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ON THE EXISTENCE OF OPTIMAL STATIONARY STATES IN CAPITAL ACCUMULATION UNDER UNCERTAINTY: A CASE OF LINEAR DIRECT UTILITY*

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I. *Introduction*

In this paper we shall consider a problem of the existence of optimal stationary states in a quasi-stationary model of capital accumulation under uncertainty. The considered model in this paper is a reduced model of capital accumulation under uncertainty, which is similar to those of Radner (1973) and Zilcha (1976). We assume the stationarity of the economy in that the probability distributions of production technologies and utility functions are the same at every period in time. However our model is quasi-stationary since future utilities are discounted. The case of a stationary model in which future utilities are not discounted was considered by Radner (1973).

The existence of optimal stationary states in general deterministic models has been proved by Khan & Mitra (1986) and McKenzie (1986). In this paper we shall prove the existence of optimal stationary states in a model with uncertainty, where the linearity of direct utility function is assumed. Our technique for proof depends on that of Khan & Mitra (1986). First we shall prove the existence of discounted golden-rule states by using their technique. In their proof Kakutani's fixed-point theorem was used. However, since uncertainty is incorporated in our economy, we have to consider an infinite dimensional space, that is, the space of essentially bounded measurable functions. To find a fixed-point in the space we shall apply the theorem by Fan (1952). Next we shall prove that any discounted golden-rule state is an optimal stationary state by using the primal approach of Khan & Mitra (1986).

This paper is formulated in the following fashion. In section II we shall construct a general reduced model of capital accumulation under uncertainty. In section III the stationarity assumptions will be introduced into the model. In section IV the existence of discounted golden-rule state of capital accumulation will be proved. In section V it will be proved that discounted golden-rule states are optimal stationary states. In Appendix the proofs of two lemmas on the fundamental properties of the sationary model will be given.

II. *A General Reduced Model*

For a preparation to construct a stationary model, first we shall present a general reduced

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model of capital accumulation where future utilities and production technologies are uncertain.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Each element in Ω denotes a possible state of nature, which may be interpreted as a stream of environments in all past, present, and future periods. Family $\mathscr{F}$ is the set of all possible events and P denotes the probability distribution of states. Let $T=\{0, 1, 2, \ldots\}$ be the space of time. The uncertainty of states is described by a filtration $\{\mathscr{F}_t | t\in T\}$, i.e., $\mathscr{F}_t$ is a family of subsets of Ω such that $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ for all $t\in T$. Each family $\mathscr{F}_t$ is interpreted as the informations about states that will become known up to period t. We assume that $(\Omega, \mathscr{F}_t, P)$ is complete for all t.

The production technology available at each period $t>0$ is described by a relation Y_t: $\Omega \rightarrow R^m \times R^m$, that is,

$$\omega \in \Omega \rightarrow Y_t(\omega) \subset R^m \times R^m,$$

where R^m denotes an m-dimensional Euclidean space. We assume that the graph of Y_t defined by

$$G(Y_t)=\{(x, y, \omega) \mid (x, y)\in Y_t(\omega)\}$$

is $\mathscr{B}(R^m)\times\mathscr{B}(R^m)\times\mathscr{F}_t$-measurable, where $\mathscr{B}(R^m)$ is the family of all Borel subsets of R^m. By $Y_t(\omega)$ we represent the possibility of transformation of capital stocks. That is, $(x, y)\in Y_t(\omega)$ means that under state ω it is possible to transform capital stock x at period $t-1$ into capital stock y at period t.

The satisfaction in the economy at each period $t>0$ is described by a utility function $u_t : G(Y_t) \rightarrow R\cup\{-\infty\}$, that is,

$$(x, y, \omega)\in G(Y_t) \rightarrow u_t(x, y, \omega)\in R\cup\{-\infty\},$$

where R denotes the real line. We assume that u_t is a $\mathscr{B}(R^m)\times\mathscr{B}(R^m)\times\mathscr{F}_t$-measurable function, which may take value $-\infty$. $u_t(x, y, \omega)$ is interpreted as a level of social welfare under state ω obtained at period t if capital stocks at periods $t-1$ and t are x and y respectively.

In order to describe a program of capital accumulation, we will use a stochastic process, i.e., a function $K: T\times\Omega \rightarrow R^m$ such that $K(t, \cdot)$ is $\mathscr{F}_t$-measurable for each $t\in T$. To denote a stochastic process $K: T\times\Omega\rightarrow R^m$, we also write as $K=\{k_t | t\in T\}$, where k_t is a function defined by $k_t(\omega)=K(t, \omega)$.

Definition 2.1: A stochastic process $K=\{k_t | t\in T\}$ is called a *program*, if k_t is an essentially bounded $\mathscr{F}_t$-measurable function on Ω to R^m for each $t\in T$.

In program $K=\{k_t | t\in T\}$, for each $t\in T$, k_t is a random variable and $k_t(\omega)$ denote a capital stock planned to accumulate at period t in state ω. We should note that by this definition we restrict ourselves to essentially bounded programs.

Definition 2.2: A program $K=\{k_t | t\in T\}$ is said to be *feasible* if for each $t>0$, $(k_{t-1}(\omega), k_t(\omega))\in Y_t(\omega)$ a.s.

Since k_t and Y_t is $\mathscr{F}_t$-measurable, production technology Y_t is perfectly known in determining capital stock k_t at period t. However, in determining k_{t-1} at period $t-1$, production technology Y_t is unknown. In this sense, uncertainty exists in production technology. Similarly, utility function u_t is perfectly known in determining capital stock k_t

at period t, but unknown in determining k_{t-1} at period $t-1$. Thus, uncertainty also exists in utility function.

To evaluate feasible programs, the so-called overtaking criterion in the following definition is used, which is a generalization of the usual maximization problem. For a feasible program $K=\{k_t | t \in T\}$, by $U_t(K)$ we denote the sum of expected utilities that will be obtained up to period t in program K, i.e.,

$$U_t(K)=\int_\Omega \left[\sum_{s=1}^{t} u_s(k_{s-1}(\omega), k_s(\omega), \omega)\right] dP(\omega)$$

Definition 2.3: A feasible program $K=\{k_t | t \in T\}$ is said to be *optimal* if there is no other feasible program $K'=\{k_t' | t \in T\}$ with $k_0=k_0'$ such that

$$\limsup_{t \to +\infty} [U_t(K')-U_t(K)]>0.$$

III. *A Stationary Model and Stationary States*

Now we introduce the stationarity assumptions into the model. First we assume the stationarity of informations and probability distributions over time.

(A.1) (stationarity of uncertainty): There is a map $\tau: \Omega \to \Omega$ such that for each t, $\tau: (\Omega, \mathscr{F}_t, P) \to (\Omega, \mathscr{F}_{t-1}, P)$ is measurability- and measure-preserving:

(1) For each t, $\tau: (\Omega, \mathscr{F}_t) \to (\Omega, \mathscr{F}_{t-1})$ is one to one and onto, and both τ and τ^{-1} are measurable.

(2) For each t, $P(\tau^{-1}(E))=P(E)$ for all $E \in \mathscr{F}_{t-1}$.

This assumption means that two measure spaces $(\Omega, \mathscr{F}_t, P)$ and $(\Omega, \mathscr{F}_{t-1}, P)$ are isomorphic to each other under transformation τ. In fact, under this assumption, $\tau(\mathscr{F}_t)=\mathscr{F}_{t-1}$ and $P(\tau(E))=P(E)$ for all $E \in \mathscr{F}_t$. Hence, $(\Omega, \tau(\mathscr{F}_t), P)$ is equivalent to $(\Omega, \mathscr{F}_{t-1}, P)$.

Under the above assumption, for any $\mathscr{F}_t$-measurable function $f: \Omega \to R^m$, there is an $\mathscr{F}_{t-1}$-measurable function $g: \Omega \to R^m$ such that $g \cdot \tau = f$. In fact, by (1) of (A.1), function g defined by $g=f \cdot \tau^{-1}$ is such a function. Namely, any $\mathscr{F}_t$-measurable function f is described as $f=g \cdot \tau$ by an $\mathscr{F}_{t-1}$-measurable function g. Thus any $\mathscr{F}_t$-measurable function f is described as $f=g \cdot \tau^t$ by an $\mathscr{F}_0$-measurable function g, where τ^t denotes the t-time composite map of τ.

Conversely, let $g: \Omega \to R^m$ be an $\mathscr{F}_0$-measurable function. Then, by (1) of (A.1), function $f: \Omega \to R^m$ defined by $f=g \cdot \tau^t$ is $\mathscr{F}_t$-measurable. In addition, by (2) of (A.1), for any $B \in \mathscr{B}(R^m)$, we have

$$\begin{aligned} P(f^{-1}(B)) &= P((g \cdot \tau^t)^{-1}(B)) \\ &= P((g \cdot \tau^{t-1})^{-1}(B)) \\ &\ \ \vdots \\ &= P(g^{-1}(B)). \end{aligned}$$

Thus, functions f and g can be regarded as the same random variable only except that periods in time are different.

Next we shall assume the stationarity of production technologies and social welfares.

(A.2) (stationarity of production set and utility function):
(1) For each t, $Y_t = Y_1 \cdot \tau^{t-1}$.
(2) There is a number $0 < \delta < 1$ such that for each t,

$$u_t(x, y, \omega) = \delta^{t-1} u_1(x, y, \tau^{t-1}(\omega))$$

for all $(x, y, \omega) \in G(Y_t)$.

Under this assumption, map $\tau: \Omega \to \Omega$ is interpreted as the time-shifting operator. Each element in Ω may be interpreted as a stream of environments at all the periods in time. Let us call it "history." Let $\omega \in \Omega$ and $\omega' = \tau(\omega)$. Then, under assumptions (A.1) and (A.2), history ω' can be regarded as the exactly same history as history ω, except that each period t in history ω corresponds to period $t-1$ in history ω'.

Now, let us make the usual assumptions of convexity, continuity, and boundedness for the model. Since the model is stationary, we only have to make them on the production technology and the utility function at period 1, that is, Y_1 and u_1. From now on, to denote Y_1 and u_1, we will write Y and u respectively.

(A.3) (convexity):
(1) For each $\omega \in \Omega$, $Y(\omega)$ is a convex subset of $R^m \times R^m$.
(2) For each $\omega \in \Omega$, $u(x, y, \omega)$ is concave in (x, y).

(A.4) (continuity):
(1) For each $\omega \in \Omega$, $Y(\omega)$ is a closed subset of $R^m \times R^m$.
(2) For each $\omega \in \Omega$, $u(x, y, \omega)$ is upper semicontinuous in (x, y), i.e., if $(x_n, y_n) \to (x_0, y_0)$, then

$$\limsup_{n \to \infty} u(x_n, y_n, \omega) \leqq u(x_0, y_0, \omega).$$

(A.5) (boundedness): There are numbers b^* and u^* having the following properties:
(1) If $(x, y) \in Y(\omega)$ and $|x| > b^*$, then $|y| < |x|$.
(2) If $(x, y) \in Y(\omega)$ and $|x| \leqq b^*$, then $u(x, y, \omega) \leqq u^*$.

Let $\mathscr{L}_\infty$ denote the set of all essentially bounded $\mathscr{F}_0$-measurable functions on Ω to R^m. By $\mathscr{L}_\infty \cdot \tau$ we denote the set of all essentially bounded $\mathscr{F}_1$-measurable functions on Ω to R^m. The capital stock at period 0 is described by using a function in $\mathscr{L}_\infty$. Also, the capital stock at period 1 is described by using a function in $\mathscr{L}_\infty \cdot \tau$, and is denoted as $f \cdot \tau$ by a function $f \in \mathscr{L}_\infty$.

The techonology to transform capital stocks at period 1 is represented by the following set.

$$\mathscr{Y} = \{(f, g) \mid f, g \in \mathscr{L}_\infty \cdot \tau, (f(\omega), g(\omega)) \in Y(\omega) \text{ a.s.}\}.$$

Thus, to describe the feasibility of capital stocks between period 0 to period 1, we can use the following set.

$$\mathscr{D} = \{(f, g) \in \mathscr{L}_\infty \times \mathscr{L}_\infty \mid (f, g \cdot \tau) \in \mathscr{Y}\}.$$

Since the model is stationary, by $\mathscr{D}$ we can define the feasibility of capital stocks between any two adjacent periods $t-1$ and t. Thus, a program $\{k_t \mid t \in T\}$ is feasible if and only if $(k_{t-1} \cdot \tau^{1-t}, k_t \cdot \tau^{-t}) \in \mathscr{D}$ for all t. Set $\mathscr{D}$ may be called "sationary transformation set."

Definition 3.1: A function $k \in \mathscr{L}_\infty$ is called a *stationary state* if $(k, k) \in \mathscr{D}$.

For each $(f, g) \in \mathscr{D}$, let us define $\psi(f, g)$, if possible, by

$$\psi(f, g) = \int_\Omega u(f(\omega), g(\tau(\omega)), \omega) dP(\omega),$$

Here $\psi(f, g)$ may be interpreted as the expected social welfare when capita stocks at periods 0 and 1 (or, $t-1$ and t) are f and g respectively.

Definition 3.2: A stationary state $k \in \mathscr{L}_\infty$ is called a *discounted golden-rule state* if $\psi(k, k) \geqq \psi(f, g)$ for all $(f, g) \in \mathscr{L}_\infty$ with $f - \delta g = (1-\delta)k$.

Definition 3.3: A stationary state $k \in \mathscr{L}_\infty$ is said to be *optimal* if program $\{k \cdot \tau^t \mid t \in T\}$ is optimal.

Here we shall state two lemmas on the fundamental properties of the model. They can be proved in a well-known and usual manner. For proof, see Appendix.

Lemma 3.1: $\mathscr{Y}$ is a convex and closed subset of $(\mathscr{L}_\infty \times \mathscr{L}_\infty) \cdot \tau$ in the weak* topology. Therefore, so is $\mathscr{D}$.

Lemma 3.2: ψ is concave and upper semicontinuous on set $\{(f, g) \in \mathscr{D} \mid \|f\| \leqq b^*\}$ in the weak* topology, i.e., for any $b \in R$, set $\{(f, g) \in \mathscr{D} \mid \|f\| \leqq b^*$ and $\psi(f, g) \geqq b\}$ is convex and weak* closed.

Remark 3.1: Let $\mathscr{L}_1$ be the set of all $\mathscr{F}_0$-integrable functions on Ω to R^m. $\mathscr{L}_\infty$ can be regarded as the set of all norm-continuous linear functions on $\mathscr{L}_1$ [see Dunford & Schwartz (1964, p. 289, Thm. 8.5)]. On the other hand, for each $g \in \mathscr{L}_1$, a linear function on $\mathscr{L}_\infty$ defined by

$$f \in \mathscr{L}_\infty \rightarrow \int g \cdot f dP \in R$$

is continuous in the norm topology. There is the weakest topology for $\mathscr{L}_\infty$ such that for all $g \in \mathscr{L}_1$ the above defined linear function is continuous. Such a topology for $\mathscr{L}_\infty$ is commonly referred to as the weak* topology and is denoted by $\sigma(\mathscr{L}_\infty, \mathscr{L}_1)$.

IV. *Existence of Discounted Golden-Rule States*

In this section we shall prove the existence of discounted golden-rule states. To do so, we assume the following:

(A.6) (monotonicity): If $(x, y) \in Y(\omega)$ and $x \leqq x'$, then $(x', y) \in Y(\omega)$ and $u(x, y, \omega) \leqq u(x', y, \omega)$.

(A.7) (expansibility): There is $(\underline{f}, \underline{g}) \in \mathscr{D}$ such that $\underline{f} - \delta \underline{g} \ll (1-\delta)k$ for any $(k, h) \in \mathscr{D}$. Here, by $f \ll g$ we mean that $f(\omega) < g(\omega)$ a.s.

(A.8) (linearity of direct utility:) There are $\mathscr{F}_1$-measurable functions $\alpha: \Omega \rightarrow R^m$ with $\alpha \neq 0$ and $\beta: \Omega \rightarrow R^m$ such that

$$u(x, y, \omega) = \sup\{\alpha(\omega)z + \beta(\omega) \mid z \geqq 0, (x-z, y) \in Y(\omega)\}$$

for each (x, y, ω).

Remark 4.1: Under assumptions (A.4), (A.5), and (A.8), we can easily show that for $(f, g) \in \mathscr{D}$, there is $c \in \mathscr{L}_\infty \cdot \tau$ with $c \geqq 0$ such that

$$(f(\omega) - c(\omega),\ g \cdot \tau(\omega)) \in Y(\omega)$$

and

$$u(f(\omega),\ g \cdot \tau(\omega),\ \omega) = \alpha(\omega)\ c(\omega) + \beta(\omega) \text{ a.s.}$$

Namely, $(f - c,\ g \cdot \tau) \in \mathscr{Y}$ and $\psi(f, g) = \int [\alpha c + \beta] dP$ [See Hildenband (1974, p. 60, Prop. 3)].

Let us define a bounded subset of $\mathscr{L}_\infty$ by

$$\mathscr{H} = \{ f \in \mathscr{L}_\infty \mid (f, g) \in \mathscr{D} \text{ and } \| f \| \leqq b^* \}.$$

Lemma 4.1: $\mathscr{H}$ is nonempty, convex, and weak* compact.

Proof: By (A.7) of expansibility, we have $\underline{f} - \delta \underline{g} \ll \underline{f} - \delta \underline{f}$, i.e., $\underline{f} \ll \underline{g}$. Hence, by (A.6) of monotonicity, $(\underline{g}, \underline{g}) \in \mathscr{D}$. Therefore, from (A.6) of monotonicity it follows that $(f', \underline{g}) \in \mathscr{D}$ for any $f' \geqq \underline{g}$. Assume that $f'(\omega) > b^*$ a.s. Then, by (A.5) of boundedness, we can conclude that $|\underline{g} \cdot \tau(\omega)| < b^*$ a.s., i.e., $\|\underline{g}\| < b^*$. Hence $\underline{g} \in \mathscr{H}$, which implies that $\mathscr{H} \neq \phi$. The convexity of $\mathscr{H}$ is obvious.

In addition, from Lemma 3.1 it follows that $\mathscr{H}$ is weak* closed. Therefore, it is weak* compact. That is because any bounded subset of $\mathscr{L}_\infty$ is relatively compact in the weak* topology by Alaoglu's theorem [see Dunford & Schwartz (1964, p. 424, Thm. V.4.2, Cor. V.4.3)]. ∎

Let define a relation $\Phi: \mathscr{H} \to \mathscr{D}$ by

$$\Phi(k) = \{ (f, g) \in \mathscr{D} \mid f - \delta g = (1 - \delta) k \}.$$

Lemma 4.2: Φ has the following properties.

(1) If $(f, g) \in \Phi(k)$, then $\|f\| \leqq b^*$ and $\|g\| \leqq b^*$.

(2) For each $k \in \mathscr{H}$, $\Phi(k)$ is nonempty, convex, and weak* compact.

Proof: Let $k \in \mathscr{H}$ and $(f, g) \in \Phi(k)$. Suppose that $\|g\| > b^*$. Since $(f, g) \in \Phi(k)$, $f - \delta g = (1 - \delta) k$. Since $\|k\| \leqq b^*$, this implies that

$$\|f\| \leqq (1 - \delta) \|k\| + \delta \|g\| < \|g\| .$$

Therefore, for some ω we can take $x > f(\omega)$ such that

$$|g \cdot \tau(\omega)| > |x| > b^* .$$

Since $(f, g \cdot \tau) \in \mathscr{Y}$, by (A.6) of monotonicity we have $(x, g \cdot \tau(\omega)) \in Y(\omega)$. This contradicts (1) in (A.5) of boundedness. Hence $\|g\| \leqq b^*$. Thus, the above inequality also implies that $\|f\| \leqq b^*$. This proves property (1).

To prove property (2), let $k \in \mathscr{H}$. By (A.7) of expansibility, $(\underline{f}, \underline{g}) \in \mathscr{D}$ and $\underline{f} - \delta \underline{g} \ll (1 - \delta) k$. Let $f' = \delta \underline{g} + (1 - \delta) k$. Then, $\underline{f} \ll f'$. Therefore, by (A.6) of monotonicity, $(f', \underline{g}) \in \mathscr{D}$. Therefore $(f', \underline{g}) \in \Phi(k)$. That is, $\Phi(k) \neq \Phi$. The convexity of $\Phi(k)$ is obvious from the definition of Φ. The weak* compactness of $\Phi(k)$ follows from property (1) and the weak* closedness of $\mathscr{D}$. ∎

Under (A.5) of boundedness, the above lemma enables us to define a relation $\Psi: \mathscr{H} \to \mathscr{D}$ by

$$\Psi(k) = \{ (f, g) \in \Phi(k) \mid \psi(f, g) \geqq \psi(f', g') \text{ for all } (f', g') \in \Phi(k) \} .$$

Lemma 4.3: If $(f^0, g^0) \in \Psi(k^0)$, then there exists $c^0 \in \mathscr{L}_\infty \cdot \tau$ with $c \geqq 0$ satisfying the following conditions:

(1) $(f^0 - c^0, g^0 \cdot \tau) \in \mathscr{Y}$.

(2) $\psi(f^0, g^0) = \int [\alpha c^0 + \beta] dP$

(3) $\int \alpha(-h + \delta g) dP \leqq \int \alpha(c^0 - f^0 + \delta g^0) dP$ for all $(h, g \cdot \tau) \in \mathscr{Y}$.

Proof: Since $(f^0, g^0) \in \Psi(k^0)$, by Remark 4.1 there exists $c^0 \in \mathscr{L}_\infty \cdot \tau$ with $c^0 \geqq 0$ which satisfies conditions (1) and (2).

Consider two subsets of $\mathscr{L}_\infty \cdot \tau$ defined by

$$H = \left\{ c - f^0 + \delta g^0 \mid c \in \mathscr{L}_\infty \cdot \tau,\ c \geqq 0,\ \int \alpha c^0 dP < \int \alpha c dP \right\}$$

and

$$C = \{ -h + \delta g \mid (h, g \cdot \tau) \in \mathscr{Y} \}\ .$$

Suppose that $H \cap C \neq \phi$. Then there exist $c \in \mathscr{L}_\infty \cdot \tau$ with $c \geqq 0$ and $(h, g \cdot \tau) \in \mathscr{Y}$ such that $\int [\alpha c^0 + \beta] dP < \int [\alpha c + \beta] dP$ and $c - f^0 + \delta g^0 = -h + \delta g$. Since $(h, g \cdot \tau) \in \mathscr{Y}$, $(h + c, g \cdot \tau) \in \mathscr{Y}$ by (A.6) of monotonicity. Therefore, $(h + c, g) \in \Phi(k^0)$. However, $\psi(f^0, g^0) < \psi(h + c, g)$. This contradicts that $(f^0, g^0) \in \Psi(k^0)$. Hence, $H \cap C = \phi$. Thus,

$$\int \alpha(-h + \delta g) dP < \int \alpha(c - f^0 + \delta g^0) dP$$

for all $(h, g \cdot \tau) \in \mathscr{Y}$ and $c \in \mathscr{L}_\infty \cdot \tau$ with $c \geqq 0$ and $\int \alpha c^0 dP < \int \alpha c dP$. Since c can be chosen arbitrarily close to c^0, this implies (3). ■

Lemma 4.4: Ψ has the following properties.

(1) Ψ is nonempty- and convex-valued.

(2) Ψ is closed, i.e., $\{(k, f, g) \in \mathscr{H} \times \mathscr{D} \mid (f, g) \in \Psi(k)\}$ is a weak* closed subset of $\mathscr{H} \times \mathscr{D}$.

Proof: Let $k \in \mathscr{H}$. Then $\Phi(k)$ is nonempty and weak* compact by Lemma 4.2. Therefore, since ψ is upper semicontinuous, $\Psi(k)$ is nonempty. In addition, since $\Phi(k)$ is convex by Lemma 4.2, the convexity of $\Psi(k)$ immediately follows from the concavity of ψ.

To prove property (2), consider a net (k^N, f^N, g^N) in $\mathscr{H} \times \mathscr{D}$, which is directed by $\geq$, and assume that it converges in the weak* topology to a point $(k_*, f_*, g_*) \in \mathscr{H} \times \mathscr{D}$ such that $(f^N, g^N) \in \Psi(k^N)$ for all N.

First we shall show that $(f_*, g_*) \in \Phi(k_*)$. Since (k^N, f^N, g^N) converges to (k_*, f_*, g_*) in the weak* topology, there exists a sequence (k_n, f_n, g_n) converging to (k_*, f_*, g_*) almost surely such that each (k_n, f_n, g_n) is a convex combinations of some elements (k^N, f^N, g^N)'s (see Sublemma in Appendix). Since $(f^N, g^N) \in \Phi(k^N)$ for each $N, f_n - \delta g_n = (1 - \delta) k_n$ for each n. Hence $f_* - \delta g_* = (1 - \delta) k_*$, which implies that $(f_*, g_*) \in \Phi(k_*)$.

Next we shall show that $(f_*, g_*) \in \Psi(k_*)$. Suppose that there were $(f_0, g_0) \in \mathscr{D}$ such that $\psi(f_*, g_*) < \psi(f_0, g_0)$ and $f_0 - \delta g_0 = (1 - \delta) k^*$. By (A.7) of expansibility, for some $0 < w < 1$ we have

$$\psi(f_*, g_*) < \psi(f_w, g_w) \text{ and } f_w - \delta g_w \ll (1-\delta)k^* ,$$

where (f_w, g_w) is defined by $f_w = w\underline{f} + (1-w)f_0$ and $g_w = w\underline{g} + (1-w)g_0$. Since ψ is upper semi-continuous and k^N converges to k^* in the weak* topology, there is N_1 such that

$$\psi(f^N, g^N) < \psi(f_w, g_w) \tag{4.1}$$

and

$$\int \alpha(f_w - \delta g_w)dP < \int \alpha(1-\delta)k^N dP \tag{4.2}$$

for all $N \geq N_1$. By Remark 4.1, there exists $c_w \in \mathscr{L}_\infty^+ \cdot \tau$ such that $(f_w - c_w, g_w \cdot \tau) \in \mathscr{Y}$ and $\psi(f_w, g_w) = \int [\alpha c_w + \beta] dP$. Also, by Lemma 4.3, for each N we have $c^N \in \mathscr{L}_\infty \cdot \tau$ with $c^N \geqq 0$ satisfying all the conditions in the lemma. Therefore, by (4.1) we have

$$\int [\alpha c^N + \beta] dP < \int [\alpha c_w + \beta] dP \ . \tag{4.3}$$

for all $N \geq N_1$. Thus, by (4.2) and (4.3) we have

$$\int \alpha(c^N - f^N + \delta g^N) dP < \int \alpha(c_w - f_w + \delta g_w) dP$$

for all $N \geq N_1$. Since $(f_w - c_w, g_w \cdot \tau) \in \mathscr{Y}$, this is a contradiction to (3) in Lemma 4.3. ■

Theorem 1: Under (A.1), . . . , (A.8), there exists a discounted golden-rule state.

Proof: Let us define a relation F: $\mathscr{H} \to \mathscr{H}$ by

$$F(k) = \{f \mid (f, g) \in \Psi(k)\} \ .$$

By Lemma 4.1, $\mathscr{H}$ is nonempty, convex, and weak* compact. By Lemma 4.4, F is nonempty- and convex-valued. Morover, since Ψ is closed, so is F. Hence, since the range of F(or, $\mathscr{H}$) is weak* compact, relation F is upper semicontinuous in the weak* topology [see Fan (1952, Lem. 2)]. Thus, by Fan's fixed-point theorem ((1952), Thm. 1), there exists $k^* \in \mathscr{H}$ such that $k^* \in F(k^*)$. By definition of F, there is $(f^*, g^*) \in \Psi(k^*)$ such that $k^* = f^*$. Since $(f^*, g^*) \in \Phi(k^*)$, $k^* - \delta g^* = (1-\delta)k^*$. Therefore $g^* = k^*$. Thus, $(k^*, k^*) \in \Psi(k^*)$, which implies that k^* is a discounted golden-rule state ■

V. *Optimal Stationary States*

In this section we shall prove that any discounted golden-rule state is an optimal stationary state.

Lemma 5.1: For any feasible program $K = \{k_t \mid t \in T\}$, there exists $(f_0, g_0) \in \mathscr{D}$ having the following properties:

(1) $(f_0, g_0) \in \Phi(k_0)$.

(2) For any $\varepsilon > 0$ there exists s_0 such that for all $s > s_0$,

$$\sum_{t=1}^{s} \delta^{t-1} \psi(f_0, g_0) \geqq U_s(K) - \varepsilon \ .$$

Proof: Define h_t by $h_t = k_t \cdot \tau^{-t}$. Also, for each $s > 0$, let us define (f_s, g_s) by

$$f_s=\sum_{t=1}^{s}\delta^{t-1}h_{t-1}/\sum_{t=1}^{s}\delta^{t-1} \text{ and } g_s=\sum_{t=1}^{s}\delta^{t-1}h_t/\sum_{t=1}^{s}\delta^{t-1} .$$

Since program K is feasible, $(h_{t-1},h_t)\in\mathscr{D}$ for all t. Therefore, by (A.3) of convexity, $(f_s,g_s)\in\mathscr{D}$ for all s. Also, since (h_{t-1},h_t) is uniformly bounded by (A.5), so is sequence (f_s,g_s). Since $0<\delta<1$, it is clear that sequence (f_s,g_s) is converging in the norm topology. Hence, by Lemma 3.1 we have a limit $(f_0,g_0)\in\mathscr{D}$. Thus, we have

$$\begin{aligned} f_0-\delta g_0 &= \lim_{s\to\infty}(f_s-\delta g_s) \\ &= \lim_{s\to\infty}(k_0-\delta^s h_s)/\sum_{t=1}^{s}\delta^{t-1} \end{aligned}$$

This implies that $f_0-\delta g_0=(1-\delta)k_0$, i.e., property (1).

Moreover, it follows from the concavity of u in (A.3) that

$$\int u(f_s,g_s\cdot,\cdot)dP\geqq\int\sum_{t=1}^{s}\delta^{t-1}u(h_{t-1},h_t\cdot\tau,\cdot)dP/\sum_{t=1}^{s}\delta^{t-1} .$$

for each s. In addition, since τ^{t-1} is measure-preserving by (A.1),

$$\begin{aligned} &\int\delta^{t-1}u(h_{t-1},h_t\cdot\tau,\cdot)dP \\ =&\int\delta^{t-1}u(h_{t-1}\cdot\tau^{t-1},h_t\cdot\tau^t,\tau^{t-1})dP \\ =&\int u_t(k_{t-1},k_t,\cdot)dP . \end{aligned}$$

Hence, for each s, we have the following inequality,

$$\int\sum_{t=1}^{s}\delta^{t-1}u(f_s,g_s\cdot\tau,\cdot)dP\geqq\int\sum_{t=1}^{s}u_t(k_{t-1},k_t,\cdot)dP ,$$

i.e.,

$$\sum_{t=1}^{s}\delta^{t-1}\psi(f_s,g_s)\geqq U_s(K) .$$

Since $\delta<1$, by virtue of Lemma 3.2, this implies property (2). ■

Theorem 2: Under (A.1), . . . , (A.5), any discounted golden-rule state is an optimal stationary state.

Proof: Let k be a discounted golden-rule state. Let us consider any feasible program $K=\{k_t\,|\,t\in T\}$ such that $k_0=k$. By Lemma 5.1, we have $(f_0,g_0)\in\mathscr{D}$ with properties (1) and (2) in the lemma. Since k is discounted golden-rule state, $\psi(k,k)\geqq\psi(f_0,g_0)$. Therefore, by property (2), for any $\varepsilon>0$, we have

$$\begin{aligned} \int\sum_{t=1}^{s}u_t(k\cdot\tau^{t-1},k\cdot\tau^t,\cdot)dP &= \int\sum_{t=1}^{s}\delta^{t-1}u(k\cdot\tau^{t-1},k\cdot\tau^t,\tau^{t-1})dP \\ &=\sum_{t=1}^{s}\delta^{t-1}\psi(k,k) \\ &\geqq\sum_{t=1}^{s}\delta^{t-1}\psi(f_0,g_0) \\ &\geqq U_s(K)-\varepsilon \end{aligned}$$

for all sufficiently large s. This implies the optimality of k. ■

Corollary 1: Under (A.1), . . . , (A.8), there exists an optimal stationary state.

Proof: This immediately follows from Theorems 1 and 2. ■

We sometimes allow that $(0,0)\in\mathscr{D}$, i.e., the zero capital stock is a stationary state. Under the natural assumption that nothing can be produced without capital, the zero capital stock is an optimal stationary state. In this case the existence of optimal stationary states is an obvious problem. However, we are not concerned with such a meaningless stationary state. In fact, the above theorem insures the existence of a nontrivial optimal stationary state. In order to show that, we assume the following condition.

(A.9): If $(0,0)\in\mathscr{D}$, then there is $(f,g)\in\mathscr{D}$ such that $f=\delta g$ and $\int u(f,g\cdot\tau,\cdot)dP>\int u(0,0,\cdot)dP$

Corollary 1: Under (A.1), . . . , (A.9), there exists an optimal stationary state k with $k\neq 0$.

Proof: The golden-rule state k proved to exist in Theorem 1 can not be 0 by (A.9). Hence Theorem 2 implies the existence of an optimal stationary state k with $k\neq 0$.

APPENDIX

In this appendix we shall prove Lemmas 3.1 and 3.2 in section 3. The technique for proof is standard [for example, see Takekuma (1980)].

Sublemma: For any net (generalized sequence) $\{f_n|n\in\Lambda\}$ in $\mathscr{L}_\infty$ converging to f_0 in $\sigma(\mathscr{L}_\infty,\mathscr{L}_1)$, there exists a sequence $\{\underline{f}_i|i=1,2,\dots\}$ in $\mathscr{L}_\infty$ converging to f_0 almost surely such that each $\underline{f}_i$ is a convex combination of some elements in $\{f_n|n\in\Lambda\}$.

Proof: Assume that a net $\{f_n|n\in\Lambda\}$ in $\mathscr{L}_\infty$ converges to f_0 in $\sigma(\mathscr{L}_\infty,\mathscr{L}_1)$. Obviously $f_n\in\mathscr{L}_1$ for each n and $f_0\in\mathscr{L}_1$. Let us regard $\{f_n|n\in\Lambda\}$ as a net in $\mathscr{L}_1$ and show that f_n converges to f_0 in $\sigma(\mathscr{L}_1,\mathscr{L}_\infty)$. Let $p\in\mathscr{L}_\infty$. Clearly $p\in\mathscr{L}_1$. Therefore, since f_n converges to f_0 in $\sigma(\mathscr{L}_\infty,\mathscr{L}_1)$, $\lim\int p\cdot f_n dP=\int p\cdot f_0 dP$. This proves that f_n converges to f_0 in $\sigma(\mathscr{L}_1,\mathscr{L}_\infty)$.

Let A be the smallest norm-closed convex subset of $\mathscr{L}_1$ including $\{f_n|n\in\Lambda\}$. Then set A is $\sigma(\mathscr{L}_1,\mathscr{L}_\infty)$-closed [see Dunford & Schwartz (1964, p. 422, Thm. 13)]. Hence $f_0\in A$, because f_n converges to f_0 in $\sigma(\mathscr{L}_1,\mathscr{L}_\infty)$. On the other hand, set A is identical with the norm-closure in $\mathscr{L}_1$ of the convex hull of $\{f_n|n\in\Lambda\}$ [see Dunford & Schwartz (1964, p. 425, Lem. 2.4 (ii))]. Therefore there exists a sequence $\{\underline{f}_i|i=1,2,\dots\}$ converging to f_0 in the norm topology of $\mathscr{L}_1$ such that each element $\underline{f}_i$ of the sequence is a convex combination of some elements in $\{f_n|n\in\Lambda\}$. Moreover, without loss of generality, we can assume that the sequence converges to f_0 almost surely. This is because the convergence in the mean implies the convergence in measure, and because any sequence of measurable functions which converges in measure has a subsequence which converges almost surely [see Dunford

& Schwartz (1964, p. 122, Thm. 6 and p. 150, Cor. 13)]. ■

Proof of Lemma 3.1: The convexity of $\mathscr{Y}$ immediately follows from (1) of (A.3). To prove the weak* closedness of $\mathscr{Y}$, take any net $\{(f_n, g_n) | n \in \Lambda\}$ in $\mathscr{Y}$ converging to a point (f_0, g_0) in the weak* topology. By the above sublemma there is a sequence $\{(\underline{f}_i, \underline{g}_i) | i = 1, 2, \dots\}$ in $\mathscr{Y}$ converging to (f_0, g_0) a.s. Therefore, since $(\underline{f}_i(\omega), \underline{g}_i(\omega)) \in Y_1(\omega)$ a.s. for each i, it follows that $(f_0(\omega), g_0(\omega)) \in Y_1(\omega)$ a.s. by (1) of (A.4). That is, $(f_0, g_0) \in \mathscr{Y}$. This proves the weak* closedness of $\mathscr{Y}$. ■

Proof of Lemma 3.2: The concavity of ψ immediately follows from (2) of (A.3). To prove the upper semicontinuity of ψ, take any net $\{(f_n, g_n) | n \in \Lambda\}$ in $\mathscr{D}$ converging to a point $(f_0, g_0) \in \mathscr{D}$ in the weak* topology, and assume that $\psi(f_n, g_n) \geqq b$ for all $n \in \Lambda$. By the above sublemma there is a sequence $\{(\underline{f}_i, \underline{g}_i) | i = 1, 2, \dots\}$ converging to (f_0, g_0) almost surely such that each $(\underline{f}_i, \underline{g}_i)$ is a convex combination of some elements in $\{(f_n, g_n) | n \in \Lambda\}$. Since function ψ is concave, $\psi(\underline{f}_i, \underline{g}_i) \geqq b$ for all i. This implies that

$$\limsup_i \int u(\underline{f}_i, \underline{g}_i \cdot \tau, \cdot) dP \geqq b .$$

Since $\|f_n\| \leqq b^*$ for all n, $\|\underline{f}_i\| \leqq b^*$ for all i. Therefore, from (2) of (A.5) it follows that $u_1(\underline{f}_i(\omega), \underline{g}_i \cdot \tau(\omega), \omega) \leqq u^*$ a.s. Hence, by Fatou's lemma, we have

$$\limsup_i \int u(\underline{f}_i, \underline{g}_i \cdot \tau, \cdot) dP \leqq \int \limsup_i u(\underline{f}_i, \underline{g}_i \cdot \tau, \cdot) dP .$$

Moreover, since $u_1(\cdot, \cdot, \omega)$ is upper semicontinuous by (2) of (A.2), $\limsup u(\underline{f}_i(\omega), \underline{g}_i \cdot \tau(\omega), \omega) \leqq u(f_0(\omega), g_0 \cdot \tau(\omega), \omega)$ a.s., which implies that

$$\int \limsup_i u(\underline{f}_i, \underline{g}_i \cdot \tau, \cdot) dP \leqq \int u(f_0, g_0 \cdot \tau, \cdot) dP .$$

Thus we can conclude that $\psi(f_0, g_0) \geqq b$. This proves the upper semicontinuity of ψ. ■

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