ON DUALITY THEORY FOR THE CONTINUOUS TIME MODEL OF CAPITAL ACCUMULATION

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I. Introduction

In their recent paper, Benveniste and Scheinkman (1982) have consider a model of capital accumulation and clarified the general structure of dynamic optimization problems. They established several useful results on duality theory for continuous time models with infinite horizon. The purpose of the present paper is to show that their results hold in more general situations. The generalization in this paper is purely mathematical, but it makes their results applicable to a much broader class of economic models.

In duality theory for infinite horizon models, the so-called transversality condition is a difficult problem. In section IV we shall reconsider this problem in a general model. First we shall prove that the optimal path can be completely characterized by the support price path and the value function. It is proved in Theorem 4.1 that a path of capital accumulation is optimal if and only if the value function becomes zero as time goes on and there exists a price path which supports both the value function and the instantaneous utility function. Since the value function is used in characterization of optimal paths, the theorem may not be so useful in checking whether a certain path is optimal. However, the theorem is true under very weak assumption, and it is a basic theorem for duality theory. In fact, by the theorem, it is proved in Theorem 4.2 that a path of capital accumulation is optimal if and only if there exists a price path supporting the instantaneous utility function and the value of capital stock becomes zero as time goes on. The second theorem holds only under specific assumption, but it is quite useful in that the value function is not used. The theorem was originally proved by Benveniste and Scheinkman (1982) under somewhat stronger assumptions. In discrete time models, a corresponding theorem has been proved by Weitzman (1973).

In section V, it is proved in Theorem 5.1 that on the optimal path the marginal value of capital is the marginal cost of investment. The theorem was also originally proved by Benveniste and Scheinkman (1982) in a less general situation. In addition, by the theorem, we can see that the argument by Benveniste and Scheinkman (1979) on the differentiability of the value function holds under more general assumptions.

II. Mathematical Notation

The real line is denoted by $R$. By $R^n$, where $n$ is an integer, we shall denote the $n$-dimensional Euclidean space. For any $x, y \in R^n$, the inner product of $x$ and $y$ is denoted by
The Euclidean norm of any \( x \in \mathbb{R}^n \) is denoted by \( ||x|| \), i.e., \( ||x|| = \sqrt{x \cdot x} \). For any \( x, y \in \mathbb{R}^n \), \( x \geq y \) means that each component of \( x \) is equal to or greater than the corresponding component of \( y \), \( x > y \) means that every component of \( x \) is greater than the corresponding component of \( y \), and \( x \geq y \) means that \( x \geq y \), but \( x \neq y \). For any subset \( U \) of \( \mathbb{R}^n \), \( \operatorname{int} U \) denotes the interior of \( U \) in \( \mathbb{R}^n \) and \( \operatorname{co} U \) denotes the convex hull of \( U \). For any convex subset \( U \) of \( \mathbb{R}^n \), \( \operatorname{rel. int} U \) denotes the relative interior of \( U \).

A function \( f: I \rightarrow \mathbb{R}^n \) defined on a closed interval \( I \subset \mathbb{R} \) to \( \mathbb{R}^n \) is said to be absolutely continuous if the restriction of \( f \) on any compact interval is absolutely continuous in the usual sense. The derivative of \( f \) is denoted by \( f' \). For any concave (or convex) function \( f: U \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \) defined on a convex subset \( U \) of \( \mathbb{R}^n \), symbol \( \partial f(x) \) denotes the set of all subgradients of function \( f \) at \( x \in U \), i.e., \( \partial f(x) = \{ p \in \mathbb{R}^n | f(x) - p \cdot x \geq (\text{or resp. } \leq) f(y) - p \cdot y \text{ for all } y \in U \} \).

A mapping \( F: U \rightarrow 2^{\mathbb{R}^1} \) defined on a subset \( U \) of \( \mathbb{R}^n \) to the family of all non-empty subsets of \( \mathbb{R}^1 \) is called a correspondence. Correspondence \( F \) is said to be lower semi-continuous at \( x_0 \in U \) if for any \( y_0 \in F(x_0) \) and any sequence \( \{ x_i \mid i = 1, 2, \ldots \} \) in \( U \) converging to \( x_0 \), there exists a sequence \( \{ y_i \mid i = 1, 2, \ldots \} \) converging to \( y_0 \) such that \( y_i \in F(x_i) \) for all \( i \). Correspondence \( F \) is said to be lower semi-continuous if \( F \) is lower semi-continuous at all \( x \in U \).

### III. The Model and A Support Price Theorem

Let \( m \) be the number of different capital goods in the economy. The technology of the economy is described by a correspondence \( Y: [0,\infty) \rightarrow 2^{\mathbb{R}^n \times \mathbb{R}^n} \), whose image \( Y(t) \) denotes the set of all feasible pairs \( (x,y) \) of capital stock \( x \in \mathbb{R}^n \) and investment level \( y \in \mathbb{R}^n \) at time \( t \in [0,\infty) \). Define a correspondence \( X: [0,\infty) \rightarrow 2^{\mathbb{R}^n} \) by \( X(t) = \{ x \in \mathbb{R}^n | (x,y) \in Y(t) \} \) for some \( y \in \mathbb{R}^n \). Also, define \( G_T = \{ (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty) \mid (x,y) \in Y(t) \} \). Social welfare at any point in time is represented by an instantaneous utility function \( u: G_T \rightarrow \mathbb{R} \), whose image \( u(x,y,t) \) denotes the indirect utility of capital stock \( x \) and investment level \( y \) at time \( t \).

**Assumption 1:**

(i) For each \( t \in [0,\infty) \), \( Y(t) \) is a convex subset of \( \mathbb{R}^n \times \mathbb{R}^n \).
(ii) For each \( t \in [0,\infty) \), \( \operatorname{int} X(t) \neq \emptyset \).
(iii) The function \( u \) is a measurable function such that, for each \( t \in [0,\infty) \), \( u(x,y,t) \) is a concave function of \( (x,y) \) on \( Y(t) \).

An absolutely continuous function \( f: [t',t''] \rightarrow \mathbb{R}^m \), where \( t', t'' \in [0,\infty) \) and \( t' < t'' \), is said to be a feasible arc if \( (f(t), f(t)) \in Y(t) \) for almost every \( t \in [t', t''] \).

**Assumption 2:** There exist a countable family of feasible arcs, say \( F \), having the following properties.

(i) For each \( t \in [0,\infty) \), define \( F_t = \{ f \in F \mid f \text{ is a function defined on } [t', t''] \text{ to } \mathbb{R}^m \text{ such that } t' \leq t < t'' (t' = t \text{ only when } t = 0) \} \) and
\[ D_t = \{(f(t), \dot{f}(t)) \in \mathbb{R}^m \times \mathbb{R}^m \mid f \in F_t\} \]

Then \( D_t \) is a dense subset of \( Y(t) \) for all \( t \in [0, \infty) \).

(ii) \[ \int_{t'}^{t''} u(f(t), \dot{f}(t), t) dt < +\infty \text{ for any } f: [t', t''] \to \mathbb{R}^m \text{ in } F. \]

The above general assumption is implied by the following simpler assumption [for proof, see Takekuma (1982, Appendix)].

Assumption 2':
(i) The correspondence \( Y \) is lower semi-continuous.
(ii) If \( f: [t', t''] \to \mathbb{R}^m \) is a feasible arc whose derivative \( \dot{f} \) is continuous, then
\[ \int_{t'}^{t''} u(f(t), \dot{f}(t), t) dt < +\infty. \]

If the function \( u \) is continuous, then Assumption 2' (ii) is automatically satisfied. Therefore, Assumption 2 holds if correspondence \( Y \) is lower semi-continuous and function \( u \) is continuous.

An absolutely continuous function \( k: [t', \infty) \to \mathbb{R}^m \), where \( t' \geq 0 \), is said to be a feasible path from time \( t' \) if \( (k(t), \dot{k}(t)) \in Y(t) \) for almost every \( t \in [t', \infty) \). For each \( (x, t) \in \mathbb{R}^m \times [0, \infty) \), let \( A(x, t) \) denote the set of all feasible paths \( k \) from time \( t \) with \( k(t) = x \).

Assumption 3: If \( k \) is a feasible path from time \( t' \), then
\[ \int_{t'}^{t''} u(k(t), \dot{k}(t), t) dt < +\infty \text{ for all } t'' \in [t', \infty). \]

Under this assumption, a criterion of optimality for feasible paths can be defined. A feasible path \( k \) from time \( t' \) is said to be overtaken by another feasible path \( k' \in A(k(t'), t') \) if there exist \( \varepsilon > 0 \) and \( t_0 \geq t' \) such that \[ \int_{t'}^{t''} u(k'(t), \dot{k'}(t), t) dt > \int_{t'}^{t''} u(k(t), \dot{k}(t), t) dt + \varepsilon \text{ for all } t'' \geq t_0. \]

A feasible path \( k \) from time \( t' \) is said to be an optimal path from time \( t' \) if \( k \) is not overtaken by any \( k' \in A(k(t'), t') \).

Let \( k \) be a feasible path from time \( 0 \). Then, we can define a function \( \tilde{u}: G_{r} \to \mathbb{R} \) by
\[ \tilde{u}(x, y, t) = u(x, y, t) - u(k(t), \dot{k}(t), t). \]

In addition, if \( \int_{t'}^{t''} u(k(t), \dot{k}(t), t) dt < -\infty \text{ for all } t', t'' \in [0, \infty) \) with \( t' \leq \) \( t'' \), then we can define a function \( V: \mathbb{R}^m \times [0, \infty) \to \mathbb{R} \cup \{ -\infty, +\infty \} \) by
\[ V(x, t') = \sup_{k' \in A(x, t')} \left[ \liminf_{t'' \to \infty} \int_{t'}^{t''} \tilde{u}(k'(t), \dot{k}'(t), t) dt \right]. \]

The function \( \tilde{u} \) should be called the normalized utility function with respect to the feasible path \( k \). Also, the function \( V \) should be called the normalized value function with respect to the feasible path \( k \), which is a generalization of the usual value function. By Assump-
tion 1 (iii), we can easily show that, for each \( t \), \( u(x, y, t) \) is a concave function of \((x, y)\) and \( V(x, t) \) is a concave function of \( x \). From now on, we should always remember that functions \( u \) and \( V \) are defined for a particular feasible path from time 0.

Now let us consider an optimal path \( k \) from time 0 satisfying the following conditions.

**Assumption 4:**

(i) \( \int_{t'}^{t''} u(k(t), k(t), t)dt > -\infty \) for all \( t', t'' \in [0, \infty) \) with \( t' \leq t'' \).

(ii) \( k(t) \in \text{int} X(t) \) for all \( t \in [0, \infty) \).

(iii) \( \partial V(k(0), 0) \neq \emptyset \), where \( \partial V(k(0), 0) \) denotes the set of all subgradients of function \( V(., 0) \) at \( k(0) \).

We have the following support price theorem for such an optimal path [for proof, see Takekuma (1979), or (1982)].

**The Support Price Theorem:** Let \( k \) be an optimal path from time 0 satisfying Assumption 4. Then, under Assumptions 1, 2 (or 2'), and 3, for any \( p \in \partial V(k(0), 0) \) there exists an absolutely continuous function \( q: [0, \infty) \to \mathbb{R}^n \) with the following properties:

(i) \( q(0) = p \).

(ii) \( q(t) \in \partial V(k(t), t) \) for all \( t \in [0, \infty) \).

(iii) \( -(q(t), q(t)) \in \partial u(k(t), k(t), t) \) for almost every \( t \in [0, \infty) \). In the above, for each \( t \in [0, \infty) \), symbols \( \partial V(., t) \) and \( \partial u(., ., t) \) denote the sets of all subgradients for functions \( V(., t) \) and \( u(., ., t) \) respectively.

It is well-known that condition (iii) is the above theorem corresponds to the Euler equation and can be formulated by a Hamiltonian equation. Define a Hamiltonian function \( H: \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \cup \{-\infty, +\infty\} \) by

\[
H(p, x, t) = \sup \{u(x, y, t) + p \cdot y | (x, y) \in Y(t)\}.
\]

It is easy to show that, under Assumption 1 (i), (iii), for each \( t \in [0, \infty) \) \( H(p, x, t) \) is a convex function of \( p \) and is a concave function of \( x \). Then, condition (iii) is equivalent to the following:

\[ k(t) \in \partial_1 H(q(t), k(t), t), \quad -(q(t), q(t)) \in \partial_2 H(q(t), k(t), t), \quad \text{and} \quad H(q(t), k(t), t) = u(k(t), k(t), t) + q(t) \cdot k(t) \]

for almost every \( t \in [0, \infty) \). Here, for each \( t \in [0, \infty) \), symbols \( \partial_1 H(., k(t), t) \) and \( \partial_2 H(q(t), ., t) \) denote the sets of all subgradients for functions \( H(., k(t), t) \) and \( H(q(t), ., t) \) respectively.

**IV. Complete Characterization of the Optimal Path**

First we shall prove a basic theorem which shows a necessary and sufficient condition for the optimality.

**Theorem 4.1:** Let \( k \) be a feasible path from time 0 satisfying Assumption 4. Then, under Assumptions 1, 2 (or 2'), and 3, \( k \) is an optimal path from time 0 if and only if

(i) \( \lim_{t \to \infty} V(k(t), t) = 0 \)
and there exists an absolutely continuous function \( q : [0, \infty) \rightarrow \mathbb{R}^m \) satisfying the following conditions:

(ii) \( q(t) \in \partial V(k(t), t) \) for all \( t \in [0, \infty) \).

(iii) \( -(q(t), q(t)) \in \partial u(k(t), \dot{k}(t), t) \) for almost every \( t \in [0, \infty) \).

In the above, \( V \) is the normalized value function with respect to path \( k \).

Proof: (Necessity) Assume that \( k \) is an optimal path from time 0. Then, by the Support Price Theorem we have an absolutely continuous function \( q \) satisfying the above conditions (ii) and (iii). Also, since \( k \) is an optimal path from time 0, by definition of \( V \), \( V(k(t), t) = 0 \) for all \( t \in [0, \infty) \). This implies condition (i).

(Sufficiency) Assume that \( k \) is a feasible path from time 0 satisfying condition (i), and that there exists an absolutely continuous function \( q \) satisfying conditions (ii) and (iii). Then, we can show that path \( k \) is agreeable, i.e.,

\[
V(k(0), 0) = \int_0^1 \tilde{u}(k(t), \dot{k}(t), t)dt + V(k(s), s) \text{ for all } s \in [0, \infty),
\]

where \( \tilde{u} \) is the normalized utility function with respect to path \( k \).

Suppose that this were not true. Then, by definition of \( V \), there is \( s_0 \in [0, \infty) \) such that

\[
V(k(0), 0) > \int_0^1 \tilde{u}(k(t), \dot{k}(t), t)dt + V(k(s_0), s_0).
\]

Therefore, again by definition of \( V \), there exist \( k' \in A(k(0), 0) \), \( \varepsilon > 0 \), and \( s_1 \in [0, \infty) \) such that, for all \( s \geq s_1 \),

\[
\int_0^1 \tilde{u}(k'(t), \dot{k}'(t), t)dt \geq \int_0^1 \tilde{u}(k(t), \dot{k}(t), t)dt + V(k(s_0), s_0) + \varepsilon.
\]

Hence, again by definition of \( V \), it follows from the above inequality that

\[
\int_0^1 \tilde{u}(k'(t), \dot{k}'(t), t)dt + V(k'(s_0), s_0) \geq \int_0^1 \tilde{u}(k(t), \dot{k}(t), t)dt + V(k(s_0), s_0) + \varepsilon.
\]

On the other hand, from condition (ii), it follows that \( u(k(t), \dot{k}(t), t) + \dot{q}(t)\cdot k(t) + q(t)\cdot \dot{k}(t) \geq u(k'(t), \dot{k}'(t), t) + \dot{q}(t)\cdot k'(t) + q(t)\cdot \dot{k}'(t) \) for almost every \( t \in [0, \infty) \). By definition of \( \tilde{u} \), \( u \) in this inequality can be replaced by \( \tilde{u} \). Therefore, by integrating the inequality over \([0, s_0]\), we have

\[
\int_0^{s_0} \tilde{u}(k(t), \dot{k}(t), t)dt + q(s_0)\cdot k(s_0) - q(0)\cdot k(0) \geq \int_0^{s_0} \tilde{u}(k'(t), \dot{k}'(t), t)dt + q(s_0)\cdot k'(s_0) - q(0)\cdot k'(0).
\]

Remember that \( k'(0) = k(0) \). Also, \( V(k(s_0), s_0) - q(s_0)\cdot k(s_0) \geq V(k'(s_0), s_0) - q(s_0)\cdot k'(s_0) \) by condition (ii). Hence, the above inequality implies that

\[
\int_0^{s_0} \tilde{u}(k(t), \dot{k}(t), t)dt + V(k(s_0), s_0) \geq \int_0^{s_0} \tilde{u}(k'(t), \dot{k}'(t), t)dt + V(k'(s_0), s_0).
\]
This contradicts (4.2). Thus, (4.1) has been proved.

Now, by (4.1) and by definition of $\bar{u}$, $V(k(0), 0) = V(k(s), s)$ for all $s \in [0, \infty)$. Hence, condition (i) implies that $V(k(0), 0) = 0$. Thus,

$$V(k(0), 0) = \int_0^\infty \bar{u}(k(t), k(t), t) dt.$$ 

This implies the optimality of path $k$. Q.E.D.

Now let us make some additional assumptions on the model in order to get a more useful theorem.

**Assumption 5:**

1. (Non-negativity of capital stock) If $(x, y) \in Y(t)$, then $x \geq 0$.
2. (Free disposability) If $(x, y) \in Y(t)$ and $y' \leq y$, then $(x, y') \in Y(t)$ and $u(x, y, t) \leq u(x, y', t)$.
3. (Possibility of inaction) $(0, 0) \in Y(t)$ and $u(0, 0, t) \geq 0$ for all $t \in [0, \infty)$.

The next theorem is a counterpart of the duality theorem which was proved by Weitzman (1973) in a discrete time model.

**Theorem 4.2:** Let $k$ be a feasible path from time 0 satisfying Assumption 4. In addition, assume that $\int_0^\infty u(k(t), k(t), t) dt$ is finite. Then, under Assumption 1, 2 (or 2'), 3, and 5, $k$ is an optimal path from time 0 if and only if there exists an absolutely continuous function $q : [0, \infty) \to \mathbb{R}^n$ satisfying the following conditions:

1. $-(q(t), k(t)) \in \partial u(k(t), k(t), t)$ for almost every $t \in [0, \infty)$.
2. $\lim_{t \to \infty} q(t) \cdot k(t) = 0$.

**Proof:** (Necessity) Assume that $k$ is an optimal path from time 0. Then, we can use Theorem 4.1, i.e., there exists an absolutely continuous function $q : [0, \infty) \to \mathbb{R}^n$ satisfying conditions (ii) and (iii) of Theorem 4.1. Hence, condition (i) of this theorem has been proved. Also, by condition (ii) of Theorem 4.1, for each $t \in [0, \infty)$, $V(k(t), t) - q(t) \cdot k(t) \geq V(x, t) - q(t) \cdot x$ for all $x \in \mathbb{R}^m$, where $V$ is the normalized value function with respect to path $k$. Therefore, by putting $x = 0$, we have

$$V(k(t), t) - q(t) \cdot k(t) \geq V(0, t) \geq 0.$$ 

On the other hand, by definition of $V$ and Assumption 5 (iii),

$$V(0, s) \geq \int_0^s \bar{u}(0, 0, t) dt \geq -\int_0^s u(k(t), k(t), t) dt \geq 0,$$

where $\bar{u}$ is the normalized utility function with respect to path $k$. Thus, by the above inequalities, we have

$$V(k(s), s) + \int_s^\infty u(k(t), k(t), t) dt \geq q(s) \cdot k(s) \text{ for all } s \geq 0.$$ 

(4.3)
Moreover, by Assumption 5(i), \(k(t) \geq 0\) for all \(t \in [0, \infty)\), and by condition (iii) of Theorem 4.1 and Assumption 5(ii), \(q(t) \geq 0\) for all \(t \in [0, \infty)\). Hence,

\[
q(t) \cdot k(t) \geq 0 \quad \text{for all } t \geq 0.
\]

In (4.3), by condition (i) of Theorem 4.1, \(\lim_{t \to \infty} V(k(t), t) = 0\). Also, since the integral \(\int_0^\infty u(k(t), k(t), t) dt\) is finite by assumption, \(\lim_{t \to \infty} \int_0^\infty u(k(t), k(t), t) dt = 0\). Therefore, by (4.4), we can conclude that \(\lim_{t \to \infty} q(t) \cdot k(t) = 0\), which is condition (ii) of this theorem.

(Sufficiency) Assume that there exists an absolutely continuous function \(q : [0, \infty) \to \mathbb{R}^n\) satisfying conditions (i) and (ii) of this theorem.

Let \(k' \in A(k(0), 0)\). Then, by condition (i) of this theorem, \(u(k(t), k(t), t) + q(t) \cdot k(t) + q(t) \cdot k(t) \geq u(k'(t), k'(t), t) + q(t) \cdot k'(t) + q(t) \cdot k'(t)\) for almost every \(t \in [0, \infty)\). Since \(k'(0) = k(0)\), by integration we have

\[
\int_0^t u(k(t), k(t), t) dt + q(s) \cdot k(s) \geq \int_0^t u(k'(t), k'(t), t) dt + q(s) \cdot k'(s) \quad \text{for all } s \geq 0.
\]

Moreover, by Assumption 5(i), \(k'(t) \geq 0\) for all \(t \in [0, \infty)\), and by condition (i) of this theorem and Assumption 5(ii), \(q(t) \geq 0\) for all \(t \in [0, \infty)\). Hence,

\[
q(t) \cdot k'(t) \geq 0 \quad \text{for all } t \geq 0.
\]

By (4.5) and (4.6), we have

\[
q(s) \cdot k(s) \geq \int_0^s u(k(t), k(t), t) dt - \int_0^s u(k'(t), k'(t), t) dt \quad \text{for all } s \geq 0.
\]

By virtue of condition (ii) of this theorem, this implies that \(k\) is not overtaken by \(k'\). Since \(k'\) is an arbitrary path in \(A(k(0), 0)\), this proves the optimality of \(k\). Q.E.D.

V. The Subgradients of the Value Function

In this section we shall prove that on the optimal path the marginal value of capital accumulated equals the marginal cost of accumulation at each point in time.

**Lemma 5.1:** Let \(t_0 \in [0, \infty)\) and \(x_0 \in \text{int } X(t_0)\). Then, under Assumption 1(i), set, \(\{y \in \mathbb{R}^m \mid (x_0, y) \in \text{rel.int } Y(t_0)\}\) is dense in set, \(\{y \in \mathbb{R}^m \mid (x_0, y) \in Y(t_0)\}\).

**Proof:** Let \(y_0 \in \mathbb{R}^m\) with \((x_0, y_0) \in Y(t_0)\). Since \(x_0 \in \text{int } X(t_0)\), there exist \((x'_0, y'_0), (x'_1, y'_1), \ldots, (x'_m, y'_m) \in Y(t_0)\) such that \(x_0 \in \text{co } \{x'_0, x'_1, \ldots, x'_m\}\). Therefore, there exist \((x''_0, y''_0), (x''_1, y''_1), \ldots, (x''_m, y''_m) \in \text{rel.int } Y(t_0)\) such that each \((x''_i, y''_i)\) is so close to \((x'_i, y'_i)\) that \(x_0 \in \text{co } \{x''_0, x''_1, \ldots, x''_m\}\). Hence, there exist \(\alpha_0, \alpha_1, \ldots, \alpha_m \geq 0\) with \(\sum_{i=0}^m \alpha_i = 1\) such that \(x_0 = \sum_{i=0}^m \alpha_i x''_i\). Define \((\bar{x}, \bar{y}) = \sum_{i=0}^m \alpha_i (x''_i, y''_i)\). Then, obviously \(\bar{x} = x_0\), and also \((x_0, \bar{y}) \in \text{rel.int } Y(t_0)\) since \((x''_i, y''_i) \in \text{rel.int } Y(t_0)\) for all \(i\). Thus, for all
β with 0 < β ≤ 1, \( (x_0, \beta y + (1 - \beta)y_0) \in \text{rel.int } Y(t_0) \). Namely, there is a point \( y \) with \( (x_0, y) \in \text{rel.int } Y(t_0) \) such that \( y \) is arbitrarily close to \( y_0 \). This proves the lemma. \( \Box \).

The following theorem outlines a relation between the value function and the utility function, which was proved by Benveniste and Scheinkman [(1982), Thm. 1] under somewhat stronger assumptions.

**Theorem 5.1:** Let \( k \) an optimal path from time 0 satisfying Assumption 4. Then, under Assumptions 1, 2 (or 2'), and 3,

\[ \partial V(k(t), t) \subseteq -\partial u(k(t), \dot{k}(t), t) \]

for almost every \( t \in [0, \infty) \),

where symbol \( \partial u(k(t), \cdot, t) \) denotes the set of all supgradients for function \( u(k(t), \cdot, t) \) for each \( t \in [0, \infty) \).

**Proof:** First we should note [see, for example, Natanson (1955, p. 255)] that for almost every \( t \in [0, \infty) \),

\[
\lim_{\theta \to 0^+} \frac{1}{\theta} \int_t^{t+\theta} u(k(\tau), \dot{k}(\tau), \tau) d\tau = u(k(t), \dot{k}(t), t)
\]

and

\[
\lim_{\theta \to 0^+} \frac{k(t+\theta) - k(t)}{\theta} = \dot{k}(t).
\]

Also, since family \( F \) in Assumption 2 is countable, for almost every \( t \in [0, \infty) \),

\[
\lim_{\theta \to 0^+} \int_t^{t+\theta} u(f(\tau), \dot{f}(\tau), \tau) d\tau = u(f(t), \dot{f}(t), t)
\]

and

\[
\lim_{\theta \to 0^+} \frac{f(t+\theta) - f(t)}{\theta} = \dot{f}(t) \quad \text{for all } f \in F_i.
\]

Let \( t \in [0, \infty) \) be a point in time such that both (5.1) and (5.2) hold. Since \( k \) is an optimal path from time 0, the restriction of \( k \) on \([t, \infty)\) is an optimal path from time \( t \). In this case we can also apply the Support Principle Theorem. Let \( p \in \partial V(k(t), t) \). Then, there exists an absolutely continuous function \( q : [t, \infty) \to \mathbb{R}^m \) such that \( q(t) = p \) and \(-q(\tau), q(\tau) \in \partial u(k(\tau), \dot{k}(\tau), \tau) \) for almost every \( \tau \in [t, \infty) \).

Let \( y \in \mathbb{R}^m \) with \( (k(t), y) \in \text{rel.int } Y(t) \). Then, by Assumption 2 (i), we can pick \( f_0, f_1, \ldots, f_n \in F_i \) with \( n \leq 2m \) such that

\[
(k(t), y) \in \text{co } \{(f_0(t), \dot{f}_0(t)), (f_1(t), \dot{f}_1(t)), \ldots, (f_n(t), \dot{f}_n(t))\}.
\]

Hence, there exist \( a_0, a_1, \ldots, a_n \geq 0 \) with \( \Sigma_{i=1}^n a_i = 1 \) such that

\[
(k(t), y) = \Sigma_{i=1}^n a_i (f_i(t), \dot{f}_i(t)).
\]

Since \( f_i \in F_i \) for all \( i \), there exists \( t' > t \) such that \( f_i \) is defined on \([t, t']\) for all \( i \). Therefore we can define a feasible arc \( f : [t, t'] \to \mathbb{R}^m \) by \( f(\tau) = \Sigma_{i=1}^n a_i f_i(\tau) \). Since \(-q(\tau), q(\tau) \in \partial u(k(\tau), \dot{k}(\tau), \tau) \) for almost every \( \tau \in [t, \infty) \), for each \( f_i \) we have

\[
u(k(\tau), \dot{k}(\tau), \tau) + q(\tau) \cdot k(\tau) + q(\tau) \cdot \dot{k}(\tau)
\]

\[
\geq u(f_i(\tau), \dot{f}_i(\tau), \tau) + q(\tau) \cdot f_i(\tau) + q(\tau) \cdot \dot{f}_i(\tau)
\]

for almost every \( \tau \in [t, t'] \).
By integration, for each \( f_t \) and for all \( \theta \) with \( 0 < \theta \leq t' - t \) we have

\[
\int_t^{t+\theta} u(k(\tau), \dot{k}(\tau), \tau) d\tau + q(t+\theta)\cdot k(t+\theta) - q(t)\cdot k(t) \\
\geq \int_t^{t+\theta} u(f_t(\tau), \dot{f}_t(\tau), \tau) d\tau + q(t+\theta)\cdot f_t(t+\theta) - q(t)\cdot f_t(t),
\]

Therefore, for all \( \theta \) with \( 0 < \theta \leq t' - t \) we have

\[
\int_t^{t+\theta} u(k(\tau), \dot{k}(\tau), \tau) d\tau + q(t+\theta)\cdot k(t+\theta) - q(t)\cdot k(t) \\
= \int_t^{t+\theta} u(k(\tau), \dot{k}(\tau), \tau) d\tau + q(t+\theta)\cdot f(t+\theta) - q(t)\cdot f(t),
\]

i.e.,

\[
\frac{1}{\theta} \int_t^{t+\theta} u(k(\tau), \dot{k}(\tau), \tau) d\tau + q(t+\theta)\cdot \frac{k(t+\theta) - k(t)}{\theta} \\
\geq \int_t^{t+\theta} u(f_t(\tau), \dot{f}_t(\tau), \tau) d\tau + q(t+\theta)\cdot \frac{f(t+\theta) - f(t)}{\theta}.
\]

Thus, since \( q(t) = p \), we have in the limit as \( \theta \to 0 \)

\[
u(k(t), \dot{k}(t), t) + p\cdot \dot{k}(t) \equiv \int_t^{t+\theta} u(f_t(\tau), \dot{f}_t(\tau), \tau) d\tau + q(t+\theta)\cdot \frac{f(t+\theta) - f(t)}{\theta}.
\]

Recall that we can choose \( f_t \) so that \((f_t(t), \dot{f}_t(t))\) is arbitrarily close to \((k(t), y)\). Therefore, since any concave function is continuous at any point in the relative interior of its domain, the above inequality implies that \( u(k(t), \dot{k}(t), t) + p\cdot \dot{k}(t) \equiv u(k(t), y, t) + p\cdot y \).

Thus, we have proved that \( u(k(t), \dot{k}(t), t) + p\cdot \dot{k}(t) \equiv u(k(t), y, t) + p\cdot y \).

By the above theorem, we can see that the differentiability of the instantaneous utility function \( u \) implies that of the value function \( V \). In fact, in Theorem 5.1, if \( u(., ., t) \) is differentiable at \((k(t), \dot{k}(t))\) for almost every \( t \in [0, \infty) \), then \( \partial u(k(t), \dot{k}(t), t) = \partial V(k(t), t) \) for almost every \( t \in [0, \infty) \), i.e., \( V(., t) \) is differentiable at \( k(t) \) for almost every \( t \in [0, \infty) \). This argument is an extension of the results on differentiability of the value function in Benveniste and Scheinkman (1979).
REFERENCES


