v_{t+1}(f, g, \cdot)dP + V_{t+1}(g) \qquad \text{for all } (f, g) \in \mathscr{D}_{t+1}.$$

In particular, if program $K^* = \{k^*_t \mid t \in T\}$ is weakly maximal, we can show that program K^* is agreeable, i.e.,

$$V_t(k_t^*) = \int v_{t+1}(k^*_t, k^*_{t+1}, \cdot)dP + V_{t+1}(k^*_{t+1}) \qquad \text{for all } t \in T.$$

For each $t \in T$, we define the effective domain of function V_t by

$$\mathscr{X}_t = \{f \in \mathscr{L}_\infty(\mathscr{F}_t) \mid V_t(f) > -\infty\}.$$

Also, for each t, we define a set by

$$\mathscr{Y}_t = \{g \mid (f, g) \in \mathscr{D}_t \quad \text{for some } f\}.$$

Furthermore, we assume the following condition for program K^*.

(C.2) $k^*_0 \in \operatorname{int} \mathscr{X}_0$ and $\operatorname{int}(\mathscr{Y}_t \cap \mathscr{X}_t) \neq \phi$ for all $t > 0$, where symbol "int" means the interior in the $\|\cdot\|_\infty$-topology for space $\mathscr{L}_\infty(\mathscr{F}_t)$.

By assumption (A.1), we can easily show that V_t is concave and $\mathscr{X}_t$ is convex for each $t \in T$. Also, for each $t > 0$, define a map Ψ_t: $\mathscr{D}_t \to R$ by

$$\Psi_t(f, g) = \int_\Omega u_t(f(\omega), g(\omega), \omega)dP.$$

Then, map Ψ_t can be shown to be concave under assumption (A.1).

For each $t \in T$, let $\mathscr{L}_\infty^*(\mathscr{F}_t)$ denote the dual space of $\mathscr{L}_\infty(\mathscr{F}_t)$, i.e., the set of all continuous linear functions on $\mathscr{L}_\infty(\mathscr{F}_t)$ to R. For each $t \in T$ and $k \in \mathscr{L}_\infty(\mathscr{F}_t)$, we define the set of subgradients of V_t at k by

$$\partial V_t(k) = \{\pi \in \mathscr{L}_\infty^*(\mathscr{F}_t) \mid V_t(k) + \pi \cdot (f - k) \geqq V_t(f) \quad \text{for all } f \in \mathscr{X}_t\}.$$

Moreover, for each $t > 0$ and $(k, k') \in \mathscr{L}_\infty(\mathscr{F}_{t-1}) \times \mathscr{L}_\infty(\mathscr{F}_t)$, we define the set of subgradients of Ψ_t at (k, k') by

$$\partial \Psi_t(k, k') = \{(\pi, \pi') \in \mathscr{L}_\infty^*(\mathscr{F}_{t-1}) \times \mathscr{L}_\infty^*(\mathscr{F}_t) \mid \Psi_t(k, k') + \pi \cdot (f - k) + \pi' \cdot (g - k') \geqq \Psi_t(f, g) \quad \text{for all } (f, g) \in \mathscr{D}_t\}.$$

Let us define prices of capital goods in program K^*.

Definition 3.1: We call $\{\pi_t \mid t \in T\}$ a *price system supporting* program K^*, if, for all $t \in T$, $\pi_t \in \partial V_t(k^*_t)$ and $(\pi_t, -\pi_{t+1}) \in \partial \Psi_{t+1}(k^*_t, k^*_{t+1})$, i.e.,

(1) $V_t(k^*_t) - \pi_t \cdot k_t^* \geqq V_t(f) - \pi \cdot f$ for all $f \in \mathscr{X}_t$.

(2) $\int u_{t+1}(k^*_t, k^*_{t+1}, \cdot)dP - \pi_t \cdot k_t^* + \pi_{t+1} \cdot k^*_{t+1}$

$\geqq \int u_{t+1}(f, g, \cdot)dP - \pi_t \cdot f + \pi_{t+1} \cdot g$ for all $(f, g) \in \mathscr{D}_{t+1}$.

Now we are ready to prove the existence of a price system supporting the weakly maximal program. First we shall prove the so-called induction lemma.

Lemma 3.1: Let $K^* = \{k^*_t | t \in T\}$ be a weakly maximal program satisfying conditions (C.1) and (C.2). Under assumptions (A.1) and (A.2), if $\pi_{t-1} \in \partial V_{t-1}(k^*_{t-1})$, then there exists $\pi_t \in \partial V_t(k^*_t)$ such that $(\pi_{t-1}, -\pi_t) \in \partial \Psi_t(k^*_{t-1}, k^*_t)$.

Proof: Assume that $\pi_{t-1} \in \partial V_{t-1}(k^*_{t-1})$, and define a number w_t by

$$w_t = \int u_t(k^*_{t-1}, k^*_t, \cdot)dP + V_t(k_t^*) - \pi_{t-1} \cdot k^*_{t-1}$$
$$= V_{t-1}(k^*_{t-1}) - \pi_{t-1} \cdot k^*_{t-1}.$$

Also, define two sets,

$$A = \{(w, g) \in R \times \mathscr{L}_\infty(\mathscr{F}_t) \,|\, w > w_t - \int u_t(f, g, \cdot)dP + \pi_{t-1} \cdot f$$
$$\text{for some } f \text{ with } (f, g) \in \mathscr{D}_t\},$$

and

$$B = \{(w, g) \in R \times \mathscr{L}_\infty(\mathscr{F}_t) \,|\, V_t(g) \geqq w\}.$$

Clearly, $(V_t(k^*_t), k^*_t) \in B$, and $(V_t(k^*_t), k^*_t) \in bd\ A$. By assumption (A.1), these sets are convex. In addition, since int $\mathscr{K}_t \neq \phi$ by condition (C.2), set B has non-empty interior.

Suppose $A \cap B \neq \phi$. Then there exists $(f, g) \in \mathscr{D}_t$ such that

$$V_t(g) > w_t - \int u_t(f, g, \cdot)dP + \pi_{t-1} \cdot f.$$

By the definition of value function V_t, we have

$$V_{t-1}(f) - \pi_{t-1} \cdot f > V_{t-1}(k^*_{t-1}) - \pi_{t-1} \cdot k^*_{t-1},$$

which implies that $\pi_{t-1} \notin \partial V_{t-1}(k^*_{t-1})$, a contradiction. Hence $A \cap B = \phi$.

By a separation theorem [Dunford & Schwartz (1964), Thm.V.2.8, p. 418], there exists a non-zero continuous linear function $(c, -\pi_t)$ on $R \times \mathscr{L}_\infty(\mathscr{F}_t)$, i.e., a number c and a function $\pi_t \in \mathscr{L}_\infty^*(\mathscr{F}_t)$ such that

$$c.w - \pi_t \cdot g \geqq c \cdot w' - \pi_t \cdot g' \quad \text{for all } (w, g) \in A \text{ and } (w', g') \in B.$$

This implies that

(3.1) $c[w_t - u_t\int(f, g, \cdot)dP + \pi_{t-1} \cdot f] - \pi_t \cdot g \geqq cV_t(g') - \pi_t \cdot g'$ for all $(f, g) \in \mathscr{D}_t$ and $g' \in \mathscr{K}_t$.

Suppose $c = 0$. (3.1) implies that $\pi_t \cdot (g' - g) \geqq 0$ for all $(f, g) \in \mathscr{D}_t$ and $g' \in \mathscr{K}_t$. Therefore, by condition (C.2), $\pi_t=0$, which is a contradiction to $(c, \pi_t)\neq 0$. Hence, since $c\geqq 0$ by the shapes of sets A and B, we can assume that $c=1$ without loss of generality.

Put $g'=k^*_t$ in (3.1). Then,

$$\int u_t(k^*_{t-1}, k^*_t, \cdot)dP-\pi_{t-1}\cdot k^*_{t-1}+\pi_t\cdot k^*_t\geqq\int u_t(f, g, \cdot)dP-\pi_{t-1}\cdot f+\pi_t\cdot g$$

for all $(f, g)\in\mathscr{D}_t$. This implies that $(\pi_{t-1}, -\pi_t)\in\partial\Psi_t(k^*_{t-1}, k^*_t)$.

Moreover, put $f=k^*_{t-1}$ and $g=k^*_t$ in (3.1). Then,

$$V_t(k_t^*)-\pi_t\cdot k^*_t\geq V_t(g')-\pi_t\cdot g' \quad \text{for all } g'\in\mathscr{K}_t,$$

which implies that $\pi_t\in\partial V_t(k_t^*)$. ■

Theorem 3.1: Let $K^*=\{k^*_t\,|\,t\in T]$ be a program satisfying conditions (C.1) and (C.2). Under assumptions (A.1) and (A.2), if program K^* is weakly maximal, there exists a price system supporting program K^*.

Proof: The theorem can be proved by an induction argument. Since $k_0^*\in\mathrm{int}\,\mathscr{K}_0$ by condition (C.2), by a separation argument there exists $\pi_0\in\partial V_0(k^*_0)$. Thus, by induction with respect to time t, Lemma 3.1 implies the existence of a price system $\{\pi_t\,|\,t\in T]$ supporting program K^*. ■

For each $t>0$ and $(k, k')\in\mathscr{L}_\infty(\mathscr{F}_{t-1})\times\mathscr{L}_\infty(\mathscr{F}_t)$, we define the set of partial subgradients of Ψ_t at (k, k') by

$$\partial_1\Psi_t(k, k')=[\pi\in\mathscr{L}_\infty^*(\mathscr{F}_{t-1})\,|\,(\pi, \pi')\in\partial\Psi_t(k, k')].$$

Then, we have the following theorem, which is usually called "the envelope theorem."

Theorem 3.2: Let $K^*=[k^*_t\,|\,t\in T]$ be a program satisfying conditions (C.1) and (C.2). Under assumptions (A.1) and (A.2), if program K^* is weakly maximal, then $\partial V_t(k^*_t)\subset\partial_1\Psi_{t+1}(k^*_t, k^*_{t+1})$ for all $t\in T$.

Proof: The theorem follows immediately from Lemma 3.1. ■

IV. *Integrable Price Systems*

For each $t\in T$, let $ba(\mathscr{F}_t)$ denote the set of all bounded finitely additive m-dimensional vector-valued measures on $\mathscr{F}_t$ absolutely continuous with respect to P. Also, let $\mathscr{L}_1(\mathscr{F}_t)$ denote the set of all integrable $\mathscr{F}_t$-measurable functions on Ω to R^m.

Remark 4.1: By a theorem [Dunford & Schwarts (1964), Thm.IV. 8.16, p. 296], $\mathscr{L}_\infty^*(\mathscr{F}_t)$ can be identified with $ba(\mathscr{F}_t)$, and for each $\pi\in\mathscr{L}_\infty^*(\mathscr{F}_t)$,

$$\pi \cdot f = \int f d\pi \qquad \text{for all } f \in \mathscr{L}_\infty(\mathscr{F}_t),$$

where π is also regarded as an element of $ba(\mathscr{F}_t)$. In addition, if π is countably additive, then, by the Radon-Nikodym theorem [Dunford & Schwartz 1964, Thm.III.10.7, p. 181], there is a unique derivative of π, say, $p \in \mathscr{L}_1(\mathscr{F}_t)$ such that

$$\int f d\pi = \int p \cdot f dP \qquad \text{for all } f \in \mathscr{L}_\infty(\mathscr{F}_t).$$

Thus, $\mathscr{L}_1(\mathscr{F}_t)$ can be regarded as a subset of $ba(\mathscr{F}_t)$, or $\mathscr{L}_\infty{}^*(\mathscr{F}_t)$.

Remark 4.2: If $\pi \in ba(\mathscr{F}_t)$ is non-negative, then, by a theorem [Yosida & Hewitt (1952), Thm.1.23, p. 52], π can be uniquely decomposed into two parts, that is, there exist unique $\pi_c \geqq 0$ and $\pi_p \geqq 0$ in $ba(\mathscr{F}_t)$ such that π_c is countably additive and π_p is purely finitely additive, and such that

$$\pi = \pi_c + \pi_p .$$

Therefore, by the Radon-Nikodym theorem, there is a unique derivative of π_c, say, $p \in \mathscr{L}_1(\mathscr{F}_t)$ such that

$$\int f d\pi_c = \int p \cdot f dP \qquad \text{for all } f \in \mathscr{L}_\infty(\mathscr{F}_t).$$

Since $\mathscr{L}_1(\mathscr{F}_t) \subset \mathscr{L}_\infty{}^*(\mathscr{F}_t)$, for $p \in \mathscr{L}_1(\mathscr{F}_t)$ we can write $p \in \partial V_t(k)$ if and only if $V_t(k) + \int p \cdot (f-k) dP \geqq V_t(f)$ for all $f \in \mathscr{K}_t$. Also, for $p \in \mathscr{L}_1(\mathscr{F}_t)$ and $p' \in \mathscr{L}_1(\mathscr{F}_{t+1})$ we can write $(p, -p) \in \partial \Psi_{t+1}(k, k')$ if and only if $\int u_{t+1}(k, k', \cdot) dP + \int p \cdot (f-k) dP - \int p' \cdot (g-k') dP \geqq \int u_{t+1}(f, g, \cdot) dP$ for all $(f, g) \in \mathscr{D}_{t+1}$.

Now, we are interested in a price system $\{p_t | t \in T\}$ such that $p_t \in \mathscr{L}_1(\mathscr{F}_t)$ for all $t \in T$.

Definition 4.1: We call $\{p_t | t \in T\}$ an *$\mathscr{L}_1$-price system supporting program K^**, if, for all $t \in T$, $p_t \in \mathscr{L}_1(\mathscr{F}_t)$, $p_t \in \partial V_t(k^*_t)$, and $(p_t, -p_{t+1}) \in \partial \Psi_{t+1}(k^*_t, k^*_{t+1})$, i.e.,

(1) $V_t(k_t^*) - \int p_t \cdot k_t^* dP \geqq V_t(f) - \int p_t \cdot f dP \quad$ for all $f \in \mathscr{K}_t$.

(2) $\int u_{t+1}(k^*_t, k^*_{t+1}, \cdot) dP - \int p_t \cdot k^*_t dP + \int p_{t+1} \cdot k^*_{t+1} dP$

$\qquad \geqq \int u_{t+1}(f, g, \cdot) dP - \int p_t \cdot f dP + \int p_{t+1} \cdot g dP \quad$ for all $(f, g) \in \mathscr{D}_{t+1}$,

where V_t is the normalized value function for program K^*,

In order to get an $\mathscr{L}_1$-price system, the following lemma is useful.

Lemma 4.1: If $(\pi, -\pi')\in\partial\Psi_{t+1}(k, k')$, $\pi\geqq 0$, and $\pi'\geqq 0$, then $(p, -p')\in\partial\Psi_{t+1}(k, k')$, where $p\in\mathscr{L}_1(\mathscr{F}_t)$ and $p'\in\mathscr{L}_1(\mathscr{F}_{t+1})$ are the derivatives of the countably additive parts π_c of π and π_c' of π' respectively.

Proof: Since $\pi\geqq 0$ and $\pi'\geqq 0$, by Remark 4.2, π and π' can be decomposed uniquely into a countable additive part and a purely finitely additive part. Namely, π is decomposed into $\pi_c\in ba(\mathscr{F}_t)$ and $\pi_p\in ba(\mathscr{F}_t)$, and π' into $\pi_c'\in ba(\mathscr{F}_{t+1})$ and $\pi_p'\in ba(\mathscr{F}_{t+1})$. Also, let us denote the derivative of π_c by p and that of π_c' by p'.

Moreover, by a theorem [Yosida & Hewitt (1952), Thm.1.22, p. 52], for π_p there is a sequence $A_n\in\mathscr{F}_t$ such that $A_n\subset A_{n+1}$ and $\pi_p(A_n)=0$ for all n, and such that $\lim_n P(A_n)=1$. Since π_p' is also a purely finitely additive measure defined on $\mathscr{F}_t$, there is a sequence $B_n\in\mathscr{F}_t$ such that $B_n\subset B_{n+1}$ and $\pi_p'(B_n)=0$ for all n, and such that $\lim_n P(B_n)=1$. Define $C_n=A_n\cap B_n$. Then, $C_n\in\mathscr{F}_t$, $C_n\subset C_{n+1}$, $\pi_p(C_n)=\pi_p'(C_n)=0$ for all n, and $\lim_n P(C_n)=1$.

Now, let $(f, g)\in\mathscr{D}_{t+1}$. For each n, define functions f_n and g_n by

$$(f_n(\omega), g_n(\omega))=\begin{cases}(f(\omega), g(\omega)) & \text{for } \omega\in C_n\\ (k(\omega), k'(\omega)) & \text{otherwise,}\end{cases}$$

Then, $(f_n, g_n)\in\mathscr{D}_{t+1}$. And, since $\pi_p(C_n)=\pi_p'(C_n)=0$, we have

$$\begin{aligned}\pi\cdot(k-f_n)&=\pi_c\cdot(k-f_n)+\pi_p\cdot(k-f_n)\\&=\int p\cdot(k-f_n)dP+\int_{C_n}(k-f_n)d\pi_p\\&=\int p\cdot kdP-\int p\cdot f_n dP\end{aligned}$$

and

$$\begin{aligned}\pi'\cdot(k'-g_n)&=\pi_c'\cdot(k'-g_n)+\pi_p'\cdot(k'-g_n)\\&=\int p'\cdot(k'-g_n)dP+\int_{C_n}(k'-g_n)d\pi_p'\\&=\int p'\cdot k'dP-\int p'\cdot g_n dP\end{aligned}$$

Thus, since $(\pi, -\pi')\in\partial\Psi_{t+1}(k, k')$, we have

$$\int u_{t+1}(k, k', \cdot)dP+\int p\cdot(f_n-k)dP-\int p'\cdot(g_n-k')dP\geqq\int u_{t+1}(f_n, g_n, \cdot)dP$$

for all n. Since $\lim_n P(C_n)=1$, we have in the limit

$$\int u_{t+1}(k, k', \cdot)dP+\int p\cdot(f-k)dP-\int p'\cdot(g-k')dP\geqq\int u_{t+1}(f, g, \cdot)dP. \quad \blacksquare$$

This completes the proof of the lemma.

In order to insure the non-negativity of prices we assume:
(A.3) (monotonicity) If $(x, y)\in D_t(\omega)$, $x\leqq x'$, and $x\neq x'$, then $(x', y)\in D_t(\omega)$ and $u_t(x, y, \omega)<u_t(x', y, \omega)$.

The above assumption is the monotonicity of utility functions with respect to initial capital stock at each period. Now, under this assumption, we are ready to prove:

Theorem 4.1: Let $K^*=\{k^*_t \mid t\in T\}$ be a program satisfying conditions (C.1) and (C.2). Under assumptions (A.1), (A.2), and (A.3), if program K^* is weakly maximal, then there exists an $\mathscr{L}_1$-price system $\{p_t \mid t\in T\}$ such that $(p_t, -p_{t+1})\in\partial\Psi_{t+1}(k^*_t, k^*_{t+1})$ for all $t\in T$.

Proof: Theorem 3.1 implies the existence of a price system $\{\pi_t \mid t\in T\}$ supporting program K^*. Assumption (A.3) implies that $\pi_t\geqq 0$ for all $t\in T$. Therefore, this theorem follows from Lemma 4.1.

In order to prove the existence of an $\mathscr{L}_1$-price system supporting a program K^*, we need to assume the interiority of the program. For each $f\in\mathscr{L}_\infty(\mathscr{F}_{t-1})$, define

$$\mathscr{Y}_t(f)=\{g\in\mathscr{L}_\infty(\mathscr{F}_t) \mid (f, g)\in\mathscr{D}_t\}.$$

At time t, given k^*_{t-1}, we choose k^*_t from set, $\mathscr{Y}_t(k^*_{t-1})$. The following condition means that k^*_t is chosen in the interior of $\mathscr{Y}_t(k^*_{t-1})$.

(C.3) (interiority) $k^*_t\in\operatorname{int}\mathscr{Y}_t(k^*_{t-1})$ for all $t>0$.

For a weakly maximal program satisfying the above interiority condition, we can prove the existence of an $\mathscr{L}_1$-price system supporting it.

Theorem 4.2: Let $K^*=\{k^*_t \mid t\in T\}$ be a program satisfying conditions (C.1), (C.2), and (C.3). Under assumptions (A.1), (A.2), and (A.3), if program K^* is weakly maximal, then there exists an $\mathscr{L}_1$-price system supporting program K^*.

Proof: By Theorem 3.1 we have a price system $\{\pi_t \mid t\in T\}$ supporting program K^*, i.e., $\pi_t\in\partial V_t(k^*_t)$ and $(\pi_t, -\pi_{t+1})\in\partial\Psi_{t+1}(k^*_t, k^*_{t+1})$ for all $t\geqq 0$. Assumption (A.3) implies that $\pi_t\geqq 0$ for all $t\in T$. Therefore, by Lemma 4.1, we have $\{p_t \mid t\in T\}$, where p_t is the derivative of the countably additive part of π_t for each t, such that $(p_t, -p_{t+1})\in\partial\Psi_{t+1}(k^*_t, k^*_{t+1})$ for all $t\geqq 0$. Therefore, it suffices only to prove that $p_t\in\partial V_t(k^*_t)$ for all $t\geqq 0$.

Suppose that π_t were not countably additive, there would exist a sequence of sets $A_n\in\mathscr{F}_t$ such that $A_n\subset A_{n+1}$ and $\pi_t(A_n)\leqq\pi_t(\cup_k A_k)-w$ for all n, where $w\in R^m{}_+$, and $w\neq 0$. Let $B_n=A_n\cup(\Omega\setminus\cup_k A_k)$. Then, $\cup_k B_k=\Omega$ and

$$\pi(B_n)\leqq\pi_t(\Omega)-w\,. \tag{4.1}$$

for all n. Define $f\in\mathscr{L}_\infty(\mathscr{F}_{t-1})$ by

$$f(\omega)=k^*_{t-1}(\omega)+\delta w\,, \tag{4.2}$$

where δ is a positive number. Also, for each n, define $g_n \in \mathscr{L}_\infty(\mathscr{F}_t)$ by

$$(4.3) \qquad g_n(\omega) = \begin{cases} k^*_t(\omega) & \text{for } \omega \in B_n \\ k^*_t(\omega) + \delta\pi_{t-1}(\Omega) & \text{otherwise}. \end{cases}$$

Here, by condition (C.3), we can choose a sufficiently small δ such that $(k^*_{t-1}, g_n) \in \mathscr{D}_t$. Therefore, by assumption (A.3), $(f, g_n) \in \mathscr{D}_t$. Moreover, by assumption (A.3), $\int u_t(k^*_{t-1}, k^*_t, \cdot)dP < \int u_t(f, k^*_t, \cdot)dP$. Therefore, since $\lim_n P(B_n) = 1$,

$$\int u_t(k^*_{t-1}, k^*_t, \cdot) < \int u_t(f, g_n, \cdot)dP.$$

for all sufficiently large n. Since $(\pi_{t-1}, -\pi_t) \in \partial u_t(k^*_{t-1}, k^*_t)$, we have, by (4.1), (4.2), and (4.3),

$$\begin{aligned} 0 &> \int u_t(k^*_{t-1}, k^*_t, \cdot)dP - \int u_t(f, g_n, \cdot)dP \\ &\geqq \pi_{t-1}\cdot(k^*_{t-1} - f) - \pi_t\cdot(k^*_t - g_n) \\ &= -\delta\pi_{t-1}(\Omega)\cdot w + \delta\pi_{t-1}(\Omega)\cdot\pi_t(\Omega\backslash B_n) \\ &= -\delta\pi_{t-1}(\Omega)\cdot w + \delta\pi_{t-1}(\Omega)\cdot(\pi_t(\Omega) - \pi_t(B_n)) \\ &\geqq -\delta\pi_{t-1}(\Omega)\cdot w + \delta\pi_{t-1}(\Omega)\cdot w = 0, \end{aligned}$$

a contradiction. This proves the countable additivity of π_t. Thus, $p_t = \pi_t$ for all $t > 0$. Therefore, $p_t \in \partial V_t(k^*_t)$ for all $t > 0$.

In particular, we have shown that $p_1 \in \partial V_1(k^*_1)$ and $(p_0, -p_1) \in \partial \Psi_1(k^*_0, k^*_1)$. Let $(f, g) \in \mathscr{D}_t$. Then,

$$V_1(k^*_1) + \int p_1\cdot(g - k^*_1)dP \geqq V_1(g)$$

and

$$\int u_1(k^*_0, k^*_1, \cdot)dP + \int p_0\cdot(f - k^*_0)dP - \int p_1\cdot(g - k^*_1)dP \geqq \int u_1(f, g, \cdot)dP$$

Hence, by the above two inequalities,

$$\int u_1(k^*_0, k^*_1, \cdot)dP + V_1(k^*_1) + \int p_0\cdot(f - k^*_0)dP \geqq \int u_1(f, g, \cdot)dP + V_1(g),$$

Thus, by definition of the value function V_0, we can conclude that

$$V_0(k^*_0) + \int p_0\cdot(f - k^*_0)dP \geqq V_0(f).$$

This proves that $p_0 \in \partial V_0(k^*_0)$. ■

V. *Complete Characterization of Weakly Maximal Programs*

First we shall prove a fundamental theorem which shows a necessary and sufficient condition for weak maximality.

Theorem 5.1: Let $K^*=\{k^*_t \mid t\in T\}$ be a program satisfying conditions (C.1) and (C.2). Then, under assumptions (A.1) and (A.2), program K^* is weakly maximal if and only if $\lim_t V_t(k^*_t)=0$ and there exists a price system $\{\pi_t \mid t\in T\}$ supporting program K^*.

Proof: (Necessity) Assume that program K^* is weakly maximal. Then, by Theorem 3.1 we have a price system $\{\pi_t \mid t\in T\}$ supporting program K^*. Also, since program K^* is weakly maximal, by definition of V_t, $V_t(k_t^*)=0$ for all $t\in T$. Therefore, $\lim_t V_t(k^*_t)=0$.

(Sufficiency) Assume that K^* is a program satisfying conditions (C.1) and (C.2), and that $\lim_t V_t(k^*_t)=0$ and there exists a price system $\{\pi_t \mid t\in T\}$ supporting program K^*. First we shall show that program K^* is agreeable, i.e.,

$$V_0(k^*_0)=\sum_{s=1}^{t}\int v_s(k^*_{s-1}, k^*_s, \cdot)dP+V_t(k^*_t) \quad \text{for all } t>0\,. \tag{5.1}$$

where v_s's are the normalized utility functions for program K^*.

Suppose that (5.1) were not true. Then, by definition of V_0, there is $t'>0$ such that

$$V_0(k^*_0)>\sum_{s=1}^{t'}\int v_s(k^*_{s-1}, k^*_s, \cdot)dP+V_t(k^*_s\cdot)\,.$$

Therefore, again by definition of V_0, there exists $K\in\mathscr{K}_0(k_0^*)$, $\varepsilon>0$, and $t''>t'$ such that

$$\sum_{s=1}^{t}\int v_s(k_{s-1}, k_s, \cdot)dP+V_t(k_t)>\sum_{s=1}^{t}\int v_s(k^*_{s-1}, k^*_s, \cdot)dP+V_t(k^*_t)+\varepsilon \quad \text{for all } t>t''\,. \tag{5.2}$$

On the other hand, since $\{\pi_t \mid t\in T\}$ is a price system supporting program K^*,

$$V_t(k_t^*)-\pi_t\cdot k^*_t\geqq V_t(k_t)-\pi_t\cdot k_t$$

and

$$\int u_t(k^*_{t-1}, k^*_t, \cdot)dP-\pi_{t-1}\cdot k^*_{t-1}+\pi_t\cdot k^*_t\geqq\int u_t(k_{t-1}, k_t, \cdot)dP-\pi_{t-1}\cdot k_{t-1}+\pi_t\cdot k_t$$

for all $t>0$. Since $k_0=k_0^*$, These inequalities imply that

$$\sum_{s=1}^{t}\int v_s(k^*_{s-1}, k^*_s, \cdot)dP+V_s(k^*_t)\leqq\sum_{s=1}^{t}\int v_s(k_{s-1}, k_s,\cdot)dP+V_t(k_t)$$

for all $t>0$. This contradicts (5.2). Thus, (5.1) has been proved.

Now, by (5.1) and by definition of functions v_t's, $V_0(k^*_0)=V_t(k^*_t)$ for all $t>0$. Therefore, since $\lim_t V_t(k^*_t)=0$, we can conclude that $V_0(k^*_0)=0$. Hence, by definition of v_t's,

$$V_0(k^*_0)=\sum_{s=1}^{+\infty}\int v_s(k^*_{s-1}, k^*_s, \cdot)dP.$$

By definition of V_0, this implies the weak maximality of program K^*. ∎

Now we can prove a useful theorem in the case where the sum of expected utilities is finite. We assume in addition the following:

(A.4) For each t, $(x, y)\in D_t(\omega)$ implies $x\geqq 0$.

(A.5) For each t and ω, $(0, 0)\in D_t(\omega)$ and $u_t(0, 0, \omega)\geqq 0$.

Assumption (A.4) means that capital stock must be non-negative. And, assumption (A.5) means possibility of inaction, namely that capital stock can be zero.

Theorem 5.2: Let $K^*=\{k^*_t \mid t\in T\}$ be a program satisfying conditions (C.1) and (C.2), and assume that $\sum_{t=1}^{+\infty}\int u_t(k^*_{t-1}, k^*_t, \cdot)dP$ exists and is finite. Then, under assumptions (A.1), (A.2), (A.3), (A.4), and (A.5), program K^* is weakly maximal if and only if there exists a price system $\{\pi_t \mid t\in T\}$ satisfying the following conditions:

(1) For each $t>0$,

$$\int u_t(k^*_{t-1}, k^*_t, \cdot)dP-\pi_{t-1}\cdot k^*_{t-1}+\pi_t\cdot k^*_t\geqq\int u_t(f, g_t, \cdot)dP-\pi_{t-1}\cdot f+\pi_t\cdot g$$

for all $(f, g)\in\mathscr{D}_t$.

(2) $\lim_{t\to+\infty} \pi_t\cdot k^*_t=0$.

Proof: (Necessity) Assume that program K^* is weakly maximal. Then, by Theorem 3.1 we have a price system $\{\pi_t \mid t\in T\}$ supporting program K^*. Therefore, for each $t\in T$, $V_t(k^*_t)-\pi_t\cdot k^*_t\geqq V_t(f)-\pi_t\cdot f$ for all $f\in\mathscr{L}(\mathscr{F}_t)$, where V_t is the normalized value function for program K^*. Hence, by putting $f=0$, we have

$$V_t(k^*_t)-V_t(0)\geqq\pi_t\cdot k^*_t \quad \text{for all } t.$$

In addition, by definition of V_t and assumption (A.5),

$$V_t(0)\geqq\sum_{s=t+1}^{+\infty}\int v_s(0, 0, \cdot)dP\geqq-\sum_{s=t+1}^{+\infty}\int u_s(k^*_{s-1}, k^*_s, \cdot)dP$$

for all $t>0$, where v_t's are the normalized utility functions for program K^*. Thus, by the above two inequalities, we have

$$V_t(k_t^*)+\sum_{s=t+1}^{+\infty}\int u_s(k^*_{s-1}, k^*_s, \cdot)dP \geqq \pi_t\cdot k^*_t \quad \text{for all } t>0.$$

By Theorem 5.1, $\lim_{t\to+\infty} V_t(k^*_t)=0$. Also, since $\sum_{t=1}^{+\infty}\int u_t(k^*_{t-1}, k^*_t, \cdot)dP$ is finite,

$$\lim_{t\to+\infty}\sum_{s=t+1}^{+\infty}\int u_s(k^*_{s-1}, k^*_t, \cdot)dP=0\,.$$

Hence, the above inequality implies that $\lim_{t\to+\infty}\sup \pi_t\cdot k^*_t \leqq 0$. On the other hand, (A.4) impies that $k^*_t \geqq 0$, and (A.3) implies that $\pi_t \geqq 0$. Hence, $\pi_t\cdot k^*_t \geqq 0$ for all $t>0$. Therefore, we can conclude that $\lim_{t\to+\infty} \pi_t\cdot k^*_t=0$.

(Sufficiency): Let $K^*=\{k^*_t \mid t\in T\}$ be a program satisfying conditions (C.1) and (C.2), and assume that there exists a price system $\{\pi_t \mid t\in T\}$ satisfying conditions (1) and (2) of this theorem.

Now, let $K=\{k_t \mid t\in T\}$ be a program such that $k_0=k^*_0$. Since $\{\pi_t \mid t\in T\}$ satisfies condition (1) of this theorem.

$$\int u_t(k^*_{t-1}, k^*_t, \cdot)dP+\pi_{t-1}\cdot(k_{t-1}-k^*_{t-1})-\pi_t\cdot(k_t-k^*_t)\geqq\int u_t(k_{t-1}, k_t, \cdot)dP$$

for all $t>0$. Since $k_0=k^*_0$, by summing up we have

$$\pi_t\cdot k^*_t-\pi_t\cdot k_t \geqq \sum_{s=1}^{t}\int u_s(k_{s-1}, k_s, \cdot)dP-\sum_{s=1}^{t}\int u_s(k^*_{s-1}, k^*_s, \cdot)dP$$

for all $t>0$. (A.3) and (A.4) imply that $\pi_t\cdot k_t \geqq 0$. Hence, the above inequality and condition (2) of this theorem imply that program K does not overtake program K^*. Namely, program K^* is weakly maximal. ■

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