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ON THE EXISTENCE OF AN EQUILIBRIUM FOR AN AGGREGATE MODEL OF STATIONARY MARKOV ECONOMY*

SHIN-ICHI TAKEKUMA**

Abstract

A dynamic economy with uncertainty is considered. In the model of economy, production and capital accumulation are incorporated. The model is an extension of the asset-pricing model of Lucas. An equilibrium associated with consumers' rational expectations is defined in a general model by using a price forecast function and a value function of expected utilities. The existence of such an equilibrium is proved in an aggregate model.

I. *Introduction*

In this paper a dynamic economy with uncertainty is considered. An equilibrium associated with consumers' rational expectations is defined and the existence of such an equilibrium for an aggregate model of the economy is proved.

It may be possible to say that there are two types of uncertainty. The first is a case where some economic agents simply do not have some informations that are useful to them. For example, consumers do not know other consumers' preferences, or firms' production technologies. Namely, informations are "unevenly distributed" in the economy. The second is a case where any economic agent cannot obtain some informations that are important to economic activities. For instance, it is impossible for anyone to know tomorrow's weather completely. That is, informations are "lacking."

In static economies, the first type of uncertainty is a main problem. Economies with unevenly distributed informations were analyzed by R. Radner (1979) and B. Allen (1981). In dynamic economies, the second type of uncertainty is more significant than the first. That is because economic agents cannot perfectly know future situations of economies, and they can only make a guess about them. This kind of economies were studied by R. Lucas (1972), (1978), W. Brock (1982), and D. Duffie, J. Geanakoplos, A. Mas-Colell, & A. McLennan (1989).

In this paper we shall consider a dynamic economy with the second type of uncertainty.

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The considered economy is as follows: There are many consumers and firms in the economy. At each period in time all economic agents can know the current state of the economy, but cannot know future states of the economy after the period. They only know the probability distributions of future states. In the model of economy, production and capital accumulation are incorporated, and therefore the model in this paper can be regarded as an extension of the asset-pricing models by R. Lucas (1978) and by W. Brock (1982). Also, the model is a generalization of the static model of the so-called Arrow-Debreu economy.

This paper is formulated as follows. In section 2, a general model of a large economy with many consumers and several firms is presented. An equilibrium for the economy is defined by using the technique of R. Lucas (1978). In section 4, the economy presented in section 2 is transformed into an aggregate economy with identical consumers. In section 5, assumptions and an equilibrium existence theorem for the aggregate economy are stated. The theorem is proved in section 6. Some lemmas used for the theorem are proved in section 7.

II. *A General Economy with Many Consumers*

In this section we shall construct a general model of an economy with many consumers and several firms. In oder to describe an economy with many consumers, we introduce an atomless measure space of consumers, which is denoted by $(A, \mathscr{A}, \nu)$. Namely, set A is the set of all the consumers, family $\mathscr{A}$ is a σ-algebra of some subsets of A, and ν is a non-atomic measure defined on $\mathscr{A}$ with $\nu(A)=1$.

In the economy there are n kinds of different commodities, and the commodity space is an n-dimensional Euclidean space R^n. The consumption sets of consumers are the same, and they are all the nonnegative orthant R_+^n of R^n. Let $\mathscr{U}$ be a set of some countinuous bounded functions from R_+^n to R. The utility functions of each consumer are uncertain, but it is assumed to be an element of set $\mathscr{U}$. To give a topology to set $\mathscr{U}$, for each $u \in \mathscr{U}$ let us define a norm $|u|$ by

$$|u| = \sup\{|u(x)| \mid x \in R_+^n\}.$$

On the other hand, it is assumed that there are finitely many firms, and that the number of firms is J. Let $\mathscr{Y}$ be a set of some non-empty closed subsets of R^n. The production sets of firms are uncertain, but they are assumed to be an element of set $\mathscr{Y}$. Set $\mathscr{Y}$ is endowed with the topology of closed convergence.

Let $\mathscr{U}^A$ denote the set of all measurable functions from A to $\mathscr{U}$. When $U \in \mathscr{U}^A$, $U(a)$ denotes the utility function of consumer $a \in A$. Each element of $\mathscr{U}^A$ is a list of utility functions of all the consumers in the economy. In order to give a topology to set $\mathscr{U}^A$, let us define a norm $|U|$ for each $U \in \mathscr{U}^A$ by

$$|U| = \sup\{\varepsilon \mid \nu(\{a \in A \mid |U(a)| > \varepsilon\}) > 0\}.$$

Let $\mathscr{Y}^J$ denote the J-fold product of $\mathscr{Y}$, i.e.,

$$\mathscr{Y}^J = \{(Y_1, \dots, Y_J) \mid Y_j \in \mathscr{Y} \quad (j=1, \dots, J)\}.$$

Each element $Y \in \mathscr{Y}^J$, which is written as $Y=(Y_1, \ldots, Y_J)$, is a list of production sets of all the firms in the economy, where Y_j denotes the production set of firm j.

Let $S=\mathscr{U}^A \times \mathscr{Y}^J$. A state of the economy described by an element of S. We shall consider a model of discrete time. Let $T=\{0, 1, 2, \ldots\}$ be the set of periods in time. The utility functions of consumers and the production sets of firms at each period are uncertain, and they are described by a random process. Let $(\Omega, \mathscr{F}, P)$ be a probability space, i.e., Ω is the set of states of nature, $\mathscr{F}$ is a family of events, and P is a probability measure. For each $t \in T$, $\mathscr{E}_t : \Omega \rightarrow \mathscr{U}^A \times \mathscr{Y}^J$ is a measurable map, which is denoted by

$$\omega \in \Omega \rightarrow (U_t, Y_t) \in \mathscr{U}^A \times \mathscr{Y}^J,$$

where $U_t: A \rightarrow \mathscr{U}$ and $Y_t=(Y_{t1}, \ldots, Y_{tJ})$.

By the above map, we mean that, if the state of nature is ω, then the utility function of consumer $a \in A$ is $U_t(a)$ and the production set of firm j is Y_{tj} at period t. We assume that (U_t, Y_t) is known at the beginning of period t.

Suppose that the state of nature is ω. For consumer $a \in A$, his utility function at period t is $U_t(a)$: $R_+^n \rightarrow R$. We assume that future utilities are discounted, and that the discount rate is the same for every consumer and is δ, where $0<\delta<1$. If the commodity consumption of consumer $a \in A$ at period t is c_t, the sum of utilities for consumer $a \in A$ is

$$\sum_{t=0}^{\infty} \delta_t U_t(a)[c_t].$$

However, consumers do not know their future utility functions and commodity consumptions. Therefore, they can maximize only the expected value of the sum of future utilities according to their expectations.

On the other hand, for each firm j, production set Y_{tj} is known at the beginning of period t, and productions are carried out within a period. Therefore, there is no uncertainty for firms, and they simply try to maximize their profits at each period.

We assume that the random process $(\mathscr{E}_t)_{t=0,1,2,\cdots}$ is a stationary Markov process. Let $\mathscr{M}(S)$ be the set of all probability measures on S, which is endowed with the weak topology. Also, let $\mathscr{B}(S)$ be the set of all Borel subsets of S.

Assumption 2.1: There exists a continuous map from S to $\mathscr{B}(S)$, which is denoted by

$$s \in S \rightarrow \mu_s \in \mathscr{M}(S),$$

and has the following property: For each $t \in T$,

$$\mu_s(B)=\text{Prob.}(\{\omega \mid \mathscr{E}_{t+1}(\omega) \in B \quad \text{and} \quad \mathscr{E}_t(\omega)=s\})$$

for all $s \in S$ and $B \in \mathscr{B}(S)$. More precisely, for each $t \in T$,

$$\int_C \mu_s(B) d(P \cdot \mathscr{E}_t^{-1})(s)=P(\mathscr{E}_{t+1}^{-1}(B) \cap \mathscr{E}_t^{-1}(C))$$

for all B, $C \in \mathscr{B}(S)$.

Measure μ_s in this assumption is called a *transition probability*. The existence of such a transition probability implies that the uncertainty at each period does not depend on the time, and that it depends only on the state at the previous period. If a state $s=(U, Y)\in S$ occurs at period t, then the uncertainty after period t depends only on state s. Therefore, we do not have to indicate period t explicitly in our arguments.

III. *An Equilibrium for the General Economy*

Let $\mathscr{L}_{\infty+}^n$ denote the set of all (equivalence classes of) essentially bounded measurable functions from A to R_+^n. To describe the amounts of commodities held by consumers at a period in time, we use a function $K\in\mathscr{L}_{\infty+}^n$, where $K(a)$ denotes the commodity holding of consumer $a\in A$ at the beginning of the period. Also, the commodity consumptions of consumers at a period in time are described by a function $C\in\mathscr{L}_{\infty+}^n$, where $C(a)$ denotes the commodity consumption of consumer $a\in A$ at the period.

Let $\mathscr{L}_{1+}^J$ denote the set of all (equivalence classes of) integrable functions from A to $R^J{}_+$. To denote the shares of firms held by consumers at a period in time, we use a function $\theta\in\mathscr{L}_{1+}^J$. The j-th coordinate of θ is denoted by θ_j, and $\theta_j(a)$ denotes the shares of the j-th firm held by consumer $a\in A$ at the beginning of the period. We assume that the total of shares of each firm is unity, i.e.,

$$\int_A \theta_j d\nu=1$$

for each j. Since we only have to consider situations in equilibrium, we can confine ourself to the set,

$$\Theta=\left\{\theta\in\mathscr{L}_{1+}^J \mid \int_A \theta d\nu=\underline{1}\right\},$$

where $\underline{1}$ denotes the unit vector in R^J.

In order to define an equilibrium for the economy, following the technique by R. Lucas (1978) we shall use two functions, by which we depict consumers' expectations. The first is a forecast function concerning prices of commodities and shares. It is a measurablefunction on $S\times\mathscr{L}_{\infty+}^n\times\Theta$ to Δ^{n+J}, which is denoted by

$$\phi : S\times\mathscr{L}_{\infty+}^n\times\Theta \rightarrow \Delta^{n+J},$$

where

$$\Delta^{n+J}=\{\nu=(\nu_1, \ldots, \nu_{n+J})\in R_+^{n+J} \mid \textstyle\sum_{i=1}^{n+J}\nu_i=1\}.$$

The first n coordinates of the value of function ϕ denote prices of commodities, and the last J coordinates denote prices of shares.

By function ϕ, it is meant that consumers forecast prices of commodities and shares at each period depending on the whole situation of the economy. A whole situation of the economy at the beginning of each period consists of state $s\in S$, commodity holdings $K\in\mathscr{L}_{\infty+}^n$, and share holdings $\theta\in\Theta$. We shall call (s, K, θ) a *macro-state* of the economy. In addition, for each $a\in A$, we shall call $(K(a), \theta(a))$ a *micro-state* of consumer a.

If the macro-state at a period in time is (s, K, θ), then consumers expect that the prices of commodities and shares at the period will be $\psi(s, K, \theta)$. Under the assumption of stationary Markov process, it is natural for consumers to expect prices in such a way, because the futures of the economy after the period depend only on macro-state (s, K, θ) at the period.

The second function is an expectation function concerning future utilities for consumers. It is a real-valued, measurable, and bounded function on $A\times R_+{}^n\times R_+{}^J\times S\times \mathscr{L}_{\infty+}{}^n\times\Theta$, which is denoted by

$$V: A\times R_+{}^n\times R_+{}^J\times S\times \mathscr{L}_{\infty+}{}^n\times\Theta \to R.$$

By function V, it is meant that consumers predict the sum of present and future utilities depending not only on macro-state, but also on their own holdings of commodities and shares. If macro-state at a period in time is (s, K, θ), and if consumer $a\in A$ has commodities $x\in R_+{}^n$ and shares $b\in R_+{}^J$, he expects that the sum of his present and future expected utilities will be $V(a, x, b; s, K, \theta)$. Such a way of expectation is natural for the same reasons as in case of function ψ. Here, we should note that $V(a, x, b; s, K, \theta)$ is meaningful even if (x, b) is not equal to the true micro-state $(K(a), \theta(a))$ of consumer a, i.e., $x\neq K(a)$ or $b\neq\theta(a)$. That is because measure space $(A, \mathscr{A}, \nu)$ is non-atomic.

We shall call ψ a *price function* and V a *value function*. Now, by using a pair $\{\psi, V\}$ of a price function and a value function, we define an equilibrium for the economy.

Definition 3.1: A pair $\{\psi, V\}$ of a price function and a value function is called *an equilibrium for the general economy*, if for all $s=(U, Y)\in S$ and $(K, \theta)\in\mathscr{L}_{\infty+}{}^n\times\Theta$ there exist $(K^*, \theta^*)\in\mathscr{L}_{\infty+}{}^n\times\Theta$, $C^*\in\mathscr{L}_{\infty+}{}^n$, and $y_j^*\in Y_j$ $(j=1, \ldots, J)$ such that the following conditions are satisfied, where $p\in R^n$, $q\in R^J$, and $\psi(s, K, \theta)=(p, q)$.

(1) Firms are maximizing their profits, i.e.,

$$p\cdot y_j^*=\sup p\cdot Y_j \quad \text{for all } j=1, \ldots, J. \tag{3.1}$$

(2) Based on their expectations, consumers are maximizing their expected utilities subject to their budget constraints, i.e., for almost all $a\in A$,

$$p\cdot(C^*(a)+K^*(a))+q\cdot\theta^*(a)\leqq p\cdot K(a)+q\cdot\theta(a)+\textstyle\sum^J{}_{j=1}\theta_j(a)p\cdot y_j^* \tag{3.2}$$

and

$$\begin{aligned} &V(a, K(a), \theta(a); s, K, \theta)\\ &\quad=U(a)[C^*(a)]+\delta\int_S V(a, K^*(a), \theta^*(a); r, K^*, \theta^*)\, d\mu_s(r)\\ &\quad\geqq U(a)[c]+\delta\int_S V(a, x, b; r, K^*, \theta^*)\, d\mu_s(r) \end{aligned} \tag{3.3}$$

for all (c, x, b) with $p\cdot(c+x)+q\cdot b\leqq p\cdot K(a)+q\cdot\theta(a)+\sum^J{}_{j=1}\theta_j(a)p\cdot y_j^*$.

(3) Commodity markets are all in equilibrium, i.e.,

$$\int_A C^*d\nu+\int_A K^*d\nu=\int_A Kd\nu+\textstyle\sum^J{}_{j=1}y_j^* \tag{3.4}$$

In the above definition, it should be noted that share markets are all in equilibrium, since $\theta^* \in \Theta$. Also, we should note that condition (2) insures the consistency between price function ψ and value function V.

One of the new features in the above definition of equilibrium is that a micro-state of each consumer is explicitly distinguished from a macro-state of the economy.

IV. *An Aggregate Economy with Many Consumers*

In this section we shall transform the general model in the previous section into a model of economy where all the consumers are the same and identical. Such a simplified model is sometimes used in macroeconomic analysis.

The utility functions of consumers are denoted by a map $U: A \rightarrow \mathscr{U}$, which is an element of set $\mathscr{U}^A$. We assume that the utility functions of all the consumers are the same. Then, map U is constant, i.e., for some $u \in \mathscr{U}$, $U(a)=u$ for all $a \in A$. Therefore, we can regard set $\mathscr{U}^A$ as set $\mathscr{U}$.

Next, we assume that consumers are all in the same situation, and that their holdings of commodities and shares are the same. The commodity holdings of consumers are described by a function $K: A \rightarrow R^n{}_+$, which is an element of set $\mathscr{L}_{\infty+}{}^n$. When consumers have all the same amounts of commodities, then function K is constant, i.e., for some $k \in R_+{}^n$. $K(a)=k$ for all $a \in A$. Therefore, we can regard set $\mathscr{L}_{\infty+}{}^n$ as set $R^n{}_+$.

The share holdings of consumers are denoted by a function $\theta: A \rightarrow R_+{}^J$, which is an element of set $\mathscr{L}_{1+}{}^J$. Since the totals of shares are assumed to be unity, when consumers have al the same amounts of shares, $\theta(a)=1$ for all $a \in A$. Thus, set Θ can be regarded as a one-point set $\{1\}$, and can be ignored.

Thus, a macro-state (U, Y, K, θ) of the economy can be simply denoted by $(u, Y, k, 1)$ in the aggregate model.

Let $S = \mathscr{U} \times \mathscr{Y}^J$. Then, the price function in the aggregate model is changed to a function on $S \times R_+{}^n$, which is denoted by

$$\psi: S \times R_+{}^n \rightarrow \Delta^{n+J}.$$

Since consumers are all in the same situation, they will also behave all in the same way. Therefore, we do not have to describe the behaviors of all the consumers, but just that of a representative consumer. Thus, the value function in the aggregate model is reduced to a function on $R_+{}^n \times R_+{}^J \times S \times R_+{}^n$, which is denoted by,

$$V: R_+{}^n \times R_+{}^J \times S \times R_+{}^n \rightarrow R.$$

Now we can define an equilibrium for the aggregate economy. Definition 3.1 is rewritten in the following fashion.

Definition 4.1: A pair $\{\psi, V\}$ of a price function ψ and a value function V is called *an equilibrium for the aggregate economy*, if for all $s=(u, Y) \in S$ and $k \in R_+{}^n$, there exist $c^* \in R_+{}^n$, $k^* \in R_+{}^n$, and $y_j^* \in Y_j$ $(j=1, \ldots, J)$ such that the following conditions are satisfied, where $p \in R^n$, $q \in R^J$, and $\psi(s, k)=(p, q)$.

(1) $p \cdot y_j^* = \sup p \cdot Y_j$ for all $j = 1, \ldots, J$.

(2) $p \cdot (c^* + k^*) + q \cdot 1 \leqq p \cdot k + q \cdot 1 + \sum^J_{j=1} p \cdot y_j^*$ (4.1)

and

$$V(k, \underline{1}; s, k) = u(c^*) + \delta \int_S V(k^*, 1; r, k^*)\, d\mu_s(r)$$

$$\geqq u(c) + \delta \int_S V(x, b; r, k^*)\, d\mu_s(r) \tag{4.2}$$

for all (c, x, b) with $p \cdot (c + x) + q \cdot b \leqq p \cdot k + q \cdot 1 + \sum^J_{j=1} p \cdot y_j^*$.

(3) $c^* + k^* = k + \sum^J_{j=1} y_j^*$.

V. *The Existence of an Equilibrium for the Aggregate Model*

In order to prove the existence of an equilibrium for the aggregate economy, we assume the following.

Assumptiop 5.1: Any utility function $u \in \mathscr{U}$ has the following properties:
(1) Function u is continuous and concave.
(2) Function u is monotone-increasing, i.e., if $c \leqq c'$ and $c \neq c'$, then $u(c) < u(c')$.
(3) $u(0) = 0$.

Assumption 5.2: Any production set $Y \in \mathscr{Y}$ has the following properties:
(1) Set Y is convex and compact.
(2) $Y \cap R_+^n = \{0\}$.

Assumption 5.3: Sets $\mathscr{U}$ and $\mathscr{Y}$ are bounded:
(1) There exists a number $\alpha > 0$ such that $|u| \leqq \alpha$ for all $u \in \mathscr{U}$.
(2) There exists a number $\beta > 0$ such that $|y| \leqq \beta$ for all $y \in Y$ with $Y \in \mathscr{Y}$.
Under these assumptions we can prove the following theorem.

Theorem 5.4: Under Assumptions 5.1, 5.2, and 5.3, there exists an equilibrium $\{\psi, V\}$ for the aggregate economy that has the following properties:
(1) Function V is continuous.
(2) For each $(x, b; s) \in R_+^n \times R_+^J \times S$, $V(x, b; s, k)$ is constant in k.
(3) For each $(s, k) \in S \times R_+^n$, $V(x, b; s, k)$ is monotone-nondecreasing and concave in (x, b).

One of the interesting properties of the value function V is property (2) in the above theorem, which implies that function V does not depend on variable k of macro-state (s, k). Thus, in the aggregate economy we can identify micro-state with macro-state, just as R. Lucas (1978) did.

IV. *The Proof of Theorem 5.4*

In this section we shall prove Theorem 5.4. All the lemmas in this section will be proved in the next section.

Let $\mathscr{C}$ be the space of all bounded continuous functions on $R_+^n \times R_+^J \times S$ to R with norm topology. For each $W \in \mathscr{C}$, define a function MW on $R_+^n \times R_+^J \times S$ to R by

$$MW(k, b; s) = \sup \left\{ u(c) + \delta \int_S W(x, b; r) d\mu_s(r) \,|\, c \in R_+^n,\ x \in R_+^n,\ c + x = k + \sum_j b_j y_j,\ y_j \in Y_j \right\},$$

where $s=(u, Y)$, $Y=(Y_1, \ldots, Y_J)$, and $b=(b_1, \ldots, b_J)$.

Lemma 6.1: For any $W \in \mathscr{C}$, function MW has the following properties:
(1) *Function* MW is continuous and bounded, i.e., $MW \in \mathscr{C}$.
(2) If $W(k, b; s)$ is monotone-nondecreasing and concave in (k, b), then so is $MW(k, b; s)$.
(3) If $W(0, b; s)=0$ for all (b, s), then $MW(0, b; s)=0$ for all $(b; s)$.

By virtue of property (1) in the above lemma, we have a map,

$$W \in \mathscr{C} \rightarrow MW \in \mathscr{C},$$

which is denoted by M: $\mathscr{C} \rightarrow \mathscr{C}$. Concerning this map, we have the following lemma.

Lemma 6.2: There exists a unique function $W^* \in \mathscr{C}$ that has the following properties:
(1) Function W^* is a fixed-point of map M, i.e., $W^* = MW^*$.
(2) For each $s \in S$, $W^*(k, b; s)$ is monotone-nondecreasing and concave in (k, b).
(3) $W^*(0, b; s)=0$ for all $(b; s)$.

Let $(s, k) \in S \times R_+^n$. Since $W^* = MW^*$, under Assumptions 5.1 and 5.2 we have c^*, $k^* \in R_+^n$, and $y_j^* \in Y_j$ $(j=1, \ldots, J)$ such that

$$W^*(k, 1; s) = u(c^*) + \delta \int_S W^*(k^*, 1; r) d\mu_s(r) \tag{6.1}$$

and

$$c^* + k^* = k + \sum_j y_j^*. \tag{6.2}$$

Now, let us define a subset of $R_+^n \times R_+^J$, $\Phi(s, k)$, by

$$\Phi(s, k) = \Big\{ (p, q) \in R^n \times R^J \,|\, (p, q) \in \Delta^{n+J} \text{ and } W^*(k, 1; s) \geqq u(c) + \delta \int_S W^*(x, b; r) d\mu_s(r) \text{ for all } (c, x, b) \in R_+^n \times R_+^n \times R_+^J \text{ with } p \cdot (c + x) + q \cdot b \leqq p \cdot k + q \cdot 1 + \textstyle\sum_{j=1}^J \sup p \cdot Y_j \Big\},$$

where $s=(u, Y)$ and $Y=(Y_1, \ldots, Y_J)$.

Lemma 6.3: For all $(s, k) \in S \times R_+^n$, $\Phi(s, k) \neq \phi$.

By this lemma we can define a correspondence,

$$(s, k) \in S \times R_+^n \rightarrow \Phi(s, k) \subset \Delta^{n+J},$$

which will be denoted by $\Phi: S \times R_+^n \rightarrow \Delta^{n+J}$. The correspondence has the following property.

Lemma 6.4: The correspondence $\Phi: S \times R_+^n \rightarrow \Delta^{n+J}$ has a closed graph.

This lemma implies that there exists a measurable function $\psi: S \times R_+^n \rightarrow \Delta^{n+J}$ such that $\psi(s, k) \in \Phi(s, k)$ for all $(s, k) \in S \times R_+^n$ [see Hildenbrand (1974), lem. 1, p. 55].

Let us define a function V on $R_+^n \times R_+^J \times S \times R_+^n$ to R by

$$V(x, b; s, k) = W^*(x, b; s). \tag{6.3}$$

Then, obviously, V is continuous and bounded. Also, by Lemma 6.2 we can easily check that function V has properties (1), (2), and (3) of Theorem 5.4. It remains to show that $\{\psi, V\}$ is an equilibrium, i.e., we only have to show that $\{\psi, V\}$ satisfies conditions (1), (2), and (3) of Definition 4.1.

Now, let $\psi(s, k) = (p, q) \in R^n \times R^J$. Then, by construction of ψ, $(p, q) \in \Phi(s, k)$. Therefore, it follows from (6.3) that

$$V(k, 1, s, k) \geqq u(c) + \delta \int_S V(x, b; r, k^*) d\mu_s(r) \tag{6.4}$$

for all $(c, x, b) \in R_+^n \times R_+^n \times R_+^J$ with

$$p \cdot (c+x) + q \cdot b \leqq p \cdot k + q \cdot 1 + \textstyle\sum_{j=1}^J \sup p \cdot Y_j \tag{6.5}$$

where $S=(u, Y)$ and $Y=(Y_1, \ldots, Y_J)$.

(6.1), (6.3), and (6.4) imply (4.2) in (2) of Definition 4.1. And (6.2) implies (3) of Definition 4.1. Also, by (6.2) we have,

$$p \cdot (c^* + k^*) + q \cdot 1 = p \cdot (k + \textstyle\sum_j y_j^*) + q \cdot 1 \leqq p \cdot k + q \cdot 1 + \sum_{j=1}^J \sup p \cdot Y_j, \tag{6.6}$$

which is (4.1) in (2) of Definition 4.1.

Suppose that strict inequality holds in (6.6). Then, condition (2) of Assumption 5.1 implies that $(c^*, k^*, 1)$ does not maximize the right hand side of (6.4) under budget constraint (6.5). This is a contradiction to (6.1) and (6.3). Therefore, equality holds in (6.6), and we have

$$\textstyle\sum_{j=1}^J p \cdot y_j^* = \sum_{j=1}^J \sup p \cdot Y_j,$$

which implies (1) of Definition 4.1. This completes the proof of Theorem 5.4.

VII. *Proofs of Lemmas*

In this section we shall prove the lemmas which were used in the previous section.

Proof of Lemma 6.1: Given $W \in \mathscr{C}$, define a function $f: R_+{}^n \times R_+{}^n \times R^J{}_+ \times S \to R$ as follows: For $(c, x, b, s) \in R_+{}^n \times R_+{}^n \times R_+{}^J \times S$, let

$$f(c, x, b, s) = u(c) + \delta \int_S W(x, b; r) d\mu_s(r),$$

where $s=(u, Y)$. Moreover, define a correspondence F: $R_+{}^J \times S \times R_+{}^n \to R_+{}^n \times R_+{}^n$ as follows: For $(b, s, k) \in R_+{}^J \times S \times R_+{}^n$, let

$$F(b, s, k) = \{(c, x) \in R_+{}^J \times R_+{}^n | c + x = k + \textstyle\sum^J{}_{j=1} b_j y_j \text{ for some } y_j \in Y_j \ (j=1, \ldots, J)\},$$

where $s=(u, Y)$ and $b=(b_1, \ldots, b_J)$. Then, by (1) of Assumption 5.2 and definition of MW, we have

$$MW(k, b; s) = \max \{f(c, x, b, s) \mid (c, x) \in F(b, s, k)\}.$$

Since map, $s \to \mu_s$, is continuous by assumption, it is easy to show that function f is continuous. It is also easy to prove that correspondence F is continuous. Therefore, by applying the so-called maximum theorem to f and F, we have the continuity of function MW. In addition, its boundedness follows from (1) of Assumption 5.3. This proves (1) of the lemma.

It can be easily shown by (2) of Assumption 5.1, (1) and (2) of Assumption 5.2 that $MW(k, b; s)$ is monotone-nondecreasing in (k, b). The concavity of $MW(k, b; s)$ in (k, b) follows from (1) of Assumption 5.1 and (1) of Assumption 5.2. Thus, (2) of the lemma is proved.

It is implied by (3) of Assumption 5.1 and (2) of Assumption 5.2 that $MW(0, b; s)=0$ for all (b, s). This completes the proof of the lemma. ∎

Proof of Lemma 6.2: Let $W_1, W_2 \in \mathscr{C}$. Then, by definition of norm $|\cdot|$, it is true that

$$W_2 - |W_1 - W_2| \leqq W_1 \leqq W_2 + |W_1 - W_2|.$$

By definition of map M, we have

$$MW_2 - \delta |W_1 - W_2| \leqq MW_1 \leqq MW_2 + \delta |W_1 - W_2|.$$

which implies that

$$|MW_1 - MW_2| \leqq \delta |W_1 - W_2|. \tag{7.1}$$

Thus, map M is shown to be a contraction map. It is well-known that such a contraction map has a fixed-point, i.e., there exists a map $W^* \in \mathscr{C}$, such that $W^* = MW^*$. Hence, (1) of Lemma 6.2 is proved.

In addition, for any $W \in \mathscr{C}$, by (7.1) we have

$$|MW - W^*| \leqq \delta |W - W^*|.$$

Therefore, it follows that

$$|M^m W - W^*| \leqq \delta \; |M^{m-1} W - W^*| \leqq, \ldots, \leqq \delta^m \; |W - W^*|,$$

where M^m denotes the m-time composition of map M. This proves that for any $W \in \mathscr{C}$,

$$\lim_{m \to \infty} M^m W = W^*. \tag{7.2}$$

Let W_0 be a function on $R_+{}^n \times R_+{}^J \times S$ to R such that $W_0(k, b; s) = 0$ for all $(k, b; s)$. Obviously, function W_0 is continuous and bounded, i.e., $W_0 \in \mathscr{C}$. Also, $W_0(k, b; s)$ is monotone-nondecreasing and concave in (k, b), and $W_0(0, b; s) = 0$ for all $(b; s)$. Therefore, by (1) and (2) of Lemma 6.1, function $M^m W_0$ has the same properties. Thus, since function $M^m W_0$ converges uniformly to function W^* according to (7.2), function W^* has also the exactly same properties. This proves (2) and (3) of Lemma 6.2. ■

Proof of Lemma 6.3: Let $(s, k) \in S \times R_+{}^n$ and define two subsets D, E of $R_+{}^n \times R_+{}^J$ by

$$D = \left\{ (v, b) \mid v = c + x, \; u(c) + \delta \int_S W^*(x, b; r) d\mu_s(r) > W^*(k, 1; s) \right\}$$

and

$$E = \{ (v, b) \mid v = k + \textstyle\sum^J{}_{j=1} y_j \quad \text{for some } y_j \in Y_j \; (j = 1, \ldots, J), \; b = 1 \},$$

where $s = (u, Y)$. (6.1) and (2) of Assumption 5.1 implies that $D \neq \phi$. Also, (2) of Assumption 5.2 (2) implies that $E \neq \phi$. The convexity of D and E follows from (1) of Assumption 5.1 and (1) of Assumption 5.2.

Suppose that $D \cap E \neq \phi$. Then, there exist c, $x \in R_+{}^n$, and $y_j \in Y_j$ $(j = 1, \ldots, J)$ such that

$$u(c) + \delta \int_S W^*(x, 1; r) d\mu_s(r) > W^*(k, 1; s)$$

and

$$c + x = k + \textstyle\sum^J{}_{j=1} y_j.$$

Since $W^* = MW^*$, this contradicts the definition of map M. Hence, $D \cap E = \phi$.

Therefore, by a separation theorem, there exists a non-zero vector $(p, q) \in R^n \times R^J$ such that

$$p \cdot v + q \cdot b \geqq p \cdot v' + q \cdot b' \quad \text{for all } (v, b) \in D, \; (v', b') \in E. \tag{7.3}$$

Since function u is monotone-increasing and function $W^*(k, b; a)$ is monotone-nondecreasing in (k, b), set D has a special shape, and $p > 0$ and $q \geqq 0$. Thus, without loss of generality, we can assume that $(p, q) \in \Delta^{n+J}$.

(7.3) implies that

$$p\cdot v+q\cdot b\geqq p\cdot k+q\cdot \underline{1}+\textstyle\sum_j \sup p\cdot y_j \quad \text{for all } (v, b)\in D \tag{7.4}$$

Suppose that equality in (7.4) holds for some $(v, b)\in D$. Then, there exist $c, x\in R_+{}^n$ such that

$$u(c)+\delta \int_S W^*(x, b; r)d\mu_s(r)>W^*(k, \underline{1}; s) \tag{7.5}$$

and

$$p\cdot(c+x)+q\cdot b=p\cdot k+q\cdot \underline{1}+\textstyle\sum_j \sup pY_j. \tag{7.6}$$

Since $p>0$, if the right-hand side of (7.6) is equal to zero, then $c=x=0$. Therefore, by virtue of (2) and (3) of Lemma 6.2, (7.5) implies that $u(0)>0$. This contradicts (3) of Assumption 5.1. On the other hand, since functions u and W^* are continuous, if the right-hand side of (7.6) is positive, then, by a small change of c or x, we still have the inequality (7.5) and

$$p\cdot(c+x)+q\cdot b<p\cdot k+q\cdot \underline{1}+\textstyle\sum_j \sup p\cdot Y_j.$$

This contradicts (7.3). Thus, we have proved that

$$p\cdot v+q\cdot b>p\cdot k+q\cdot \underline{1}+\textstyle\sum_j \sup p\cdot Y_j \quad \text{for all } (v, b)\in D,$$

which implies that $(p, q)\in\Phi(s, k)$. ■

Proof of Lemma 6.4: Define a function g: $R_+{}^n\times R_+{}^n\times R_+{}^J\times S\to R$ as follows: For $(c, x, b, s)\in R_+{}^n\times R_+{}^n\times R_+{}^J\times S$, let

$$g(c, x, b, s)=u(c)+\delta \int_S W^*(x, b; r)d\mu_s(r)$$

where $s=(u, Y)$. Moreover, define a correspondence G: $S\times R_+{}^n\times\Delta^{n+J}\to R_+{}^n\times R_+{}^n\times R_+{}^J$ as follows: For $(s, k, p, q)\in S\times R_+{}^n\times \Delta^{n+J}$, let

$$G(s, k, p, q)=\{(c, x, b)\in R_+{}^n\times R_+{}^n\times R_+{}^J \mid p\cdot(c+x)+q\cdot b\leqq p\cdot k+q\cdot \underline{1}+\textstyle\sum_j \sup p\cdot Y_j\},$$

where $s=(u, Y)$. It is easy to show that function g and correspondence G are continuous.

Define a function h: $S\times R_+{}^n\times \Delta^{n+J}\to R$ as follows: For $(s, k, p, q)\in S\times R_+{}^n\times\Delta^{n+J}$, let

$$h(s, k, p, q)=\sup \{g(c, x, b, s) \mid (c, x, b)\in G(s, k, p, q)\}.$$

By applying the so-called maximum theorem to g and G, we have the continuity of function h. Thus, map,

$$(s, k, p, q) \to h(s, k, p, q)-W^*(k, \underline{1}; s),$$

is continuous. Hence, we have the following closed set,

$$\{(s, k, p, q) \mid h(s, k, p, q) \leqq W^*(k, \underline{1}; s)\},$$

which is easily shown to be the graph of correspondence Φ. ■

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