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EQUITY AND EFFICIENCY IN THE PUBLIC GOODS ECONOMY: SOME COUNTEREXAMPLES*

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I. *Introduction*

In dividing a fixed amount of homogeneous cake among several individuals, an apparently equitable as well as efficient method is to divide the whole amount into equal pieces. Furthermore, this equitable and efficient division can be realized by ordering individuals arbitrarily and letting each individual cut out a piece in this order subject to the condition that he should receive the piece himself if no succeeding individual voluntarily accepts it. Therefore, in this problem of a cake division, not only does there exist an equitable and efficient division, but also there exists a well-defined procedure for actually attaining it.

An essentially similar remark can be made concerning the problem of allocating several divisible commodities in fixed supply among several individuals. Following Foley (1967), Pazner and Schmeidler (1974), and Varian (1974), let us call an allocation *equitable* if and only if no individual feels better off with the commodity bundle that any other individual receives than with what he himself receives. Then there exists an equitable as well as efficient allocation that can be attained by a well-defined mechanism; that is, an equitable and efficient allocation can be brought forth by dividing the aggregate commodity bundle equally among individuals and, then, letting them exchange their initial allotments in the competitive markets if they so wish. By virtue of the second welfare theorem, the resulting allocation is efficient, while the equity thereof is guaranteed by the identity of individual budget sets. The importance of this theorem, which is due originally to Foley, lies in the fact that the authority, which is in charge of the equity and efficiency of the *final* allocation, may leave the task to the impersonal market mechanism if only the equal *initial* allocation could be somehow secured.

The viewpoint adopted in this paper is that we may examine the relative performance of the alternative public goods allocation mechanisms by asking whether or not they allow the analogue of Foley's theorem to go through in the public goods economy, thereby enrich-

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ing our understanding of the working thereof. Our verdicts will be largely negative, which will be substantiated by a series of counterexamples.

II. *Equity, Efficiency, and Fairness*

It is useful to have several basic concepts in a fairly general framework at hand. Let there be s individuals, m private goods, and n public goods in the economy, which are indexed by $i \in S := \{1, 2, ..., s\}$, $j \in M := \{1, 2, ..., m\}$, and $k \in N := \{1, 2, ..., n\}$, respectively. Each individual $i \in S$ is characterized by his initial endowment of private goods $\omega_i \in R^m_+$ and his (differentiable and concave) utility function $U_i(X_i, Y)$ with positive marginal utilities everywhere, which is defined on R^{m+n}_+ with values in R, where $X_i \in R^m_+$ and $Y \in R^n_+$ denote, respectively, the private goods vector alloted to individual i and the public goods vector that is common to all individuals.[1] The differentiable and convex production possibility frontier of the economy is denoted by

$$(1) \quad F(\sum_{i \in S} X_i, Y, \sum_{i \in S} \omega_i) = 0,$$

which says that $\sum_{i \in S} X_i$ amount of private goods and Y amount of public goods can be produced by putting $\sum_{i \in S} \omega_i$ amount of initially held private goods into pıoductive use. We assume that each and every individual has free access to the production technology, so that each individual $i \in S$ can secure by himself the bundle $(X_i, Y_i) \in R^{m+n}_+$ satisfying $F(X_i, Y_i, \omega_i) = 0$.

In what follows, an allocation is an assignment of the private and public goods vectors to each and every individual, whereas a *feasible allocation* is an allocation $[\{X_i\}_{i \in S}, Y] \in R^{sm+n}_+$ that satisfies (1). A feasible allocation $[\{X_i\}_{i \in S}, Y] \in R^{sm+n}_+$ is said to be *equitable* if and only if no individual envies the allotment to any other individual, that is, $U_i(X_i, Y) \geqq U_i(X_j, Y)$ holds true for all $i, j \in S$. A feasible allocation $[\{X_i\}_{i \in S}, Y] \in R^{sm+n}_+$ is said to be *efficient* if and only if there exists no other feasible allocation that improves the welfare position of at least one individual without harming anybody else. Finally, a feasible allocation that is equitable as well as efficient is said to be *fair*.

Recollect that, for any efficient and feasible allocation $[\{X_i\}_{i \in S}, Y] \in R^{sm+n}_+$, there exist positive numbers $\lambda_i > 0$ $(i \in S)$ and $\mu > 0$ such that

$$(2) \quad \forall i \in S, \forall j \in M: \lambda_i \frac{\partial U_i}{\partial X_{ij}} - \mu \frac{\partial F}{\partial X_{ij}} \leqq 0,$$

$$(3) \quad \forall k \in N: \sum_{i \in S} \lambda_i \frac{\partial U_i}{\partial Y_k} - \mu \frac{\partial F}{\partial Y_k} \leqq 0.$$

and

[1] R denotes the set of all real numbers, and R^l denotes the l-fold Cartesian product of R. For any $x = (x_1, \ldots, x_l)$ and $y = (y_1, \ldots, y_l)$ in R^l, $x \geqq y$ [resp. $x > y$] means that $x_h \geqq y_h$ [resp. $x_h > y_h$] holds true for all $h = 1, \ldots, l$. Finally, the non-negative orthant of R^l is defined by $R^l_+ = \{x \in R^l | x \geqq 0\}$, whereas the positive orthant of R^l is defined by $R^l_{++} = \{x \in R^l | x > 0\}$.

$$(4)\quad \sum_{i\in S}\sum_{j\in M}\left(\lambda_i \frac{\partial U_i}{\partial X_{ij}} - \mu \frac{\partial F}{\partial X_{ij}}\right) X_{ij} + \sum_{k\in N}\left(\sum_{i\in S} \lambda_i \frac{\partial U_i}{\partial Y_k} - \mu \frac{\partial F}{\partial Y_k}\right) Y_k = 0$$

hold true.

III. *On the Existence of Fair Allocations*

III.1. Non-Existence Example

Our first order of business is to ascertain whether or not there is a general guarantee of the existence of fair allocations in the public goods economy. A simple modification of the pioneering work by Pazner and Schmeidler (1974) settles this problem in the negative.

Example 1

Consider an economy where there are two individuals 1 and 2, one private consumption good X, and one public good Y. In addition, each individual is endowed only with 1 unit of labor, which can be retained and consumed as leisure. Let X_i and L_i denote, respectively, the private good and leisure consumed by individual $i=1, 2$. Assume that the utility functions and the production possibility frontier are such that

$$(5)\quad U_1(L_1, X_1, Y) = L_1 + \frac{11}{10}X_1 + \frac{1}{2}Y$$

$$(6)\quad U_2(L_2, X_2, Y) = L_2 + 2X_2 + \frac{1}{2}Y$$

$$(7)\quad \sum_{i=1}^{2} X_i + Y = 1 - L_1 + \frac{1}{10}(1 - L_2),$$

where $0 \leqq L_i \leqq 1$ $(i=1, 2)$. Consider any feasible, efficient and equitable allocation $[\{X_i, L_i\}_{i=1}^2, Y] \in R_+^5$, assuming the existence thereof. If $Y>0$ is the case, we decrease Y by ε_1 such that $0<\varepsilon_1 \leqq Y$ and increase X_1 and X_2 by $\varepsilon_1/2$ to obtain $\Delta U_1 = \varepsilon_1/20 > 0$ and $\Delta U_2 = \varepsilon_1/2 > 0$ in contradiction with efficiency. Therefore, $Y=0$ must be true. Suppose, next, that $L_1>0$. We may then decrease L_1 by ε_2 and increase X_1 by ε_2, where $0<\varepsilon_2 \leqq L_1$, to obtain $\Delta U_1 = \varepsilon_2/10 > 0$ and $\Delta U_2 = 0$ in contradiction with efficiency. Therefore, $L_1=0$ is needed for efficiency. Third, suppose that $L_2<1$ and $X_2>0$ are true. We then choose ε_3 so that $0<\varepsilon_3 \leqq \min\{1-L_2, 10X_2\}$. By increasing L_2 by ε_3 and decreasing X_2 by $\varepsilon_3/10$, we may secure $\Delta U_1 = 0$ and $\Delta U_2 = 4\varepsilon_3/5 > 0$ in contradiction with efficiency. Therefore, $X_2>0$ cannot but imply $L_2=1$. Fourth, we show that $X_2>0$ is in fact true. Suppose, to the contrary, that $X_2=0$. Then we obtain $U_2(0, X_1, 0) = 2X_1 > U_2(L_2, 0, 0) = L_2$ by virtue of the feasibility constraint:

$$X_1 = 1 + \frac{1}{10}(1 - L_2),\ 0 \leqq L_2 \leqq 1.$$

Therefore, individual 2 envies individual 1 in contradiction with equity. Therefore $X_2>0$, hence $L_2=1$ is true. Putting all pieces together, we are assured that any feasible, efficient and equitable allocation $[\{X_i, L_i\}_{i=1}^2, Y] \in R_+^5$, if one exists, must satisfy:

$$(8)\quad X_1 + X_2 = 1,$$

(9) $\frac{11}{10}X_1 \geqq 1+\frac{11}{10}X_2,$

(10) $1+2X_2 \geqq 2X_1.$

It is easy to verify, however, that (8), (9) and (10) are incompatible. ||

III.2. Economies with Fair Allocations

It is clearly illegitimate to accuse a mechanism for its "failure" to locate a fair allocation if the economic environment is such that there exists no fair allocation wheresoever. Therefore, care should be taken with the sense in which we talk about the analogue of Foley's theorem in the public goods economy in view of the lack of general guarantee of the existence of fair allocations. Accordingly, our next task is to construct several economies that are assured of the existence of fair allocations, which will be arranged in the order of their later appearance.

Model A

Consider an economy with two private goods, one public good, and two individuals, 1 and 2, whose preferences are specified by

(11) $U_i(X_i, Y)=\log X_{i1}+\theta_i \log(X_{i2}Y+1) \quad (i=1, 2),$

where $X_i=(X_{i1}, X_{i2}) \in R^2_+$ and $\theta_i>0$ $(i=1, 2)$. Assume also that the production technology is specified by

(12) $\sum_{i=1}^{2}\sum_{j=1}^{2} X_{ij}+Y=\omega:=\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}.$

For a feasible allocation $[\{X_i\}_{i=1}^2, Y] \in R^5_{++}$ to be fair in this economy,

(13) $\log X_{i1}+\theta_i \log(X_{i2}Y+1) \geqq \log X_{j1}+\theta_i \log(X_{j2}Y+1) \qquad (i \neq j;\ i, j=1, 2),$

(14) $\sum_{i=1}^{2}\frac{\theta_i X_{i1}X_{i2}}{X_{i2}Y+1}=1,$

and

(15) $\frac{\theta_1 X_{11}}{X_{12}Y+1}=\frac{\theta_2 X_{21}}{X_{22}Y+1}$

must be satisfied. Solving (14) and (15) for X_{i1}, we obtain

(16) $X_{i1}=\frac{X_{i2}Y+1}{\theta_i(X_{12}+X_{22})} \quad (i=1, 2).$

Substitution of (16) into (12) and (13) yields

(17) $\frac{\sum_{i=1}^{2}\frac{1}{\theta_i}(X_{i2}Y+1)}{\sum_{i=1}^{2}X_{i2}}+\sum_{i=1}^{2}X_{i2}+Y=\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}$

and

(18) $(1+\theta_i)\log(X_{i2}Y+1)\geqq\log\dfrac{\theta_i}{\theta_j}+(1+\theta_i)\log(X_{j2}Y+1)\qquad(i\neq j;\ i,j=1,2).$

Assume now that our economy satisfies $\theta_1=\theta>0$, $\theta_2=1$, and that the half open interval

(19) $$B=\left(\max\left\{1,\frac{\sqrt{\theta}}{\theta},\frac{2}{1+\sqrt{\theta}}\right\},\frac{(\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij})^2+8}{4(2+\frac{\sqrt{\theta}}{\theta}(1+\theta))}\right]$$

is non-empty.

We now set about proving that this economy has infinitely many fair allocations. As an auxiliary step, define

(20) $a=X_{12}Y+1,\ b=X_{22}Y+1,$

and rewrite (17) and (18) as follows:

(21) $(1+\theta)\log a\geqq\log\theta+(1+\theta)\log b,$

(22) $2\log b\geqq-\log\theta+2\log a,$

(23) $$\frac{Y}{a+b-2}\left(\frac{a}{\theta}+b\right)+\frac{a+b-2}{Y}+Y=\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}.$$

Our task boils down to verifying the existence of $a,b>1$ and $Y>0$ satisfying (21), (22) and (23). With this purpose in mind, we take any $b\in B$ and define

(24) $a(b)=\sqrt{\theta}\,b.$

By definition of the interval B, $(a(b), b)$ thus defined satisfies $a(b)>1$ and $b>1$. Furthermore, $(a(b), b)$ satisfies (22) with equality. To show that $(a(b), b)$ satisfies (21) as well, we have only to notice that

$$(1+\theta)\log a(b)-\{\log\theta+(1+\theta)\log b\}=\frac{1+\theta}{2}\log\theta-\log\theta=\frac{\theta-1}{2}\log\theta\geqq0$$

holds true for all $\theta>0$. Consider now the following quadratic equation in Y:

(25) $$\left\{1+\frac{\alpha(b)}{\beta(b)}\right\}Y^2-\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\right)Y+\beta(b)=0,$$

where

(26) $\alpha(b)=\left(1+\dfrac{\sqrt{\theta}}{\theta}\right)b,\ \beta(b)=(1+\sqrt{\theta})b-2.$

By definition of the interval B, the discriminant of (25):

(27) $$D(b)=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\right)^2-4\left\{\left(2+\frac{\sqrt{\theta}}{\theta}(1+\theta)\right)b-2\right\}$$

is non-negative, so that (25) has a real root

$$(28)\quad Y(b)=\frac{\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}+\sqrt{D(b)}}{2\left\{1+\dfrac{(1+\sqrt{\theta}/\theta)b}{(1+\sqrt{\theta})b-2}\right\}},$$

which is positive. By construction, $(a(b), b, Y(b))$ satisfies (23), as may easily be verified.

Solving (20) for $X_{12}(b)$ and $X_{22}(b)$, and substituting the results into (16) to obtain $X_{11}(b)$ and $X_{21}(b)$, we may finally assert that the 5-tuple

$$[X_{11}(b), X_{12}(b), X_{21}(b), X_{22}(b), Y(b)]$$
$$=\left[\frac{\sqrt{\theta}\,bY(b)}{\theta\{(1+\sqrt{\theta})b-2\}}, \frac{\sqrt{\theta}\,b-1}{Y(b)}, \frac{bY(b)}{(1+\sqrt{\theta})b-2}, \frac{b-1}{Y(b)}, Y(b)\right]$$

is a fair allocation for any $b\in B$. Therefore, our economy is assured of the existence of infinitely many fair allocations.

In passing, we may derive from (15) and (18) a necessary condition for a fair allocation:

$$(29)\quad -\frac{1}{2}\log\theta \geqq \log\frac{X_{11}}{X_{21}} \geqq -\frac{\theta}{1+\theta}\log\theta,$$

which will be invoked later on. ||

Model B(1)

There are two commodities, one private and one public, and three individuals, 1, 2 and 3, whose preferences are given by

$$(30)\quad U_i(X_i, Y)=\alpha_i X_i+\log Y,$$

where $\alpha_i>0$ $(i=1, 2, 3)$. The production possibility frontier is given by

$$(31)\quad \sum_{i=1}^{3} X_i+Y=\omega:=\sum_{i=1}^{3}\omega_i.$$

For a feasible allocation $[\{X_i\}_{i=1}^{3}, Y]\in R_{++}^{4}$ to be fair in this economy, it is necessary and sufficient that

$$(32)\quad X_i=X_j\ (i\neq j;\ i, j=1, 2, 3) \text{ and } Y=\sum_{i=1}^{3}\frac{1}{\alpha_i}$$

hold true. Therefore, if α_i and ω_i $(i=1, 2, 3)$ satisfy

$$(33)\quad \sum_{i=1}^{3}\left(\omega_i-\frac{1}{\alpha_i}\right)>0,$$

then this economy has a fair allocation defined by $X_i=\frac{1}{3}\sum_{j=1}^{3}(\omega_j-\alpha_j^{-1})$ and $Y=\sum_{j=1}^{3}\alpha_j^{-1}$. ||

Model B(2)

Consider an economy with two private goods, one public good, and two individuals, 1 and 2, having the following utility functions:

$$(34)\quad U_i(X_i, Y)=\alpha_{i1}X_{i1}+\alpha_{i2}\log X_{i2}+\log Y,$$

where $\alpha_{ij}>0$ $(i, j=1, 2)$. As in Model A, the production possibility frontier is given by (12).

It is assumed that ω_{ij} and α_{ij} are such that

$$(35)\quad \alpha_{11}\alpha_{22} \geqq \alpha_{12}\alpha_{21} \text{ and } \omega := \sum_{i=1}^{2}\sum_{j=1}^{2} \omega_{ij} > \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}}\ \frac{\alpha_{22}}{\alpha_{21}} + \log \frac{\alpha_{11}\alpha_{22}}{\alpha_{12}\alpha_{21}}$$

hold true.

Note that, for a feasible allocation $[\{X_i\}_{i=1}^{2}, Y] \in R_{++}^{5}$ to be fair in this economy, it is necessary and sufficient that

$$(36)\quad \alpha_{11}X_{11} + \alpha_{12}\log X_{12} \geqq \alpha_{11}X_{21} + \alpha_{12}\log X_{22},$$

$$(37)\quad \alpha_{21}X_{21} + \alpha_{22}\log X_{22} \geqq \alpha_{21}X_{11} + \alpha_{22}\log X_{12},$$

and

$$(38)\quad \frac{\alpha_{12}}{\alpha_{11}X_{12}} = \frac{\alpha_{22}}{\alpha_{21}X_{22}} \text{ and } Y = \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{21}}$$

hold true, along with (12).

Let an open interval Γ be defined by

$$(39)\quad \Gamma = \left(0, \frac{1}{2}\left\{\omega - \left(\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{21}} + \frac{\alpha_{22}}{\alpha_{21}}\log\frac{\alpha_{11}\alpha_{22}}{\alpha_{12}\alpha_{21}}\right)\right\}\right)$$

which is non-empty by virtue of (35). Take any $x \in \Gamma$ and define $X_{11}(x)$, $X_{12}(x)$, $X_{21}(x)$ and $X_{22}(x)$ by

$$(40)\quad X_{11}(x) = x + \frac{\alpha_{22}}{\alpha_{21}}\log\frac{\alpha_{11}\alpha_{22}}{\alpha_{12}\alpha_{21}},$$

$$(41)\quad X_{12}(x) = \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}}X_{22}(x),$$

$$(42)\quad X_{21}(x) = x$$

$$(43)\quad X_{22}(x) = \frac{\alpha_{11}\alpha_{22}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}\left[\omega - X_{11}(x) - x - \left(\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{21}}\right)\right].$$

We may then observe that the 5-tuple $[\{X_{ij}(x)\}_{i,j=1}^{2}, \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{21}}] \in R_{++}^{5}$ satisfies (12), (36), (37) and (38), hence is a fair allocation, for any $x \in \Gamma$.

For the sake of later use, we derive from (36) and (37):

$$(44)\quad \frac{\alpha_{11}}{\alpha_{12}}(X_{11} - X_{21}) \geqq \log\frac{X_{22}}{X_{12}} \geqq \frac{\alpha_{21}}{\alpha_{22}}(X_{11} - X_{21}),$$

which is a necessary condition for an allocation to be equitable. ||

Model C

Let there be two private goods, one public good, and two individuals, 1 and 2, whose preferences are specified by

$$(45)\quad U_i(X_i, Y) = \sum_{j=1}^{2}\alpha_{ij}\log X_{ij} + \log Y,$$

where $X_i = (X_{i1}, X_{i2})$ is private goods vector consumed by individual i and Y denotes public good. We assume that $\alpha_{ij}>0$ $(i, j=1, 2)$. As before, the production possibility frontier is given by (12).

Let $[\{X_i\}_{i=1}^{2}, Y]\in R_{+}^{5}$ be a fair allocation, the existence of which is to be ascertained. Clearly, then, we have $X_{ij}>0$ $(i, j=1, 2)$, $Y>0$, and they satisfy

$$(46)\quad \sum_{j=1}^{2} \alpha_{1j} \log X_{1j} \geqq \sum_{j=1}^{2} \alpha_{1j} \log X_{2j},$$

$$(47)\quad \sum_{j=1}^{2} \alpha_{2j} \log X_{2j} \geqq \sum_{j=1}^{2} \alpha_{2j} \log X_{1j},$$

$$(48)\quad \sum_{j=1}^{2} \frac{X_{i1}}{\alpha_{i1}} = Y,$$

and

$$(49)\quad \frac{\alpha_{12}X_{11}}{\alpha_{11}X_{12}} = \frac{\alpha_{22}X_{21}}{\alpha_{21}X_{22}}.$$

Coupled with (48), (49) entails

$$(50)\quad X_{11} = \frac{\alpha_{11}\alpha_{22}X_{12}Y}{\alpha_{12}X_{22}+\alpha_{22}X_{12}} \text{ and } X_{21} = \frac{\alpha_{21}\alpha_{12}X_{22}Y}{\alpha_{12}X_{22}+\alpha_{22}X_{12}}.$$

Substituting (50) into (46) and (47) and simplifying, we obtain

$$(51)\quad \frac{\alpha_{11}}{\alpha_{11}+\alpha_{12}}(\log \alpha_{11}\alpha_{22} - \log \alpha_{12}\alpha_{21}) + \log X_{12} \geqq \log X_{22},$$

and

$$(52)\quad \frac{\alpha_{21}}{\alpha_{21}+\alpha_{22}}(\log \alpha_{21}\alpha_{12} - \log \alpha_{11}\alpha_{22}) + \log X_{22} \geqq \log X_{12},$$

whereas, substituting (50) into (12), we obtain

$$(53)\quad \left(1+\frac{\alpha_{11}\alpha_{22}X_{12}+\alpha_{21}\alpha_{12}X_{22}}{\alpha_{12}X_{22}+\alpha_{22}X_{12}}\right) Y + \sum_{j=1}^{2} X_{i2} - \omega = 0.$$

It is easy to check that (50), (51), (52), and (53) constitute a set of conditions that is not only *necessary* but also *sufficient* for a feasible allocation $[\{X_i\}_{i=1}^{2}, Y]\in R_{++}^{5}$ to be fair.

Let us now define an open interval

$$(54)\quad \Delta = \left(0, \frac{\omega}{1+a}\right), \text{ where } a = \left(\frac{\alpha_{21}\alpha_{12}}{\alpha_{11}\alpha_{22}}\right)^{\frac{\alpha_{21}}{\alpha_{21}+\alpha_{22}}}.$$

Pick any $x\in\Delta$ and define $X_{12}(x)$ by

$$(55)\quad \log X_{12}(x) = \log x + \frac{\alpha_{21}}{\alpha_{21}+\alpha_{22}} (\log \alpha_{21}\alpha_{12} - \log \alpha_{11}\alpha_{22}).$$

We may verify that

$$\frac{\alpha_{11}}{\alpha_{11}+\alpha_{12}} (\log \alpha_{11}\alpha_{22} - \log \alpha_{21}\alpha_{12}) + \log X_{12}(x)$$

$$=\log x+\frac{\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12}}{(\alpha_{11}+\alpha_{12})(\alpha_{21}+\alpha_{22})}(\log \alpha_{11}\alpha_{22}-\log \alpha_{21}\alpha_{12})\geqq \log x$$

holds true, so that $(X_{12}, X_{22})=(X_{12}(x), x)$ satisfies, not only (52), but also (51) for any $x\in\Delta$. Note also that (54) and (55) imply $X_{12}(x)=ax$.

Substituting this into (53), we obtain

$$(56)\quad Y(x)=\frac{(\alpha_{12}+a\alpha_{22})\{\omega-(1+a)x\}}{(1+\alpha_{21})\alpha_{12}+a\alpha_{22}(1+\alpha_{11})},$$

which is positive for any $x\in\Delta$. Finally, we define $X_{11}(x)$ and $X_{21}(x)$ by

$$(57)\quad X_{11}(x)=\frac{\alpha_{11}\alpha_{22}X_{12}(x)Y(x)}{\alpha_{12}x+\alpha_{22}X_{12}(x)} \text{ and } X_{21}(x)=\frac{\alpha_{21}\alpha_{12}xY(x)}{\alpha_{12}x+\alpha_{22}X_{12}(x)},$$

both of which are positive by construction. We are now assured that $[X_{11}(x), X_{12}(x), X_{21}(x), x, Y(x)]\in R^5_{++}$ is a fair allocation for any $x\in\Delta$. Therefore, our economy has at least one, and indeed infinitely many, fair allocations.

Note, in passing, that (51) and (52) yield

$$(58)\quad \left(\frac{\alpha_{11}\alpha_{22}}{\alpha_{21}\alpha_{12}}\right)^{\frac{\alpha_{21}}{\alpha_{21}+\alpha_{22}}}\leqq\frac{X_{22}}{X_{12}}\leqq\left(\frac{\alpha_{11}\alpha_{22}}{\alpha_{21}\alpha_{12}}\right)^{\frac{\alpha_{11}}{\alpha_{11}+\alpha_{12}}}$$

as a necessary condition for a fair allocation, which proves useful later. ||

IV. *Analogues of Foley's Theorem in the Public Goods Economy*

So much for preliminaries. We are now ready to examine whether or not several plausible analogues of Foley's theorem go through in the public goods economy with a view toward deepening our understanding of the performance of some public goods allocation mechanisms. In place of the competitive price mechanism in the theorem of Foley, we will examine the *public competitive equilibrium* [Foley (1967)], the *Lindahl equilibrium* [Foley (1970), Johansen (1963) and Lindahl (1967)], the *Groves-Ledyard mechanism* [Groves and Ledyard (1977) and (1980)], the *Zeuthen-Nash bargaining scheme* [Harsanyi (1956) and (1977, Chapter 8), Luce and Raiffa (1957, Chapter 6) and Nash (1950)], the *Kalai-Smorodinsky arbitration scheme* [Kalai and Smorodinsky (1975)], the *Shapley value allocation scheme* [Champsaur (1975), Ichiishi (1983, Chapter 6) and Shapley and Shubik (1969)], and the *Perles-Maschler super-additive solution scheme* [Perles and Maschler (1983)] in turn.

IV.1. The Public Competitive Equilibrium

A straightforward extension of the competitive equilibrium paradigm to situations involving public goods is the public competitive equilibrium due to Foley (1967). According to this scenario, an agent called government takes charge of the efficient provision of public goods and taxes consumers to cover the cost thereof, whereas consumers buy the private goods bundles they most prefer subject to the after-tax budget constraints, taking the private goods *prices* and the public goods *quantities* as given. A public competitive equilibrium obtains if demand and supply balance for each private good, and the allocation as a whole is efficient.

Suppose that the government tax rule is such that individual i shares β_i percentage of the cost $C(Y)$, which is measured in terms of the numeraire private good, of producing public goods Y. If individuals are initially equal in endowments as well as in tax share, the after-tax budgets of all individuals become identical and the public goods analogue of Foley's theorem holds just for the same reason.[2] This observation, although trivial, serves us twice. First, it enables us to reduce the problem of the existence of a fair allocation in the public goods economy to that of the existence of a public competitive equilibrium, about which we know much by now. See, among others, Greenberg (1977) and Richter (1975). Second, it shows that our desideratum on the public goods provision mechanism is *not* will-o'-the-wisp, as it *can* be satisfied. Will it be satisfied, it is now legitimate to ask, by other celebrated mechanisms?

IV.2. Lindahl Equilibrium

Presumably, the most well-known public good provision mechanism is that of Lindahl (1967), which is also a natural extension of the competitive market mechanism for private goods economy to situations including public goods. A single market price is specified for each private good, while a separate price is specified to each individual for each public good, the sum of which over all individuals is equated to the price received by the producer thereof. Given these prices, each agent is engaged in the usual maximization exercise, and the Lindahl equilibrium obtains when the demand and supply balance for each private good, and each individual demands the same amount of public good as is produced.[3,4]

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[2] If, instead of determining the efficient level of public goods provision Y and collecting tax from individual $i\in S$ by the amount $\beta_i C(Y)$, the government levies on individual $i\in S$ the wealth tax by the amount $\gamma_i(p\cdot\omega_i)$, where $\gamma_i\geqq 0$, $\Sigma_{i\in S}\gamma_i=1$, and $p=(p_1, p_2, \ldots, p_m)\in R^m_+$ denotes the price vector of private goods, and determines the level of public goods so as to satisfy $C(Y)=\Sigma_{i\in S}\gamma_i(p\cdot\omega_i)$, this statement is not necessarily true. Even when $\gamma_i=\gamma(=s^{-1})$ and $\omega_i=\omega$ for all $i\in S$ hold true, the resulting allocation $[\{X_i\}_{i\in S}, Y]\in R^{sm+n}_+$ need not be efficient. Indeed, in Model C economy, the allocation is efficient only when γ is so stipulated as to satisfy

$$\gamma=\frac{\sum_{i=1}^2\{s\sum_{j=1}^2\omega_j/\sum_{j=1}^2\alpha_{ij}\}}{s\sum_{j=1}^2\omega_j+\sum_{i=1}^2\{s\sum_{j=1}^2\omega_j/\sum_{j=1}^2\alpha_{ij}\}},$$

a very special requirement indeed.

[3] It was Myrdal (1953) who first posed the problem of equity and/or justice of the Lindahl equilibrium allocation. Observe that, at the Lindahl equilibrium allocation, the *marginal* rate of substitution between a public and a private (numeraire) goods is equal, for each individual, to the individualized price of a public good, but "it would be the net increase in *total* utility, and not the marginal quantity, which is relevant for consideration of justice [Myrdal (1953), p. 184]." Our Example 2 settles this problem of equity of the Lindahl equilibrium allocation in the negative, thereby substantiating Myrdal's classic and well-taken criticism.

[4] We could have exemplified the possible lack of fairness of the Lindahl equilibrium allocation by using a much simpler model involving only two goods, one private and one public, and two individuals whose preferences are represented by log-linear utility functions. In that case, the no-envy equity requirement reduces to requiring that individuals consume the identical physical amount of private good. Furthermore, the log-linear utility implies that private good and public good are separable in consumer's preferences. In order to show that the pathology of the Lindahl equilibrium allocation is not due to these special features, we presented our Example 2 that is free therefrom. But the simpler example has its use; it shows that Kaneko's (1977) *ratio equilibrium* fares no better in our arena, since the ratio equilibrium and the Lindahl equilibrium coincide in our example, where there is only one private good and that the cost function for public good is linear.

Example 2

Consider our Model A economy. In view of the linear production possibility frontier (12), the market price of the second private good and that of public good in terms of the numeraire (=the first private good) must be unity. Let h_i be the individualized tax-price of public good for individual i, which satisfies $h_i \geqq 0$ $(i=1, 2)$ and $\sum_{i=1}^{2} h_i = 1$. Maximizing the utility function (11) subject to the budget constraint $\sum_{j=1}^{2} X_{ij} + h_i Y_i = \sum_{j=1}^{2} \omega_{ij}$, we may verify that individual i's demand functions for private good $X_{ij}(h_i)$ $(j=1, 2)$ and that for public good $Y_i(h_i)$ satisfy

(59) $X_{i2}(h_i) = h_i Y_i(h_i)$.

If $X_{i2}(h_i) > 0$, hence $Y_i(h_i) > 0$, then $X_{i1}(h_i)$ and $Y_i(h_i)$ satisfy

(60) $$X_{i1}(h_i) = \frac{1 + h_i \{Y_i(h_i)\}^2}{\theta_i Y_i(h_i)}$$

and

(61) $$h_i(1+2\theta_i)\{Y_i(h_i)\}^2 - (\sum_{j=1}^{2} \omega_{ij})\theta_i Y_i(h_i) + 1 = 0,$$

whereas if $X_{i2}(h_i)=0$, hence $Y_i(h_i)=0$, then

(62) $$X_{i1}(h_i) = \sum_{j=1}^{2} \omega_{ij}.$$

If the tax-price vector $(h_1, h_2) \in R_+^2$ satisfies

(63) $$D_i(h_i) = (\sum_{j=1}^{2} \omega_{ij})^2 \theta_i^2 - 4(1+2\theta_i)h_i \geqq 0$$

for $i=1, 2$, then (61) may be solved for positive real root

(64) $$Y_i(h_i) = \frac{(\sum_{j=1}^{2} \omega_{ij})\theta_i + \sqrt{D_i(h_i)}}{2h_i(1+2\theta_i)},$$

which, in turn, determines $X_{ij}(h_i)$ *via* (59) and (60).

Suppose now that $\theta_1 = \frac{1}{2}$, $\theta_2 = 1$, $\sum_{j=1}^{2} \omega_{ij} = 6$ $(i=1, 2)$ are the case, which guarantee that the interval B defined by (19) is non-empty. Solving $Y_1(h_1^L) = Y_2(h_2^L)$ and $h_1^L + h_2^L = 1$, we obtain

(65) $$h_1^L = \frac{3(\sqrt{321} - 11)}{50}, \quad h_2^L = \frac{83 - 3\sqrt{321}}{50},$$

which satisfy (63) for $i=1, 2$.

Although an *equitable* allocation $[(X_{11}, X_{12}), (X_{21}, X_{22}), Y] = [(6, 0), (6, 0), 0] \in R_+^5$ is a candidate for the Lindahl equilibrium allocation, it actually fails to qualify as such, since we obtain by computation

(66) $$U_i(X_{i1}(h_1^L), X_{i2}(h_i^L), Y(h_1^L, h_2^L)) > U_i(6, 0, 0)$$

for $i=1, 2$, where $Y(h_1^L, h_2^L) = Y_1(h_1^L) = Y_2(h_2^L)$.

To examine the fairness of the Lindahl equilibrium allocation $[(X_{11}(h_1^L), X_{12}(h_1^L)), (X_{21}$

$(h_2{}^L)$, $X_{22}(h_2{}^L))$, $Y(h_1{}^L, h_2{}^L)]\in R^5_{++}$, we now invoke the necessary condition (29) for fairness, which reads in our example as follows:

(67) $$\frac{1}{2}\log 2 \geqq \log \frac{X_{11}}{X_{21}} \geqq \frac{1}{3}\log 2.$$

By computation, we obtain $X_{11}(h_1{}^L)/X_{21}(h_2{}^L)=1.5$, so that we have

(68) $$\log \frac{X_{11}(h_1{}^L)}{X_{21}(h_2{}^L)}=0.4055>\frac{1}{2}\log 2=0.3466,$$

which implies that the condition (67) is violated. Therefore, even when individuals are equally endowed with private goods to begin with, the resulting Lindahl equilibrium allocation may fail to be fair. ||

IV.3. Groves-Ledyard Mechanism

Our next order of business is to examine the Groves-Ledyard public goods allocation scheme, in which the private goods are allocated through competitive markets and the public goods according to the government allocation and taxation rules that depend on the information concerning individual preferences communicated by themselves. These rules are so designed that (i) individuals find it in their self-interest to reveal their true preferences for the public good even when they are allowed to be free-riders if they so wish, and that (ii) the equilibria thereby attained are efficient. Let us exemplify now that, for all its nice performance on the incentival ground, the Groves-Ledyard mechanism is deficient on the fairness arena.[5]

Example 3

Consider our Model B(1) economy. We introduce an agent called government that is in charge of the public good provision. Each individual i sends to the government a message $m_i\in R_+$, together forming a message vector $m=(m_1, m_2, m_3)\in R^3_+$, in response to which the government provides the public good $Y(m)$ and taxes individual i by the amount $T_i(m)$ according to the following rules:

(69) $$Y(m)=\sum_{i=1}^{3} m_i,$$

(70) $$T_i(m)=\delta_i Y(m)+\frac{\gamma}{2}\left\{\frac{2}{3}(m_i-\bar{m}_{-i})^2-\sigma^2_{-i}\right\},$$

where $\delta_i\geqq 0$, $\sum_{i=1}^{3}\delta_i=1$ and $\gamma>0$ are parameters, and

(71) $$\bar{m}_{-i}=\frac{1}{2}\sum_{j\neq i} m_j,\ \sigma^2_{-i}=\sum_{j\neq i}(m_j-\bar{m}_{-i})^2$$

[5] It is true that the Groves-Ledyard mechanism leaves individuals with no incentive to misrepresent their preferences for public goods. Whether or not it bestows incentives on individuals to participate in the public goods allocation game thereby defined in the first place is a different matter altogether. Indeed, our Example 3 is meant to suggest that the Groves-Ledyard mechanism does not fare no better than the Lindahl mechanism on this incentive-to-participate side of the coin.

for all $i=1, 2, 3$.

Let us further specify our economy by assuming that $\omega_i=2$ $(i=1, 2, 3)$, $\alpha_1=1$, $\alpha_2=\alpha_3=\frac{1}{2}$, $\delta_i=\frac{1}{3}$ $(i=1, 2, 3)$ and $\gamma=2$. Note, first, that the condition (33) is thereby satisfied and, second, that individuals are equally situated in endowments as well as in tax shares.

It is easy to verify that, when the message vector $m=(m_1, m_2, m_3)$ prevails, individual 1's utility is given by

$$(72)\quad U_1(X_1(m), Y(m))=2-\frac{1}{3}\sum_{i=1}^{3} m_i-\left\{\frac{2}{3}\left(m_1-\frac{m_2+m_3}{2}\right)^2-\frac{1}{2}(m_2-m_3)^2\right\}+\log\left(\sum_{i=1}^{3} m_i\right).$$

In choosing his message m_1, individual 1 maximizes (72) taking m_2 and m_3 as given to obtain

$$(73)\quad \left(\sum_{i=1}^{3} m_i\right)\left\{1+4\left(m_1-\frac{m_2+m_3}{2}\right)\right\}=3.$$

By similar reasoning, we may derive

$$(74)\quad \left(\sum_{i=1}^{3} m_i\right)\left\{1+4\left(m_2-\frac{m_1+m_3}{2}\right)\right\}=6$$

and

$$(75)\quad \left(\sum_{i=1}^{3} m_i\right)\left\{1+4\left(m_3-\frac{m_1+m_2}{2}\right)\right\}=6.$$

The Groves-Ledyard allocation is given by $[\{X_j(m^*)\}_{j=1}^{3}, Y(m^*)]\in R^4_{++}$, where

$$(76)\quad X_j(m^*)=2-\frac{1}{3}\sum_{i=1}^{3} m_i^*-\left\{\frac{2}{3}\left(m_j^*-\frac{m_k^*+m_l^*}{2}\right)^2-\frac{1}{2}\left(m_k^*-m_l^*\right)^2\right\}$$
$$(j\neq k\neq l\neq j;\ j, k, l=1, 2, 3)$$

$$(77)\quad Y(m^*)=\sum_{i=1}^{3} m_i^*$$

and $m^*=(m_1^*, m_2^*, m_3^*)$ satisfies (73), (74) and (75) simultaneously. Solving (73), (74) and (75), we obtain $m_1^*=\frac{8}{5}$ and $m_2^*=m_3^*=\frac{17}{10}$, so that we obtain $X_1(m^*)=\frac{49}{150}$ and $X_2(m^*)=\frac{101}{300}$. Clearly, then, the Groves-Ledyard allocation is not fair. ||

IV.4. Zeuthen-Nash Bargaining Scheme

We now turn to the bargaining-theoretic, in contrast with the market-like, mechanism for allocating public goods. Specifically, we will be concerned with the bargaining scheme proposed by Zeuthen and Nash, which is contended to be a "fair" division reflecting the "reasonable expectancies" of "rational" individuals.[6] Does it fare any better in preserving *initial* equality into *final* fairness?

To begin with, let u_i^0 $(i\in S)$ be defined by

[6] See Harsanyi [(1977), Chapter 8], Luce and Raiffa [(1957), Chapter 6] and Nash (1960).

(78) $u_i^0 = U_i(X_i^0, Y_i^0)$,

where

(79) $(X_i^0, Y_i^0) = \arg\max U_i(X_i, Y_i)$ s.t. $F(X_i, Y_i; \omega_i) = 0$.

Clearly, individual i will not voluntarily participate in the bargaining with others unless he may thereby secure no less utility than u_i^0. Next, let $u_1 = \phi(u_2, \ldots, u_s)$ denote the utility possibility frontier, which is defined by

(80) $$\phi(u_2, \ldots, u_s) = \max \left\{ U_1(X_1, Y) \,\middle|\, \begin{matrix} \forall i \in S \setminus \{1\} : U_i(X_i, Y) \geqq u_i \\ \& \\ F(\sum_{i \in S} X_i, Y; \sum_{i \in S} \omega_i) = 0 \end{matrix} \right\}$$

The Zeuthen-Nash bargaining solution is then given by such $\{u_i^N\}_{i \in S}$ as to satisfy

(81) $$\prod_{i \in S} (u_i^N - u_i^0) = \max_{u_1 = \phi(u_2, \ldots, u_S)} \prod_{i \in S} (u_i - u_i^0).$$

Example 4

Consider our Model B(2) economy where $\alpha_{11} = \theta$, $\alpha_{12} = \alpha_{21} = \alpha_{22} = 1$, and $\omega_{ij} = 1$ $(i, j = 1, 2)$. Note that the condition (35) is thereby satisfied as far as $(3\theta - 1)/\theta > \log \theta \geqq 0$ holds true. It is easy to verify that

(82) $u_1^0 = 2(\theta - 1 - \log \theta)$ and $u_2^0 = 0$

hold true as far as $\theta > 1$, which we assume hereafter. Turning to the derivation of the utility possibility frontier $u_1 = \phi(u_2)$, we maximize $U_1(X_1, Y)$ subject to $X_{21} + \log X_{22} + \log Y = u_2$ and $\sum_{i=1}^{2} \sum_{j=1}^{2} X_{ij} + Y = 4$. Paying due attention to the non-negativity of the variables, we obtain the following three cases:

Case 1 $[X_{12} < \frac{1}{\theta}$, $X_{11} = 0$ and $Y = 1 + X_{12}]$

In this case, the frontier $u_1 = \phi(u_2)$ is defined implicitly by

(83) $$\begin{cases} u_1(x) = \log x + \log(1 + x) \\ u_2(x) = 2(1 - x) + \log(1 + x). \end{cases}$$

where the parameter x runs over an open interval $\left(0, \frac{1}{\theta}\right)$. Behind the scene, resources are so allocated that

(84) $X_{11}(x) = 0$, $X_{12}(x) = x$, $X_{21}(x) = 2(1 - x)$, $X_{22}(x) = 1$, $Y(x) = 1 + x$,

where $x \in \left(0, \frac{1}{\theta}\right)$.

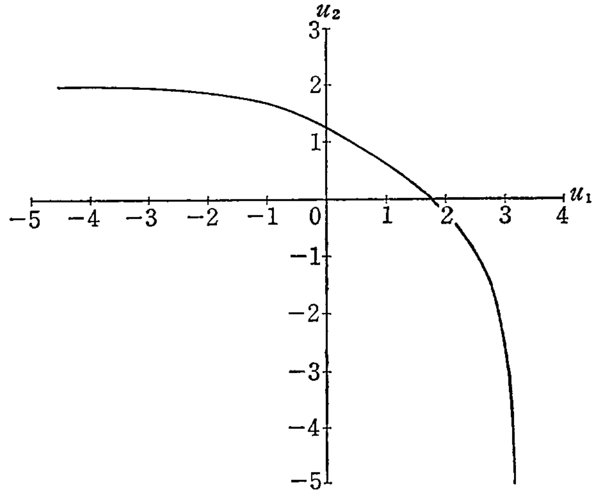
Case 2 $\left[X_{12} = \frac{1}{\theta}, Y < 1 + X_{12} \text{ and } X_{21} = 0\right]$

In this case, the parametric representation of $u_1 = \phi(u_2)$ is given by

$$(85)\quad \begin{cases} u_1(y)=\theta\left(4-\dfrac{2}{\theta}-2y\right)-\log\theta+\log\left(y+\dfrac{1}{\theta}\right) \\ u_2(y)=\log y+\log\left(y+\dfrac{1}{\theta}\right), \end{cases}$$

FIGURE 1 UTILITY POSSIBILITY FRONTIER IN EXAMPLE 4

$$u_1=\phi(u_2)\leftrightarrow\begin{cases} u_1(x)=\log x(1+x), \\ u_2(x)=2(1-x)+\log(1+x), \\ \quad\text{where } x\in(0,\,2/3). \\ u_1=-(3/2)u_2+1+\log 2+(5/2)\log 5-(7/2)\log 3, \\ \quad\text{where } u_2\in[\log(5/3),\,2/3+\log(5/3)]. \\ u_1(x)=4-3x+\log(2/3)(x+2/3), \\ u_2(x)=\log x(x+2/3), \\ \quad\text{where } x\in(0,\,1). \end{cases}$$



where the parameter y runs over an open interval (0, 1), whereas resources are so allocated that

$$(86)\quad X_{11}(y)=4-\frac{2}{\theta}-2y,\ X_{12}(y)=\frac{1}{\theta},\ X_{21}(y)=0,\ X_{22}(y)=y,\ Y(y)=y+\frac{1}{\theta},$$

where $y\in(0,1)$.

Case 3 $\left[X_{12}=\dfrac{1}{\theta},\ Y=1+X_{12}\right]$

In this case, $u_1=\phi(u_2)$ may be written implicitly as

$$(87)\quad \begin{cases} u_1(z)=\theta z-\log\theta+\log\dfrac{\theta+1}{\theta} \\ u_2(z)=\dfrac{2(\theta-1)}{\theta}-z+\log\dfrac{\theta+1}{\theta}, \end{cases}$$

whereas resource allocation may be stipulated as

(88) $X_{11}(z)=z,\ X_{12}(z)=\frac{1}{\theta},\ X_{21}(z)=\frac{2(\theta-1)}{\theta}-z,\ X_{22}(z)=1,\ Y(z)=\frac{\theta+1}{\theta},$

where the parameter z runs over a closed interval $\left[0, \frac{2(\theta-1)}{\theta}\right]$. Assume now that $\theta=\frac{3}{2}$, which satisfies all the stipulated conditions on θ, is the case. Then the foregoing three branches of $u_1=\phi(u_2)$ may be smoothly connected as in the Figure 1 and the Zeuthen-Nash solution in the utility space is given by

(89) $(u_1(z^N),\ u_2(z^N))=\left(\frac{2}{3}z^N+\log\frac{5}{2}-2\log\frac{3}{2},\ \frac{3}{2}-z^N+\log\frac{5}{3}\right),$

where $z^N\in\left[0, \frac{2}{3}\right]$ maximizes the Nash product:

(90) $N(z)=\left(\frac{3}{2}z+\log\frac{5}{2}-1\right)\left(\frac{3}{2}-z+\log\frac{5}{3}\right).$

Explicitly, we have

(91) $z^N=\frac{1}{3}\left(2+\frac{1}{2}\log 5-\frac{3}{2}\log 3+\log 2\right).$

We now invoke the condition (44), which now reads as follows:

(92) $\frac{3}{2}(X_{11}-X_{21})\geqq\log\frac{X_{22}}{X_{12}}\geqq X_{11}-X_{21}.$

Making use of (88), we may easily verify that $X_{11}(z^N)-X_{21}(z^N)=2z^N-\frac{2}{3}=\frac{1}{3}(2+\log 5-3\log 3+2\log 2)=0.5666>0.4055=\log\frac{3}{2}=\log\frac{X_{22}(z^N)}{X_{12}(z^N)}$, so that the Zeuthen-Nash allocation violates the equity condition (92). Therefore, the analogue (in the public goods economy) of Foley's theorem does not go through with respect to the Zeuthen-Nash bargaining scheme. ||

IV.5. Kalai-Smorodinsky Arbitration Scheme

Several criticisms have been made of the Zeuthen-Nash bargaining scheme.[7] In particular, the insensitivity of the Zeuthen-Nash scheme to the "aspiration levels" of individuals in the bargaining situation has been construed to be a major defect thereof. In sharp contrast, the fifth mechanism of our concern, i.e. the Kalai-Smorodinsky arbitration scheme, assigns a critical role to the aspiration levels in arbitrating the two-person bargaining situation.

Let $u_1=\phi(u_2)$ and $u^0=(u_1^0, u_2^0)$ be the utility possibility frontier and the threat point, both of which are defined in IV.4 above. Let the aspiration level of individual i, u_i^*, be defined by $u_i^*=\sup\{u_i|u_1=\phi(u_2)\}$ $(i=1, 2)$. Then, the Kalai-Smorodinsky outcome is defined by the intersection between the frontier $u_1=\phi(u_2)$ and the line connecting $u^0=(u_1^0, u_2^0)$ and $u^*=(u_1^*, u_2^*)$.

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[7] Some of these criticisms are succinctly stated and commented on by Luce and Raiffa [(1957), pp. 128–134].

Example 5

Consider our Model C economy with $\alpha_{11}=2$, $\alpha_{12}=\alpha_{21}=\alpha_{22}=1$, and $\omega_{ij}=\frac{1}{2}$ $(i, j=1, 2)$. Fixing individual 2's utility at u_2, we maximize $U_1(X_1, Y)$ subject to (12) and $U_2(X_2, Y)=u_2$ to derive $u_1=\phi(u_2)$, which is defined implicitly by

$$(93)\quad \begin{cases} u_1(x)=3\log(2-3x)+\log(2+x)-6\log 2, \\ u_2(x)=2\log x+\log(2+x)-2\log 2, \end{cases}$$

where $x\in\left(0, \frac{2}{3}\right)$. Behind the scene, resources are so allocated that

$$(94)\quad \begin{cases} X_{11}(x)=\frac{1}{2}(2-3x),\ X_{12}(x)=\frac{1}{4}(2-3x). \\ X_{21}(x)=X_{22}(x)=x,\ Y(x)=\frac{1}{2}+\frac{1}{4}x, \end{cases}$$

hold true, where $x\in\left(0, \frac{2}{3}\right)$. We may easily calculate individual i's threat utility level as follows $(i=1, 2)$:

$$(95)\quad u_1^0=-6\log 2 \text{ and } u_2^0=-3\log 3.$$

On the other hand, the aspiration levels are calculated as

$$(96)\quad \begin{cases} u_1^*=\sup\limits_{x\in(0,\frac{2}{3})} u_1(x)=-2\log 2, \\ u_2^*=\sup\limits_{x\in(0,\frac{2}{3})} u_2(x)=3(\log 2-\log 3). \end{cases}$$

The line passing through $u^0=(u_1^0, u_2^0)$ and $u^*=(u_1^*, u_2^*)$ is then given by

$$(97)\quad u_1=\frac{4}{3}u_2+2(2\log 3-3\log 2).$$

The intersection between the line (97) and the frontier $u_1=\phi(u_2)$ defined by (93) determines $x^K\in\left(0, \frac{2}{3}\right)$, which, in turn, determines the resource allocation *via* (94).

Substituting $u_1(x)$ and $u_2(x)$ in (93) into (97), we may ascertain that $x^K\in(0, 2/3)$ is a solution to the following equation:

$$(98)\quad f(x)=(2-3x)^9-\frac{3^{12}}{2^8}\cdot(2+x)\cdot x^8=0,\ x\in\left(0, \frac{2}{3}\right).$$

On the other hand, the necessary condition (58) for an allocation corresponding to this x^K to be equitable reads in this case as follows:

$$(99)\quad \frac{1}{2}\log 2\leqq\log\frac{X_{22}(x^K)}{X_{12}(x^K)}\leqq\frac{2}{3}\log 2.$$

Taking (94) into consideration, (99) reduces into

(100) $\frac{2\cdot 2^{1/2}}{4+3\cdot 2^{1/2}} \leqq x^K \leqq \frac{2\cdot 2^{2/3}}{4+3\cdot 2^{2/3}}$

To show that any solution of (98) cannot in fact satisfy (100), we observe first that

(101) $f'(x) = -3^3\cdot(2-3x)^8 - \frac{3^{12}}{2^8}\cdot x^7\cdot(9x+16) < 0$ for all $x \in \left(0, \frac{2}{3}\right)$.

Furthermore, $f(0)=2^9>0$ and $f\left(\frac{2}{3}\right) = -3^3\cdot 2^3 < 0$, so that $f(x)=0$ has a unique solution $x^K \in \left(0, \frac{2}{3}\right)$ by the theorem of intermediate values. Note, however, that

$$\begin{aligned} f\left(\frac{2\sqrt{2}}{4+3\sqrt{2}}\right) &= f(6-4\sqrt{2}) \\ &= 2^2\cdot(3-2\sqrt{2})^8\cdot 3^{12}\cdot\left\{2^8\cdot\left(\frac{2}{3}\right)^{12}\cdot(3\sqrt{2}-4)-(2-\sqrt{2})\right\} \\ &= 2^2\cdot(3-2\sqrt{2})^8\cdot 3^{12}\cdot(0.4787-0.5858) \\ &< 0 \end{aligned}$$

holds true, so that x^K cannot satisfy the condition (100). Therefore, the Kalai-Smorodinsky scheme does not ensure the fairness of the outcome even if the initial position is perfectly egalitarian. ||

IV.6. Shapley Value Allocation

The value approach regards the public goods allocation problem as a cooperative game, and looks for an "equitable" compromise that imputes the fruits of cooperation among individuals in such a way as to take "fair" account of the contribution by each individual to each possible cooperative venture. A value allocation is a feasible allocation in the economy that gives rise to the Shapley (λ-transfer) value of the associated cooperative game. Is this "fair" cooperative solution concept compatible with the no-envy equity?

Example 6

Consider our Model B(2) economy with $\alpha_{11}=2$, $\alpha_{12}=\alpha_{21}=\alpha_{22}=1$ and $\omega_{ij}=1$ $(i, j=1, 2)$, which satisfy the condition (35). Our first order of business is to define the characteristic function of the associated cooperative game: For each coalition $T \subset S$, $V(T)$ is a convex, closed, and non-empty subset of R^t, where $t=\# T$, representing the feasible utility vectors for the coalition T. It is customary to require that $u^1 \in V(T)$, $u^2 \in R^t$ and $u^1 \geqq u^2$ imply $u^2 \in V(T)$, and $V(T^1 \cup T^2) \supset V(T^1) \times V(T^2)$ for T^1 and T^2 disjoint.

For singleton coalitions, it is natural to assume that $V(\{i\}) = (-\infty, u_i^0]$ for $i=1, 2$, i.e.

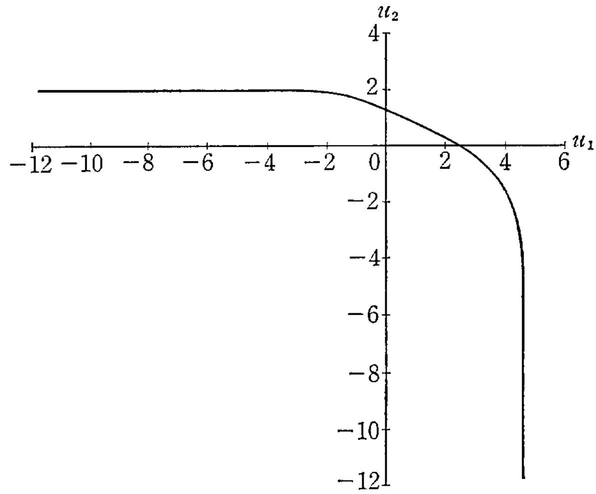
(102) $V(\{1\}) = (-\infty, 2(1-\log 2)]$ and $V(\{2\}) = (-\infty, 0]$.

In order to define $V(\{1, 2\})$, we derive the utility possibility frontier $u_1=\phi(u_2)$. There are three effective branches of the frontier, which correspond to the following cases:

Case 1 $\left[X_{12}=\frac{1}{2} \text{ and } X_{22}=1\right]$

FIGURE 2 UTILITY POSSIBILITY FRONTIER IN EXAMPLE 6

$$u_1=\phi(u_2) \leftrightarrow \begin{cases} u_1(x)=2x+\log(4-x)-2\log 2, \\ u_2(x)=\log(3-x)+\log(4-x)-2\log 2, \\ \quad \text{where } x\in(1,3). \\ u_1=-2u_2+2+3\log 3-4\log 2, \\ \quad \text{where } u_2\in[\log 3-\log 2,\ 1+\log 3-\log 2]. \\ u_1(x)=\log(2-x)+\log(4-x)-2\log 2, \\ u_2(x)=x+\log(4-x)-\log 2, \\ \quad \text{where } x\in(1,2). \end{cases}$$



In this case, the frontier is implicitly defined by

$$(103)\quad \begin{cases} u_1(x)=2x+\log 3-2\log 2 \\ u_2(x)=1-x+\log 3-\log 2, \end{cases}$$

where the parameter x runs over a closed interval [0, 1]. Corresponding to each $x\in[0, 1]$, resources are so allocated that

$$(104)\quad X_{11}(x)=x,\ X_{12}(x)=\frac{1}{2},\ X_{21}(x)=1-x,\ X_{22}(x)=1,\ Y(x)=\frac{3}{2}.$$

As a matter of fact, (103) may be written explicitly as

$$(105)\quad u_1=-2u_2+2+3\log 3-4\log 2,$$

where $u_2\in[\log 3-\log 2,\ 1+\log 3-\log 2]$.

Case 2 $\left[X_{12}=\frac{1}{2},\ X_{21}=0 \text{ and } X_{22}<1\right]$

In this case, the frontier may be parametrized by $y\in(1, 3)$ as follows:

$$(106)\quad \begin{cases} u_1(y)=2y+\log(4-y)-2\log 2 \\ u_2(y)=\log(3-y)+\log(4-y)-2\log 2, \end{cases}$$

whereas resources are correspondingly allocated as follows:

(107) $X_{11}(y)=y,\ X_{12}(y)=\frac{1}{2},\ X_{21}(y)=0,\ X_{22}(y)=\frac{3-y}{2},\ Y(y)=\frac{4-y}{2}.$

Case 3 $\left[X_{11}=0,\ X_{12}<\frac{1}{2} \text{ and } X_{22}=1\right]$

In this case, the parametric representation of the frontier is given by

$$(108)\quad \begin{cases} u_1(z)=\log(2-z)+\log(4-z)-2\log 2 \\ u_2(z)=z+\log(4-z)-\log 2, \end{cases}$$

whereas resources are so allocated as

(109) $X_{11}(z)=0,\ X_{12}(z)=\frac{2-z}{2},\ X_{21}(z)=z,\ X_{22}(z)=1,\ Y(z)=\frac{4-z}{2},$

where $z\in(1, 2)$.

It may be verified that these branches may be smoothly connected to generate the overall frontier $u_1=\phi(u_2)$ as is described in the Figure 2. We may now define $V(\{1, 2\})$ by:

(110) $V(\{1, 2\})=\{v\in R^2 \mid v\leqq u \text{ for some } u=(u_1, u_2) \text{ such that } u_1=\phi(u_2)\}.$

Next, we take any non-negative, non-zero vector $\lambda=(\lambda_1, \lambda_2)\in R^2$ and define the λ-transfer characteristic function $v^\lambda(T)$, $T\subset S$, as follows:[8]

(111) $v^\lambda(\{1\})=2(1-\log 2)\lambda_1,\ v^\lambda(\{2\})=0,$

and

$$(112)\quad v^\lambda(\{1, 2\})=\begin{cases} 2\lambda_2 & \text{if } \lambda_1=0 \\ 2(\lambda_2-\lambda_1)+\log\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)^{\lambda_1+\lambda_2} & \text{if } 0<2\lambda_1<\lambda_2, \\ \left(2+\log\frac{27}{16}\right)\lambda_1, & \text{if } 2\lambda_1=\lambda_2, \\ 2(3\lambda_1-\lambda_2)+\log\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)^{\lambda_1+\lambda_2}\cdot\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda_2} & \text{if } 0<\lambda_2<2\lambda_1, \\ 2(3-\log 2)\lambda_1 & \text{if } \lambda_2=0. \end{cases}$$

It is now time we define the λ-transfer value $\phi^\lambda=(\phi_1^\lambda, \phi_2^\lambda)$, which reads in our example as follows:

[8] Given the characteristic function $V(T)$, $T\subset S$, and an s-vector $\lambda\geqq 0$, $\lambda\neq 0$, the λ-transfer characteristic function $v^\lambda(T)$, $T\subset S$, is defined by

$$v^\lambda(T)=\sup_{u\in V(T)} \sum_{i\in T} \lambda_i u_i$$

which boils down to (111) and (112) in our example.

$$(113)\quad \phi_1^{\lambda}=\begin{cases} \lambda_2 & \text{if } \lambda_1=0,\\ \frac{1}{2}\left\{2(\lambda_2-\lambda_1\log 2)+\log\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)^{\lambda_1+\lambda_2}\right\} & \text{if } 0<2\lambda_1<\lambda_2,\\ \left(2+\log\frac{3\sqrt{3}}{8}\right)\lambda_1 & \text{if } 2\lambda_1=\lambda_2,\\ 2\{(4-\log 2)\lambda_1-\lambda_2\}+\log\left(\frac{\lambda_1+\lambda_2}{4\lambda_1}\right)^{\lambda_1+\lambda_2}\cdot\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda_2} & \text{if } 0<\lambda_2<2\lambda_1,\\ 2(2-\log 2)\lambda_1 & \text{if } \lambda_2=0, \end{cases}$$

and

$$(114)\quad \phi_2^{\lambda}=\begin{cases} \lambda_2 & \text{if } \lambda_1=0,\\ \lambda_2-(2-\log 2)\lambda_1+\frac{1}{2}\log\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)^{\lambda_1+\lambda_2} & \text{if } 0<2\lambda_1<\lambda_2,\\ \left(\log\frac{3\sqrt{3}}{2}\right)\lambda_1 & \text{if } 2\lambda_1=\lambda_2,\\ 2\{(2+\log 2)\lambda_1-\lambda_2\}+\log\left(\frac{\lambda_1+\lambda_2}{4\lambda_1}\right)^{\lambda_1+\lambda_2}\cdot\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda_2} & \text{if } 0<\lambda_2<2\lambda_1,\\ 2\lambda_1 & \text{if } \lambda_2=0. \end{cases}$$

The Shapley value of the game is a utility vector $u^S\in V(\{1, 2\})$ such that there exists a non-negative, non-zero vector $\lambda=(\lambda_1, \lambda_2)\in R^2$ such that $\lambda_i u_i^S=\phi_i^{\lambda}$ for all $i=1, 2$. It is easy to verify that *a* Shapley value is provided by

$$(115)\quad (u_1(x^S), u_2(x^S))=\left(2+\log\frac{3\sqrt{3}}{8}, \frac{1}{2}\log\frac{3\sqrt{3}}{2}\right),$$

where $x^S=1+\frac{1}{2}\log\frac{\sqrt{3}}{2}\in[0, 1]$, and the corresponding Shapley value allocation is given by

$$(116)\quad \begin{cases} X_{11}(x^S)=1+\frac{1}{2}\log\frac{\sqrt{3}}{2},\ X_{12}(x^S)=\frac{1}{2},\\ X_{21}(x^S)=-\frac{1}{2}\log\frac{\sqrt{3}}{2},\ X_{22}(x^S)=1,\ Y(x^S)=\frac{3}{2}. \end{cases}$$

Note, however, that $X_{11}(x^S)-X_{21}(x^S)=1+\log\frac{\sqrt{3}}{2}>\log\{X_{22}(x^S)/X_{12}(x^S)\}=\log 2$, so that the equity condition (44) is violated by the Shapley value allocation. ||

IV.7. Perles-Maschler Super-Additive Solution

Finally, we examine another solution to the Nash bargaining problem, which is recommendable "[t]o a society that wishes to adopt a fair division procedure to settle future disputes as they arise; such a society may have an interest in adopting a procedure which will permit the disputants to agree on precisely when the procedure should be employed [Perles and

Maschler (1983), p. 190)]." A crucial axiom characterizing this solution concept is that of super-additivity, which assures that the expected value from arbitration in the compound game should be at least as high as the sum of the expected settlements from the two independent component games, that is because this solution concept is called the super-additive solution.

Example 7

Consider our Model C economy with $\alpha_{11}=2$, $\alpha_{12}=\alpha_{21}=\alpha_{22}=1$ and $\omega_{ij}=\frac{1}{2}$ $(i, j=1, 2)$, which is identical with the economy considered in our Example 5. Recollect that the utility possibility frontier in this economy is defined implicitly by (93), the threat point $u^0=(u_1^0, u_2^0)$ by (95) and the underlying resource allocation by (94). Let x^* and x^{**} be defined, respectively, by

$$(117) \quad x^{*2}(x^*+2)=\frac{4}{27},\ x^*\in\left(0, \frac{2}{3}\right)$$

and

$$(118) \quad 3\log(2-3x^{**})+\log(2+x^{**})=0,\ x^{**}\in\left(0, \frac{2}{3}\right).$$

It is easy to verify that such x^* and x^{**} uniquely exist and satisfy $0<x^*<\frac{1}{3}<x^{**}<\frac{2}{3}$. Furthermore, $u(x)=(u_1(x), u_2(x))$ moves from $u(x^*)$ to $u(x^{**})$ as x increases from x^* to x^{**} along the utility possibility frontier. The Perles-Maschler solution is then defined by such $x^M\in[x^*, x^{**}]$ as to satisfy

$$(119) \quad \int_{x^M}^{x^{**}}\sqrt{-\frac{du_1}{dx}\frac{du_2}{dx}}dx=\int_{x^*}^{x^M}\sqrt{-\frac{du_1}{dx}\frac{du_2}{dx}}dx,$$

that is to say,

$$(120) \quad \int_{x^M}^{x^{**}}\frac{3x+4}{(x+2)\sqrt{x(2-3x)}}dx=\int_{x^*}^{x^M}\frac{3x+4}{(x+2)\sqrt{x(2-3x)}}dx.$$

To solve (120) for $x^M\in[x^*, x^{**}]$, we define

$$(121) \quad F(x):=\int\frac{3x+4}{(x+2)\sqrt{x(2-3x)}}dx$$
$$=\arctan\frac{\sqrt{x(2-3x)}}{2x}-\sqrt{3}\arcsin(1-3x).$$

Then x^M satisfies (120) if and only if

$$(122) \quad F(x^M)=\frac{1}{2}\{F(x^*)+F(x^{**})\},\ x^M\in[x^*, x^{**}]$$

holds true.

As in the Example 5, the resource allocation corresponding to x^M is equitable only if

$$(123) \quad \frac{1}{2}\log 2\leqq\log\frac{X_{22}(x^M)}{X_{12}(x^M)}\leqq\frac{2}{3}\log 2,$$

or

$$\frac{2 \cdot 2^{1/2}}{4+3 \cdot 2^{1/2}} \leqq x^M \leqq \frac{2 \cdot 2^{2/3}}{4+3 \cdot 2^{2/3}} \tag{124}$$

is satisfied.

By computation, we may verify that $x^M=0.2306$, whereas $2 \cdot 2^{1/2}/(4+3 \cdot 2^{1/2})=0.3431$, so that the critical inequality (124) is violated by the Perles-Maschler solution.[9] Therefore, the allocation corresponding to the Perles-Maschler solution fails to guarantee the fairness of the outcome. ||

V. *Concluding Remarks*

In this paper, we have examined several public goods allocation mechanisms by making use of the no-envy equity concept as a test criterion. Our verdicts are largely negative. It is shown that all but one mechanisms we have examined fail to yield an equitable and efficient allocation even when the initial situation is that of complete equality among individuals, the exceptional mechanism being the public competitive equilibrium. Therefore, the public authority must remain forever in order to monitor the fairness of the resulting allocation if only such resource allocation mechanisms as the Lindahl equilibrium, the Groves-Ledyard mechanism, the Zeuthen-Nash bargaining scheme, the Kalai-Smorodinsky arbitration scheme, the Shapley value allocation scheme, and the Perles-Maschler super-additive solution scheme are instituted. For all their niceties in other respects, we seem to be in need for making reservations on the use of these celebrated mechanisms. The purpose of this paper is served if we could be successful in substantiating this modest claim.

A final remark is in due. The fact that the public competitive equilibrium successfully clears our hurdle does not necessarily imply that it is a "good" public goods allocation mechanism. The reason is that our test criterion is a very weak one; if it fails to be satisfied, it is a quite damaging verdict, whereas even if it is satisfied, there is not much to be jubilated for. This is the way we want our results to be interpreted.

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[9] We are grateful to Ms. Fumiko Uno of Hitotsubashi University, who computed the value of x^M for this example by using the Program Package MSL II.

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