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CONDITION FOR THE AGGREGATION OF INDUSTRIAL SECTORS INTO THE CAPITAL-GOODS SECTOR AND THE CONSUMPTION-GOODS SECTOR

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I. *Introduction*

The author has presented a Leontief-type multi-sector model based on the following assumption and has analyzed its characteristics from the standpoint of capital theory [Ara (1975)].

(1) Labor is the only primary factor of production, and wages are paid on deferred terms.

(2) There exist n industrial sectors, with each industrial sector producing a different type of product, which are utilized as consumption-goods or as liquid capital goods, and once utilized as liquid capital goods, their value is totally transferred to other products.

(3) Production coefficients (labor coefficient and capital coefficient) are all given and fixed, and no technical progress exists.

(4) The labor coefficient vector is positive, while the capital coefficient matrix is non-positive and indecomposable.

(5) The wage and profit rates are uniform in all industrial sectors, and the demand and supply in each industrial sector are balanced.

In this paper, the author adopts these assumptions as fundamental, but does not assume that the capital coefficient matrix is indecomposable. When the existence of the consumption-goods industry is taken into consideration, which produces only consumption-goods as the final product, the capital coefficient matrix must be appearently decomposable. The object of this paper is to analyze a condition for the aggregation of industrial sectors which contain a plural number of capital-goods sectors and a plural number of consumption-goods sectors into a single capital-goods sector and a single consumption-goods sector.

II. *A Multi-sector Model*

We define the following symbols:

m_{ij}=the capital coefficient in the j-th sector from the i-th sector,

l_i=the labor coefficient in the i-th sector,

p_i=market price of the i-th sector's output.

We further define the following matrix and vectors:

M=the capital coefficient matrix whose elements are m_{ij},

l=the labor coefficient vector whose elements are l_i (column vector),

p=the price vector whose elements are p_i (column vector).

Assuming that m capital goods sectors and $(n-m)$ consumption goods sectors exist, we can divide the capital coefficient matrix M as

$$M=\left[\begin{array}{ccc|ccc} m_{11} & \cdots & m_{1m} & m_{1m+1} & \cdots & m_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{m1} & \cdots & m_{mm} & m_{mm+1} & \cdots & m_{mn} \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]=\left[\begin{array}{c|c} M_{kk} & M_{kc} \\ \hline 0 & 0 \end{array}\right]$$

It can be seen that M is a decomposable matrix. M_{kk} is the capital coefficient matrix between the capital-goods sectors, and M_{kc} is the capital coefficient matrix in the consumption-goods sector from the capital-goods sector. M_{kk} is a square matrix, but M_{kc} is not necessarily a square matrix. Next, corresponding to this division, we can also divide l and p as

$$l'=((l_1, \ldots, l_m), (l_{m+1}, \ldots, l_n))\equiv(l'_k, l'_c),$$
$$p'=((p_1, \ldots, p_m), (p_{m+1}, \ldots, p_n))\equiv(p'_k, p'_c),$$

where prime (′) is a symbol indicating transpotion, l_k is the labor coefficient vector in the capital goods sector, l_c is the labor coefficient vector in the consumption-goods sector, p_k is the price vector in the capital goods sector, and p_c is the price vector in the consumption-goods sector.

Using these symbols, we can express the model we have adopted by the following equations:

(1) $p'_k=(1+r)p'_kM_{kk}+wl'_k,$

(2) $p'_c=(1+r)p'_kM_{kc}+wl'_c,$

where r=the rate of profit, w=the rate of wages. The implication of these equation systems is obvious, and need not be discussed here in detail. In what follows, we assume that $w=1$.

Supposing that λ_k denotes the Frobenius root of the matrix M_{kk}, we assume it is smaller than unity. That λ_k is smaller than unity is a condition that must be satisfied for the capital-goods sector to be "productive." Thus, if we write

$$1>\lambda_k\equiv\frac{1}{1+r_k}>0,$$

then r_k denotes the maximum value of the rate of profit that the capital-goods sectr can seek. Here, it is not necessary that M_{kk} be indecomposable, but we assume that the left hand Frobenius vector (the eigen-vector corresponding to the Frobenius root) of M_{kk} is a positive vector. And as has previously been shown [Ara (1978), p. 36], it is obvious that

(3) $p'_k=l'_k[E_m-(1+r)M_{kk}]^{-1}>0 \quad \text{for} \quad r_k>r\geq 0,$

where E_m denotes the unit matrix of order m, and the inverse matrix multiplied to l'_k in the right hand side of this equation is an increasing function of r.

What is important at this point is to determine the condition for the relative prices in p_k in (3), which will allow them to remain constant in spite of all possible changes in r. In regard to this problem, the author has previously shown that the following theorem holds true [Ara (1978), p. 37]:

Theorem (1)

For the relative prices in p_k to be independent of changes in r, it is necessary and sufficient that l_k is the left hand Frobenius vector of M_{kk}.

As has been mentioned elsewhere [Ara (1980), p. 440], that l_k is the left and Frobenius vector of M_{kk} is a necessary and sufficient condition for the "organic composition of capital" to be uniform in all capital-goods sectors in a Marxian political economy. Therefore, only when the organic composition of capital is uniform in all capital-goods sectors, are the relative prices in p_k indepenednt of changes in r.

We have elucidated a condition in which the relative prices of capital goods in the capital-goods sector remain constant. Next, let us examine the case of the consumption-goods sector. As is clear from (2), only when l'_c is directly proportional to p'_kM_{kc}; namely

(4) $p'_kM_{kc}\approx l'_c$,

are the relative prices in p_c independent of changes in r. This relationship shows that the "capital intensity of labor" in the consumption-goods sector is uniform in all sectors producing consumption-goods.

Assuming that the organic composition of capital in the capital goods sector is uniform in all sectors, it follows that

(5) $p_k\approx l_k$.

Substituting l_k for p_k in (4), we write

$$l'_kM_{kc}=\lambda_c l'_c,$$

where λ_c is a proportional constant relating $l_k'M_{kc}$ to l_c'. As will be mentioned below, the relative size of λ_c and λ_k plays an important role in determining how the capital coefficient in the consumption-goods sector varies with changes in the rate of profit.

III. *Condition for the Aggregation of Industrial Sectors (1)*

We have analyzed a condition for the relative prices in the capital-goods sector and the consumption-goods sector, respectively, to remain constant in spite of changes in the rate of profit. As mentioned above, the condition can be expressed as

(6) $l'_kM_{kk}=\lambda_k l'_k$,

(7) $l'_kM_{kc}=\lambda_c l'_c$.

Let L denote the diagonal matrix whose diagonal elements are composed of the elements in l_k and l_c, namely

$$L\equiv\begin{bmatrix} l_1\cdots\cdots 0 & 0\cdots\cdots 0 \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ 0\cdots\cdots l_m & 0\cdots\cdots 0 \\ 0\cdots\cdots 0 & l_{m+1}\cdots 0 \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ 0\cdots\cdots 0 & 0\cdots\cdots l_n \end{bmatrix}.$$

And let B denote the matrix obtained by the regular transformation of the capital coefficient matrix M by L, namely

$$(8)\quad LML^{-1}=L\left[\begin{array}{c|c} M_{kk} & M_{kc} \\ \hline 0 & 0 \end{array}\right]L^{-1}\equiv\left[\begin{array}{c|c} B_{kk} & B_{kc} \\ \hline 0 & 0 \end{array}\right]\equiv B,$$

then, the total sums of each column's elements in B_{kk} are all equal to λ_k, while the total sums of each column's elements in B_{kc} are all equal to λ_c. This will be proved in the following section.

First, it can be shown that the total sums of each column's elements in B_{kk} are all equal to λ_k. By the assumption, the left hand Frobenius vector of the matrix M_{kk} is l_k. Let L_k denote the diagonal matrix whose diagonal elements are composed of the elements in l_k. Then, according to the theorem presented previously [Ara (1980), pp. 394–395], the total sums of each column's elements in the matrix obtained by the regular transformation of M_{kk} by L_k are all equal to the Frobenius root of M_{kk} (i.e. λ_k). However, it is clear from the calculations that the matrix obtained by the regular transformation of M_{kk} by L_k is B_{kk} in (8). Thus, the total sums of each column's elements in B_{kk} are all equal to λ_k. Therefore, assuming e_m denotes the row vector whose m elements are all unity, we have

$$(9)\quad e_mL_kM_{kc}L_k^{-1}=e_mB_{kk}=\lambda_ke_m,$$

which is the desired result.

Next, it can be shown that the total sums of each column's elements in B_{kc} are all equal to λ_c. Assuming again that L_c denotes the diagonal matrix whose diagonal elements are composed of the elements in l_c, and e_{n-m} is the row vector whose $(n-m)$ elements are all unity, (7) becomes

$$e_mL_kM_{kc}=\lambda_ce_{n-m}L_c.$$

Post-multiplying by L_c^{-1}, we have

$$e_mL_kM_{kc}L_c^{-1}=\lambda_ce_{n-m}.$$

However, it is clear from the calculation that

$$L_kM_{kc}L_c^{-1}=B_{kc}.$$

Therefore

$$(10)\quad e_mB_{kc}=\lambda_ce_{n-m}.$$

Thus we obtain the desired result.

Here we show a simple example. We assume M and l as the following:

$$M=\left[\begin{array}{cc|cc} 0.4 & 0.2 & 0.2 & 0.4 \\ 0.1 & 0.5 & 0.3 & 0.2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad l=\left(\begin{array}{c} 1 \\ 2 \\ \hline 1 \\ 1 \end{array}\right)$$

First, from the characteristic equation of M_{kk}

$$f(\lambda)\equiv\left|\begin{array}{cc} \lambda-0.4 & -0.2 \\ -0.1 & \lambda-0.5 \end{array}\right|=\lambda^2-0.9+0.18$$
$$=(\lambda-0.6)(\lambda-0.3)=0,$$

the Frobenius root is $\lambda_k=0.6$. From this we can obtain the left hand Frobenius vector (normalized one) as

$$(1/3,\ 2/3)\begin{bmatrix} \lambda_k-0.6 & -0.2 \\ -0.1 & \lambda_k-0.5 \end{bmatrix}=(0,\ 0)$$

Thus, it is obvious that $l'_k=(1,\ 2)$ is the Frobenius vector. Furthermore, it is also clear that

$$\begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}^{-1}=\begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}\equiv B_{kk}.$$

Namely, the sums of each column's elements in B_{kk} are both equal to $\lambda_k=0.6$.

Next, we have

$$\begin{aligned} l'_kM_{kc}&=(1,\ 2)\begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}=(0.8,\ 0.8) \\ &=0.8\ (1,\ 1)=\lambda_c l'_c \end{aligned}$$

Thus, from the above, it follows that

$$LML^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.4 \\ 0.1 & 0.5 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.4 \\ 0.2 & 0.5 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}=\left[\begin{array}{c|c} B_{kk} & B_{kc} \\ \hline 0 & 0 \end{array}\right]=B.$$

Mamely, the sums of each column's elements in B_{kk} are both equal to $\lambda_k=0.6$, and the sums of each column's elements in B_{kc} are both equal to $\lambda_c=0.8$.

It should be noted that if the labor coeffieicnt vector is the left hand Frobenius vector of the capital coefficient matrix M; for example, if

$$l'=\left(1,\ 2,\ \frac{4}{3},\ \frac{4}{3}\right),$$

and assuming that F denotes the diagonal matrix whose diagonal elements are composed of the elements in l', we have

$$FMF^{-1}=\begin{bmatrix} 0.4 & 0.1 & 0.15 & 0.3 \\ 0.2 & 0.5 & 0.45 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the capital coefficient is equal to the Frobenius root, i.e. 0.6, in all industrial sectors. This shows the case where the organic composition of capital is uniform in all capital goods and consumption-goods sectors, which is a special case in this paper.

IV. *Condition for the Aggregation of Industrial Sectors (2)*

It has been previously proposed by Hicks that economic analysis can be performed by regarding the commodities, whose relative prices remain constant, as a single commodity [Hicks (1939), p. 312]. Following this suggestion, let us elucidate a condition for the aggregation of the industrial sectors with constant relative prices.

Let x represent the output vector before aggregation, ξ the final demand vector, and M the capital coefficient matrix, as before. Then, under the condition of balanced demand and supply, we have

$$(11)\quad x=Mx+\xi.$$

It should be noted that while our purpose is to obtain a condition for the aggregation of outputs of industrial sectors, it is not a rational approach to deal with a single value obtained by the simple summation of the elements in x, since the elements in x are composed of the outputs measured in different physical units; for example, if we assign the value 20 to the summation of 10 tons of rice and 10 tractors. The aggregated value is significant only if the outputs are measured in common units. Considering this, we adopt the evaluation at the equilibrium price. Let P denote a diagonal matrix whose elements are composed of equilibrium prices, namely

$$P=\begin{bmatrix} p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n \end{bmatrix}.$$

And if we write

$$(12)\quad Px=y,$$

y is a vector whose elements are composed of the values of quantity $\times$ price. Let us write

$$(13)\quad PMP^{-1}\equiv A.$$

A corresponds to the input efficient matrix in the normal type of Leontief system. Generally speaking, P can vary with the rate of profit, so that even if the matrix M (physical coefficients) does not change, A itself can change with changes in the rate of profit.

Let $P\xi=\zeta$, where ζ is the final demand vector whose elements are composed of values. Thus pre-multiplying both sides of (11) by P, we can obtain

$$(14)\quad y=Ay+\zeta,$$

It should be noted here that while we have assumed the Frobenius root of M to be smaller than unity, as long as the Frobenius root of M is equivalent to that of A—since the regular transformation of M by P does not affect the characteristic root—, we can present

$$(15)\quad y=[E_n-A]^{-1}\zeta$$

where E_n denotes the unit matrix of order n.

Let S denote the aggregation matrix composed of row vectors which assign unity to the places of the industrial sectors to be aggregated and zero to the rest. For example, if we intend to aggregate the first and the second sector and the third and the fourth sector,

$$S=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Writing

$$Sy=y^* \quad S\zeta=\zeta^*,$$

let us express the Leontief system consistent with the system of y^* as

$$(16) \quad y^*=A^*y^*+\zeta^*,$$

where A^* denotes the input coefficient matrix in the Leontief system after the aggregation of industrial sectors. Thus, assuming the Frobenius root of A^* is smaller than unity, from (16),

$$(17) \quad y^*=[E_s-A^*]^{-1}\zeta^*$$

holds, where E_s denotes the unit matrix whose order equals the number of industries after aggregation.

In the system prior to aggregation, let the final demand vector ζ change *arbitrarily* by $\Delta\zeta$. Then from (15), we obtain

$$(18) \quad \Delta y=[E_n-A]^{-1}\Delta\zeta.$$

Similarly, from (17),

$$(19) \quad \Delta y^*=[E_s-A^*]^{-1}\Delta\zeta^*$$

holds. Therefore, $\Delta\zeta^*=S\Delta\zeta$. If the relationship

$$(20) \quad S\Delta y=\Delta y^*$$

holds for *all possible* $\Delta\zeta$, we can say that the aggregation of industrial sectors is 'acceptable.' The problem here is to determine a condition for the acceptable aggregation of industrial sectors. Regarding this problem, it has already been made clear that for the aggregation to be acceptable,

$$(21) \quad SA=A^*S$$

is necessary and sufficient [Hatanaka (1952), p. 302]. The equivalent condition was formulated as the following theorem [Ara (1959), p. 260]:

Theorem (2)

In aggregation n *industrial sectors into s industrial sectors, let A_{ij} denote the sub-matrix made by the industrial sectors to be aggregated into a single industrial sector, and let us show the input coefficient matrix before aggregation, A, in explicit form,*

$$A=\begin{bmatrix} A_{11} & A_{12}\cdots\cdots A_{1s} \\ A_{21} & A_{22}\cdots\cdots A_{2s} \\ \vdots & \vdots \quad\quad \vdots \\ A_{s1} & A_{s2}\cdots\cdots A_{ss} \end{bmatrix}$$

Then, for the aggregation of industrial sectors to be acceptable it is necessary and suf-

ficient that the total sums of each column's elements in respevtice sub-matrices A_{ij} be equal.

As an example, we show the input coefficient matrix composed of 5 industrial sectors,

$$A=\left[\begin{array}{cc|cc|c} 0.2 & 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.2 & 0.1 \\ \hline 0.1 & 0.0 & 0.0 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.1 & 0.1 \\ \hline 0.2 & 0.2 & 0.2 & 0.2 & 0.1 \end{array}\right]$$

According to the theorem above, A can be aggregated into 3 blocks, following division lines. Namely, we can aggregate the first and the second sector and the third and the fourth sector. The aggregation matrix S is

$$S=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$A^*=\begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.1 \end{bmatrix},$$

and by calculation,

$$SA=A^*S$$

can be confirmed.

V. *Condition for the Aggregation of Industrial Sectors (3)*

For our primary purpose, let us return to the model shown in section II and examine a condition for the aggregation of capital-goods sectors and consumption-goods sectors, respectively, into a single sector.

As has been discussed, for the relative prices of capital goods to remain constant in spite of changes in the rate of porfit, it is necessary and sufficient that l_k be the left hand Frobenius vector of M_{kk}. Assuming that this condition holds, then, it has been confirmed [Ara (1978), p. 37] that

$$(22) \quad p_k=l_k\left\{\frac{r_k+1}{r_k-r}\right\}$$

holds. Or, for simplicity, writing

$$(23) \quad \left\{\frac{r_k+1}{r_k-1}\right\}\equiv\pi_k(r),$$

we have

(24) $p_k = \pi_k(r) l_k$.

Of course, $\pi_k(r)$ is an increasing function in r (note that $r_k > r \geq 0$).

Next we showed that for the relative prices of consumption-goods to remain constant in spite of changes in the rate of profit, it is necessary that

(25) $l'_k M_{kc} = \lambda_c l'_c$

holds. To repeat, λ_c is a proportional constant relating $l'_k M_{kc}$ to l'_c. Let us return to (2) above,

(26) $p'_k = (1+r) p'_k M_{kc} + l'_c$

(it is assumed that $w=1$). Using (22) and (25), we have

(27) $p_c = \pi_c(r) l_c$,

where

$$\pi_c(r) \equiv [(1+r)\pi_k(r)\lambda_c + 1],$$

which is also an increasing function in r.

Let $\Pi(r)$ denote the diagonal matrix whose diagonal elements are composed of $\pi_k(r)$, in the first m (number of capital-goods sectors) places, and $\pi_c(r)$, in the other $(n-m)$ (number of consumption-goods sectors) places. Then, combining (24) and (27) together, we have

(28) $P = \Pi(r) L$.

It should be noted that in the last section we set

(29) $PMP^{-1} \equiv A$.

Substituting (28) for P in (29), and using (8), we can obtain

(30) $\Pi(r) LML^{-1} \Pi(r)^{-1} \equiv \Pi(r) B \Pi(r)^{-1} = A$,

or, extending this in detail,

(31) $\Pi(r) \left[\begin{array}{c|c} B_{kk} & B_{kc} \\ \hline 0 & 0 \end{array} \right] \Pi(r)^{-1} = \left[\begin{array}{c|c} B_{kk} & \pi_k/\pi_c \, B_{kc} \\ \hline 0 & 0 \end{array} \right] = A$

As we have already elucidated, the total sums of each column's elements in B_{kk} are all equal to λ_k, while the total sums of each column's elements in B_{kc} are all equal to λ_c. In the case of the regular transformation of the matrix B by $\Pi(r)$, the total sums of each column's elements in B_{kk} are obviously independent of r. In contrast, the total sums of each column's elements in B_{kc} are independent of r, provided that $\pi_k = \pi_c$, while otherwise they vary with changes in the rate of profit (of course, in spite of this statement, the condition still holds that the capital coefficient in the consumption-goods sector is uniform in all sectors producing consumpion-goods).

In the above analyses, we have shown that if (6) and (7) hold, under the condition of acceptability, we can aggregate the capital-goods sectors and consumption-goods sectors, respectively, into a single industrial sector, which was the objective of the paper. Here we denote further points of interest.

First, it can be seen that

$$(32)\quad \operatorname{sign}\frac{d}{dr}\left(\frac{\pi_k}{\pi_c}\right)=\operatorname{sign}\frac{d}{dr}\left[\frac{r_k+1}{(1+r)(r_k+1)\lambda_c+r_k-r}\right]$$
$$=\operatorname{sign}(\lambda_k-\lambda_c),$$

where both λ_k and λ_c are parameters physically determined by the matrix M and the labor coefficient vector l. If

$$\lambda_k>\lambda_c,$$

then the rise in the rate of profit increases the capital coefficient in the consumption-goods sector, while inversely, if

$$\lambda_c>\lambda_k,$$

then the rise in the rate of profit decreases the capital coefficient in the consumption-goods sector.

The above description concerns the capital coefficient. For the labor coefficient, it is noteworthy that the labor coefficient, after aggregation of the industrial sectors, depends similarly upon the level of the rate of profit. In the vector x, let x_k denote the vector corresponding to the capital-goods sector, x_c the vector corresponding to the consumption-goods sector. Then, in regard to the system after aggregation, the labor coefficient in the capital-goods sector is expressed by

$$\frac{l'_k x_k}{p_k' x_k}.$$

However, from (24), we have $p_k=\pi_k(r)l_k$. Thus

the labor coefficient in the capital-goods sector

$$=\frac{1}{\pi_k(r)}=\frac{r_k-r}{r_k+1}$$

holds. Obviously, this is a decreasing function in r. Similarly, the labor coefficient in the consumptilon-goods sector is expressed by

$$\frac{l'_c x_c}{p'_c x_c},$$

which, via $p_c=\pi_c(r)l_c$ given by (27), leads to

the labor coefficient in the consumption-goods sector

$$\frac{1}{\pi_c(r)}=\frac{1}{(1+r)\pi_k(r)\lambda_c+1}$$

which is also a decreasing function in r.

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