Hitotsubashi Journal of Economics 25 (1984) 155-159. © The Hitotsubashi Academy

# ON MULTIVARIATE LEFT ORTHOGONAL INVARIANT DISTRIBUTIONS

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## I. *Introduction*

In this note, we have considered generalizations of some of the results on spherical distributions to the case of left spherical matrix variate distributions. First, it is shown that the independence of $n$ rows of an $n\times p$ left spherical random matrix implies multivariate normality. Secondly, most results in Eaton (1981) are extended to the matrix variate case.

Several authors have investigated various properties of spherical (elliptical), isotropic or radial distributions. For some details of these investigations, the reader is referred to Schoenberg (1938), Kelker (1970), Chmielewski (1980), Eaton (1982) and Letac (1981). Applications of these results in studying the robustness of test procedures have been studied by Kariya and Eaton (1977), Jensen (1979), Chmielewski (1980) and others. However, not much work has been done on properties of left orthogonally invariant (left $\mathcal{O}(n)$ invariant) or left spherical distributions of $n\times p$ matrix variates. Based on Dempster (1969), Dawid (1977) investigated some basic properties of these distributions. Eaton (1983) Chapter 7, Eaton and Kariya (1981) and Kariya (1981) also treated these distributions.

In this note, generalizations of some results on spherical distributions to the case of left orthogonally invariant distributions are attempted. In Section 2, it is shown that the independence of $n$ rows of an $n\times p$ left orthogonally invariant random matrix implies multivariate normality. This provides an alternative or formal proof to the multivariate case. Section 3 treats some generalizations of Eaton's (1981) results.

## II. *Condition for Normality*

Let $X$ be an $n\times p$ random matrix whose distribution is denoted by $\mathscr{L}(X)$. We call $X$ left orthogonally invariant if $\mathscr{L}(PX)=\mathscr{L}(X)$ for all $P\in\mathcal{O}(n)$, where $\mathcal{O}(n)$ denotes the set of $n\times n$ orthogonal matrices. Let $\mathscr{P}\equiv\mathscr{P}(n,p)$ be the set of left orthogonally invariant distributions. Let the characteristic function of $X$ be

$$\psi(A)=E\,[\exp(\mathrm{itr}X'A)]\quad(A:n\times p)\tag{2.1}$$

and let $A'=[a_1,\ldots,a_n]$ and $X'=[x_1,\ldots,x_n]$. If $\mathscr{L}(X)\in\mathscr{P}$, it is easy to see that $\psi(A)=\phi(A'A)$ for some real valued function $\phi$ defined on $\mathscr{S}_0(p)$ where $\mathscr{S}_0(p)$ denotes the set of $p\times p$ nonnegative definite matrices. Further, setting $a_2=\ldots=a_n=0$ in (2.1) shows that all marginal distributions of $x_1,\ldots,x_n$ are identical with characteristic function $\phi(a_ia_i')(i=1,\ldots,n)$. A main result in this section is stated as

*Theorem 2.1.* Let $\mathscr{L}(X)\in\mathscr{P}(n,p)$ with $n\geq p$ and $n\geq 2$. Then the independence of $n$ rows of $X$ implies that $x_1, \ldots, x_n$ are i.i.d. and $\mathscr{L}(x_1)=N(O,\Sigma)$, normal distribution with mean 0 and a covariance matrix $\Sigma$.

*Proof:* The i.i.d. part is clear. To show the normality, first note that the independence of $x_i$'s implies

$$\phi(A'A)=\prod_{i=1}^{n}\phi(a_ia_i') \text{ for all } A: n\times p \tag{2.2}$$

Here setting $a_1=a_2$ and $a_3=\ldots=a_n=0$ yields

$$\phi(2a_1a_1')=\phi^2(a_1a_1')\geq 0 \text{ for any } a_1\in R^p \tag{2.3}$$

since $\phi(0)=1$. This implies $\phi(a_ia_i')\geq 0$ $(i=1, \ldots, n)$. Suppose for some $\bar{a}_1\in R^p$, $\phi(\bar{a}_1\bar{a}_1')=0$. Then from (2.2)

$$\begin{aligned}0&=\phi(n\bar{a}_1\bar{a}_1/n)=[\phi(\bar{a}_1\bar{a}_1/n)]^n\\&=[\phi(n\bar{a}_1\bar{a}_1'/n^2)]^n=[\phi(\bar{a}_1\bar{a}_1'/n)]^2,\end{aligned}$$

implying $\phi(\bar{a}_1\bar{a}_1'/n^2)=0$, since $\phi$ is real. Repeating this procedure provides $\phi(\bar{a}_1\bar{a}_1'/n^k)=0$ for any positive integer $k$. But, this contradicts the condition $\phi(0)=1$ since $\phi(aa')$ is the characteristic function of $x_1$. This shows that $0<\phi(aa')\leq 1$ for any $a\in R^p$.

Next define

$$\bar{\Phi}(A'A)=-2\log\phi(A'A). \tag{2.4}$$

Then $\phi(A'A)\geq 0$ is continuous and (2.2) implies

$$\bar{\Phi}\left(\sum_{i=1}^{n} a_ia_i'\right)=\sum_{i=1}^{n}\bar{\Phi}(a_ia_i'). \tag{2.5}$$

and $\bar{\Phi}(0)=0$. From (2.5), $\bar{\Phi}(maa')=m\bar{\Phi}(aa')$ for $m\leq n$ and $a\in R^p$, which implies $n\bar{\Phi}(aa')=\bar{\Phi}(m\frac{n}{m}aa')=m\bar{\Phi}(m\frac{n}{m}aa')$. On the other hand, $n\bar{\Phi}(m\frac{m}{n}aa')=\bar{\Phi}(maa')=m\bar{\Phi}(aa')$. This together with continuity of $\bar{\Phi}$ as a function of $a\in R^p$ implies $\bar{\Phi}(\alpha aa')=\alpha\bar{\Phi}(aa')$ for all $\alpha\geq 0$, which in turn together with (2.5) implies

$$\bar{\Phi}(\alpha A'A)=\alpha\phi(A'A) \text{ for all } \alpha\geq 0. \tag{2.6}$$

Now we extend the domain $\mathscr{S}_0(p)$ of $\bar{\Phi}$ to the vector space of $p\times p$ symmetric matrices, say $\mathscr{S}(p)$. Let

$$\mathscr{A}=\{S\mid S=S_1-S_2 \text{ for some } S_1, S_2 \text{ in } \mathscr{S}_0(p)\} \tag{2.7}$$

Then it is easy to see $\mathscr{A}=\mathscr{S}(p)$. Define a real valued function $\Phi$ defined on $\mathscr{S}(p)$ by

$$\Phi(S)=\bar{\Phi}(S_1)-\bar{\Phi}(S_2) \tag{2.8}$$

when $S=S_1-S_2$ with $S_i\in\mathscr{S}_0(p)$ $(i=1,2)$. To see this is well-defined, let $S=S_1-S_2=V_1-V_2$ with $S_i\in\mathscr{S}_0(p)$ and $V_i\in\mathscr{S}_0(p)$ $(i=1,2)$. Then since $S_1+V_2=S_2+V_1$, $\bar{\Phi}(S_1)+\bar{\Phi}(V_2)=\bar{\Phi}(S_2)+\bar{\Phi}(V_1)$, which shows that $\Phi(S_1-S_2)=\Phi(V_1-V_2)$. Further, $\Phi$ is a linear functional on the vector space $\mathscr{S}(p)$ such that $\Phi=\bar{\Phi}$ on $\mathscr{S}_0(p)$. Therefore $\Phi(S)=\mathrm{tr}\Sigma S$ for some unique symmetric matrix $\Sigma$. For $S=A'A$, $0\leq\Phi(S)=\phi(S)=\mathrm{tr}\Sigma A'A=\mathrm{tr}A\Sigma A'$ where $A$ is any $n\times p$ matrix. This implies $\Sigma$ is nonnegative definite. Since $\bar{\Phi}(aa')=a'\Sigma a$ implies $\phi(aa')=\exp(-\frac{1}{2}a\Sigma a)$ and since $\phi(aa')$ is the characteristic function of $x_1$, this completes the proof.

We first remark that in the above proof, $n$ is fixed. Secondly, it is noted that neither the existence of pdf nor $P(X=0)=0$ is assumed. Thirdly, the nonsingularity of $\Sigma$ in Theorem 2.1 does not necessarily follow.

## III. *Some Generalizations of Results in Vector Case*

In this section, some results of Eaton (1981) are generalized to a matrix variate case. Let $\mathscr{X}=\{X{:}n\times p\,|\,\mathrm{rank}(X)=p\}$ and let $\mathscr{Q}(n,p)=\{P\in\mathscr{P}(n,p)\,|\,P(X\in\mathscr{X})=1\}$. Let $U{:}n\times p$ have a uniform distribution on Stiefel manifold $\mathscr{U}_n=\{u{:}n\times p\,|\,u'u=I_p\}$. Then it is well known that $\mathscr{L}(X)\in\mathscr{Q}(n,p)$ if and only if $X=UV$ for some $V\in\mathscr{S}_+(p)$, where $\mathscr{S}_+(p)$ denotes the set of $p\times p$ positive definite matrices (e.g., Eaton (1979) Chapter 7).

Now let $\tilde{X}$ be an $(n+m)\times p$ random matrix such that $\mathscr{L}(X)\in\mathscr{Q}(n+m,p)$ and let $X_{(n)}$ be the upper $n\times p$ submatrix of $\tilde{X}$, where $n$ and $m$ are positive integers. Clearly $\mathscr{L}(X_{(n)})\in\mathscr{Q}(n,p)$. We shall call $\mathscr{L}(X)$ the $n$-marginal of $\mathscr{L}(\tilde{X})$ and denote the class of $n$-marginals of $\mathscr{Q}(n+m,p)$ by $\mathscr{Q}^{n+m}(n,p)(\subset\mathscr{Q}(n,p))$. The following result is a generalization of Theorem 1 of Eaton (1981), and characterizes the elements of $\mathscr{Q}^{n+m}(n,p)$.

*Theorem 3.1.* Let $\mu$ be a probability measure on $R^{np}$. Then the following are equivalent:

(a) $\mu\in\mathscr{Q}^{n+m}(n,p)$

(b) $\mu$ has a density with respect to Lebesgue measure on $R^{np}$, say $f$, given by $F(x)=h(x'x)$, where

$$h(t)=\int_{\mathscr{S}_+(p)}\phi(r^{-1}tr^{-1}\mid n,\,mp)\mid r\mid^{-n}G(dr) \tag{3.1}$$

$G$ is a distribution function on $\mathscr{S}_+(p)$ and

$$\phi(t\mid n,\,m,\,p)=C(n,\,m,\,p)\mid I_p-t\mid^{(m-p-1)/2}I(t) \tag{3.2}$$

Here, $I(t)=1$ if $I_p-t\in\mathscr{S}_+(p)$ and $I(t)=0$ otherwise, and $C(n,m,p)=c(n+m,p)/c(n,p)$, $c(r,p)=\pi^{-pr/2}p(r/2)$ and $p(r/2)=\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma(r-j+1/2)$.

*Proof.* Suppose $\mu\in\mathscr{Q}^{n+m}(n,p)$. Then there exists on $(n+m)\times p$ random matrix $\tilde{X}$ such that $\mathscr{L}(\tilde{X})\in\mathscr{Q}(n+m,p)$ and $\mathscr{L}(X_{(n)})=\mu$. Since $\tilde{X}=\tilde{U}V$ and $X_{(n)}=U_{(n)}V$ where $U_{(n)}$ is the upper $n\times p$ submatrix of $\tilde{U}$, and since $U$ is uniform on $\mathscr{U}_{n+m}$, using a result of Khatri (1970), the density of $U_{(n)}$ is $\phi(u'u\,|\,n,m,p)$. To show that $f$ is a density for $\mu$, let $C\subset R^{np}$ be a Borel set and let $G$ be the distribution function of $V$. Then

$$\begin{aligned}\mu(C)&=P(X_{(n)}\in C)=P(U_{(n)}V\in C)\\&=\int_{\mathscr{S}_+(p)}\int_{R^{np}}I_c(ur)\phi(u'u\mid n,\,m,\,p)\,du\;G\,(dr)\\&=\int_{\mathscr{S}_+(p)}\int_{R^{np}}I_c(u)\phi(r^{-1}u'ur^{-1}\mid n,\,m,\,p)\mid r\mid^{-n}du\;G\,(dr).\\&=\int_{R^{np}}I_c(u)f(u)du.\end{aligned}$$

The converse is straightforward.

The next result is a generalization of Theorem 2 in Eaton (1981) and gives a condition for $\mu\in\mathscr{Q}^{n+m}(n,p)$ to be a normal mixture.

*Theorem 3.2.* The following are equivalent.

(a) $\mu\in\mathscr{Q}^{n+m}(n,p)$ for all $m\geq 1$

(b) $\mu$ has a density for $\mu$ given by

$$f(u)=\int_{\mathscr{S}_+(p)}\exp\left(-\frac{1}{2}tru'u\Sigma^{-1}\right)(\sqrt{2\pi})^{-np}\mid\Sigma\mid^{-n/2}H(d\Sigma) \tag{3.3}$$

where $H$ is a distribution function on $\mathscr{S}_+(p)$.

*Proof.* ($b\Rightarrow$(a) is obvious. To show (a)$\Rightarrow$(b), suppose (a) holds. Then by Theorem 3.1, $\mu$ has a density $f(u)$ of the form

$$F(u)=\int_{\mathscr{S}_+(p)}\phi(r^{-1}u'ur^{-1}\mid n,m,p)\mid r\mid^{-n}G_m(dr)$$

$$\int_{\mathscr{S}_+(p)}\phi(r^{-1}u'ur^{-1}/m\mid n,m,p)(\sqrt{m})^{-np}\mid r\mid^{-n}\tilde{G}_m(dr)$$

where $\tilde{G}_m(dr)=G_m(dr/\sqrt{m})$. Here it is easy to see that $\lim_{m\to\infty}\phi(r^{-1}u'ur^{-1}/m\mid n,m,p)(\sqrt{m})^{-np}=(\sqrt{2\pi})^{-np}\exp(-\frac{1}{2}\mathrm{tr}u'u\Sigma^{-1})$ where $\Sigma=r^2$. Therefore, arguing exactly in the same way as in Eaton (1981) and changing $r$ to $\Sigma=r^2$ in the final step, the result follows.

Finally we generalize Theorem 3 of Eaton (1981). Let $Z:1\times p$ have a symmetric distribution such that $P(Z=0)=0$. We shall call a random matrix $X:n\times p$ an $n$-dimensional version of $Z$ if for any $a\in R^n$, there exists $c(a)\geq 0$ such that $c(a)>0$ for $a\neq 0$ and $\mathscr{L}(a'X)=\mathscr{L}(c(a)Z)$. Let $\mathscr{F}(n,p)$ denote the set of all $p$-dimensional distribution which admit $n$-dimensional versions ($n\geq 2$). Clearly $\mathscr{Q}^n(1,p)\subset\mathscr{F}(n,p)$ for $n\geq 2$.

*Theorem 3.3.* Suppose $\mathscr{L}(Z)\in\mathscr{F}(n,p)$ and cov($Z$) exists and it is positive definite. Then every $n$-dimensional version of $Z$ is given by $AX_0$, where $A$ is an $n\times n$ nonsingular matrix and $X_0$ is an $n$-dimensional version of $Z$ such that for any $P\in\mathscr{O}(n)$, $PX_0$ is also an $n$-dimensional version of $Z$ and $\mathscr{L}(a'PX_0)=\mathscr{L}(a'X_0)$ for all $a\in R^n$.

*Proof.* Let $X:n\times p$ be an $n$-dimensional version of $Z$. Then, by definition, there exists $c(a)\geq 0$ for $a\in R^n$ such that $\mathscr{L}(a'X)=\mathscr{L}(c(a)Z)$. This implies $\mathscr{L}(a'Xb)=\mathscr{L}(c(a)Zb)$ for any $b\in R^p$. Hence $E(a'Xb)^2=(c(a))^2b'\Sigma b$ where $\Sigma=\mathrm{cov}(Z)$, and so $c(a)=[b'E(X'aa'X)b/b'\Sigma b]^{\frac{1}{2}}$ for $b\neq 0$. Since $c(a)$ is independent of $b$, this implies $\Sigma^{-\frac{1}{2}}E(X'aa'X)\Sigma^{-\frac{1}{2}}=\gamma(a)I$. Therefore $\gamma(a)=\mathrm{tr}E(X'aa'X)/\mathrm{tr}\Sigma=a'\Omega a/\mathrm{tr}\Sigma$ where $\Omega=EXX'$, and $c(a)=\gamma(a)^{\frac{1}{2}}$. Since $c(a)>0$ for $a\neq 0$, $\Omega\in\mathscr{S}_+(n)$. Let $X_0=(\mathrm{tr}\Sigma)^{\frac{1}{2}}\Omega^{-\frac{1}{2}}X$. Then

$$\mathscr{L}(a'X_0)=\mathscr{L}[((\mathrm{tr}\Sigma)^{\frac{1}{2}}\Omega^{-\frac{1}{2}}a)'X]=\mathscr{L}[c((\mathrm{tr}\Sigma)^{\frac{1}{2}}\Omega^{-\frac{1}{2}}a)Z]=\mathscr{L}(||a||Z),$$

which implies that $X_0$ is an $n$-dimensional version of $Z$ and $\mathscr{L}(a'PX_0)=\mathscr{L}(a'X_0)$ for any $P\in\mathscr{O}(n)$. Set $A_0=(\mathrm{tr}\Sigma)^{-\frac{1}{2}}\Omega^{\frac{1}{2}}$ to complete the proof.

We remark that unlike in Theorem 3 of Eaton (1981), the left orthogonal invariance of $X_0$ does not follow here. Therefore $\mathscr{L}(Z)\in\mathscr{O}^n(1,p)$ does not follow either unless $p=1$.

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