EXPECTED-UTILITY-MAXIMIZING SEARCH

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I. Introduction

Since Stigler's [9] pioneering work, a basic assumption in search theory has been that job seekers maximize their expected returns from their searches, while price searchers minimize their expected costs for one unit of the searched for commodities. (McCall's [5] paper on sequential search rules follows this tradition. Lippman-McCall [3,5], and Rothschild [6] provide good surveys of the literature.) This assumption makes the analysis simple, but it does not pay enough attention to searchers' budget constraints. Moreover, such an analysis assumes that consumers purchase only one unit of the searched for commodities regardless of the prices they observe. This is inconsistent with basic economic theory.

But a few works (e.g., Kohn-Shavell [2], Harstad-Postlewaite [1]) studied search theory by assuming that searchers maximize their expected utility subject to their budget constraints (though Kohn-Shavell is vague with respect to searchers' budget constraints). In this paper we will build basic models of expected-utility-maximizing search in several different situations and compare a few of their properties with those of expected-cost-minimizing search. The situations we will consider are set up by assuming whether or not recall is allowed, whether or not the searcher has perfect information about the distribution of prices, and whether or not he decides to accept or reject prices before he revises his beliefs and recalculates the reservation price. We will compare such properties of the two approaches as the existence of the reservation price property and changes of the reservation price by costs of search, the number of times the searcher can observe prices, and increasing risk in the sense of Rothschild-Stiglitz [8]. It will turn out that expected-cost-minimizing search and expected-utility-maximizing search can have the same properties, but in most cases this is true only when some assumptions are made about the utility function of the searcher.

In the rest of this introduction we will give the basic assumptions common to all of the models in this paper. In Section II we will consider the situation in which the searcher knows the distribution of prices. In II A recall is not allowed, but in II B it is allowed. In Section III we will consider the situation in which the searcher has imperfect information about the distribution of prices. In III A he decides to accept or reject prices before he revises his beliefs and recalculates the reservation price, while in III B he does so after re-calculation. Section IV concludes this paper.

The basic assumption about the searcher is the same as that in Harstad-Postlewaite. He has the finite leisure endowment I and he exchanges leisure, which is the numeraire good, for a consumption good. He has to pay c units of leisure to elicit a price quotation. There are a finite number of prices:
where $0 < p_1 < p_{k+1}$.

When the searcher has decided to cease searching upon observing the price $p$, the quantity to purchase for this price is determined by the problem:

$$u(p, I) = \max_{x, L} w(x, L) \text{ subject to } px + L \leq I,$$

where $w$ is the direct concave utility function of the purchased good $x$ and leisure $L$, and $I$ is the "current income" or $I$ minus leisure foregone for search. We note that $u(p, I)$ is decreasing in $p$ and increasing in $I$. (In this paper we will not use such words as nonincreasing or nondecreasing. We say $u(p, I)$ is decreasing in $p$ if $p_1 \leq p_2$ implies $u(p_1, I) \leq u(p_2, I)$. Similarly we say $u(p, I)$ is increasing in $I$ if $I_1 \leq I_2$ implies $u(p, I_1) \leq u(p, I_2)$.)

We say that a search strategy exhibits the reservation price property if there is a critical price such that when the searcher observes a price lower than or equal to that price, he stops searching, and when he observes a price higher than that price, he continues searching. The critical price is called the reservation price. When recall is allowed, this definition must be slightly changed as will be shown in II B. Thus the reservation price property with the reservation price $\xi$ has the following simple form:

- stop searching if $p \leq \xi$,
- continue searching if $p > \xi$.

The reservation price property provides a very simple description of the searcher's behavior.

## II. Search with a Known Distribution of Prices

In this section we consider search problems in which the searcher knows the distribution of the prices and he has a finite time horizon. Lippman-McCall [3] showed that in this situation expected-cost-minimizing search either with or without recall exhibits the reservation price property, the reservation price is a decreasing function of the number of times he can observe prices, and that the reservation price with recall is not higher than that without recall. We will see if expected-utility-maximizing search has the same properties.

### A. Search without Recall

We consider first the case in which recall is not allowed. To calculate the optimal stopping rule let $V_n(p, I)$ be the maximal expected utility for the searcher who can search $n$ times without recall, has just observed price $p$, and has current income $I$ after the cost for this observation has been paid ($n$ includes the observation he has just made). Note that $n$ depends not only upon the searcher's current income but also upon such an exogenous condition as the availability of public transformation. Then by the definition of $u(p, I)$,

$$V_1(p, I) = u(p, I)$$

and

$$V_n(p, I) = \max [u(p, I), \sum_{i} V_{n-1}(p_i, I-c)f(p_i)] \quad (n \geq 2),$$

where $f(p_i)$ is the probability that he observes price $p_i$. The first term in the maximization
of (1) is the maximal utility he can attain by ceasing search, while the second term is the maximal expected utility he can attain by continuing search.

Thus the optimal stopping policy for \( n \geq 2 \) is to

stop searching if
\[
 u(p, I) \geq \sum_{t} V_{n-1}(p_t, I-c)f(p_t) 
\]
and

continue searching if
\[
 u(p, I) < \sum_{t} V_{n-1}(p_t, I-c)f(p_t). 
\]

If \( n = 1 \), the searcher must stop searching no matter which price he observes. Let
\[
 \xi^1 = p_m 
\]
and
\[
 \xi^n = \max \{ p \in P : u(p, I) \geq \sum_{t} V_{n-1}(p_t, I-c)f(p_t) \} \quad (n \geq 2). 
\]

We would like to prove that the above stopping rule, (2), exhibits the reservation price property and that the reservation price is \( \xi^n \). In other words we want to show that for \( n \geq 2 \) we have
\[
 \{ p \in P : u(p, I) \geq \sum_{t} V_{n-1}(p_t, I-c)f(p_t) \} = \{ p \in P : p \leq \xi^n \}. 
\]

If \( n = 1 \), it is obvious that the above stopping rule exhibits the reservation price property. For the proof we first show that \( \xi^n \) is well-defined, i.e., \( \xi^n \) exists for all \( n \). Then it can easily be shown that \( \xi^n \) is the reservation price. (Since some of the lemmas we will use are intuitively clear and can be proved relatively easily, their proofs will be omitted).

**Lemma 1**: \( V_n(p, I) \) is decreasing in \( p \) for each \( n \) and \( I \).

**Lemma 2**: \( V_n(p, I) \) is increasing in \( I \) for each \( n \) and \( p \).

**Lemma 3**: \( \xi^n \) is well-defined, i.e., \( \xi^n \) exists for all \( n \).

**Proof**: Since \( \xi^1 = p_m \), we want to prove that
\[
 \{ p \in P : u(p, I) \geq \sum_{t} V_{n-1}(p_t, I-c)f(p_t) \}
\]
is non-empty for all \( n \geq 2 \). To prove that the above set is non-empty we have only to show that
\[
 u(p_1, I) \geq \sum_{t} V_{n-1}(p_t, I-c)f(p_t)
\]
for \( n \geq 2 \). We show this by induction.

(i) \( n = 2 \).
\[
 \sum_{t} V_{1}(p_t, I-c)f(p_t) = \sum_{t} u(p_t, I-c)f(p_t) \leq u(p_1, I-c) \leq u(p_1, I).
\]

(ii) Suppose
\[
 u(p_1, I) \geq \sum_{t} V_{n-2}(p_t, I-c)f(p_t).
\]

Then
\[
 \sum_{t} V_{n-1}(p_t, I-c)f(p_t) \leq V_{n-1}(p_1, I-c) \leq V_{n-1}(p_1, I)
\]

\[
 = \max \{ u(p_1, I), \sum_{t} V_{n-2}(p_t, I-c)f(p_t) \} = u(p_1, I).
\]
The first inequality follows from Lemma 1, the second inequality from Lemma 2, and the last equality from the induction hypothesis. Q.E.D.

If we use Lemma 3 above, we can easily show that stopping rule (2) exhibits the reservation price property.

**Proposition 1:** The above stopping rule, (2), exhibits the reservation price property and the reservation price is \( \xi^n \) for all \( n \).

**Proof:** By Lemma 3 \( \xi^n \) exists for all \( n \). It is obvious that \( \xi^1 \) is the reservation price for \( n = 1 \). So we consider the case in which \( n \geq 2 \). We note that \( \sum_i V_{n-1}(p_i, I-c)f(p_i) \) does not depend on the price the searcher has just observed. Since \( u(p, I) \) is decreasing in \( p \) and \( \xi^n \) is defined in (3) as the highest price such that

\[
u(p, I) \geq \sum_i V_{n-1}(p_i, I-c)f(p_i),
\]

it is obvious that \( p \leq \xi^n \) if and only if \( u(p, I) \geq \sum_i V_{n-1}(p_i, I-c)f(p_i) \), i.e., (4) holds. Q.E.D.

We now consider how search cost, the number of times the searcher can observe prices, and increasing risk affect the reservation price.

**Lemma 4:** \( V_n(p, I) \geq V_{n-1}(p, I) \) for all \( n \geq 2, p \), and \( I \).

**Proposition 2:** If search cost increases, the reservation price increases.

**Proof:** Since \( u(p, I) \) is decreasing in \( p \), we have only to show that the second term in the maximization of (1) decreases as \( c \) increases. But this is obvious by Lemma 2. (\( \xi^1 \) remains unchanged.) Q.E.D.

**Proposition 3:** The reservation price is a decreasing function of the number of times the searcher can observe prices (\( \xi^n \leq \xi^{n-1} \)), i.e., the more a searcher can observe, the lower the reservation price.

**Proof:** Because \( \xi^1 \) is the highest price, it is larger than or equal to any reservation price for \( n \geq 2 \). Since \( u(p, I) \) is decreasing in \( p \), it suffices to show that

\[
\sum_i V_n(p_i, I-c)f(p_i) \geq \sum_i V_{n-1}(p_i, I-c)f(p_i).
\]

But this inequality follows from Lemma 4. Q.E.D.

**Proposition 4:** Assume that \( u(p, I) \) is a convex function of \( p \) for each \( I \). Then increasing risk in the sense of Rothschild-Stiglitz lowers the reservation price.

**Proof:** Suppose \( f' \) is riskier than \( f \). Since \( u(p, I) \) is decreasing in \( p \), we have only to show

\[
A = \sum_i \tilde{V}_n(p_i, I)f'(p_i) - \sum_i V_n(p_i, I)f(p_i) \geq 0
\]

for all \( n \) and \( I \), where \( \tilde{V}_n \) is the maximal expected utility associated with \( f'(\xi^1 \) remains unchanged).

It will suffice to prove the proposition when \( f \) and \( f' \) differ by a single mean preserving spread, so we shall consider four prices \( p_1' < p_2' < p_3' < p_4' \) such that \( f'(p_i) = f(p_i) \) for \( i \in \{1', 2', 3', 4'\} \),

\[
\tilde{f}(p_1') - f(p_1') = f(p_2') - f(p_2') = Q > 0,
\]

and...
\[ f(p_5) - f(p_3') = f(p_4') - f(p_4') = R > 0, \]  

(6)

where

\[ Q(p_5' - p_5) + R(p_3' - p_4') = 0. \]  

(7)

We use mathematical induction to prove \( A \geq 0. \)

(i) \( n = 1. \)

\[ A = \sum u(p_i, I)f(p_i) - \sum u(p_i, I)f(p_i). \]

Using (5) and (6), we have

\[ A = Q[p_5' - p_5] \left[ \frac{u(p_i', I) - u(p_2', I)}{p_2' - p_1'} - \frac{u(p_2', I) - u(p_4', I)}{p_4' - p_3'} \right] \geq 0. \]

The second equality follows from (7) and the inequality from the convexity.

(ii) Suppose

\[ \sum V_{n-1}(p_i, I) f'(p_i) - \sum V_{n-1}(p_i, I) f(p_i) \geq 0. \]

Then

\[ A = \sum \left[ \max \left\{ u(p_i, I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i) \]

\[ - \sum \left[ \max \left\{ u(p_i, I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i) \]

\[ \geq \sum \left[ \max \left\{ u(p_i, I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i) \]

\[ - \sum \left[ \max \left\{ u(p_i, I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i) \]

\[ = \sum_{i=1}^n \left[ \max \left\{ u(p_i', I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i) \]

\[ - \sum_{i=1}^n \left[ \max \left\{ u(p_i', I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i). \]

The inequality follows from the induction hypothesis.

Let \( \xi^* \) be the reservation price for the stopping rule that can be derived from

\[ V_n(p, I) = \max \left\{ u(p, I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\}. \]

We know from Proposition 1 that \( \xi^* \) exists. Define

\[ v_n = \sum_j V_{n-1}(p_j, I - c)f(p_j). \]

We consider five cases.

(a) \( \xi^* \geq p_4' \).

\[ A \geq \sum_{i=1}^n u(p_i', I)f(p_i') - \sum_{i=1}^n u(p_i', I)f(p_i') \geq 0. \]

The last inequality is due to the proof in (i).

(b) \( p_4' > \xi^* \geq p_3' \).

\[ A \geq \sum_{i=1}^n u(p_i', I)f(p_i') - \sum_{i=1}^n u(p_i', I)f(p_i') + R_v \]

\[ = \sum_{i=1}^n \left[ \max \left\{ u(p_i', I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i') \]

\[ - \sum_{i=1}^n \left[ \max \left\{ u(p_i', I), \sum_j V_{n-1}(p_j, I - c)f(p_j) \right\} \right] f(p_i'). \]
\[
\begin{align*}
&\geq \sum_{i=1}^{3} u(p_{i}, I) f(p_{i}) - \sum_{i=1}^{3} u(p_{i}, I) f(p_{i}) + Ru(p_{4}, I) \\
&= \sum_{i=1}^{4} u(p_{i}, I) f(p_{i}) - \sum_{i=1}^{4} u(p_{i}, I) f(p_{i}) \\
&\geq 0.
\end{align*}
\]

(c) \( p_{0} > \xi \geq p_{1} \).
\[
A \geq \sum_{i=1}^{3} u(p_{i}, I) f(p_{i}) - \sum_{i=1}^{3} u(p_{i}, I) f(p_{i}) - R(\nu_{n} - \nu_{n}) \\
= Q[u(p_{1}, I) - u(p_{2}, I)] \\
\geq 0.
\]

(d) \( p_{p} > \xi \geq p_{1} \).
\[
A \geq Q[u(p_{1}, I) - \nu_{n}] - R(\nu_{n} - \nu_{n}) \geq Q(\nu_{n} - \nu_{n}) = 0.
\]

(e) \( p_{p} > \xi \).
\[
A \geq Q(\nu_{n} - \nu_{n}) - R(\nu_{n} - \nu_{n}) = 0.
\]

Q.E.D.

We have used convexity as a sufficient condition to guarantee that increasing risk lowers the reservation price when the searcher maximizes his expected utility. The meaning of this assumption is that decrease in utility due to additional increase in the price is larger when price is lower. If \( u(p, I) \) is twice-differentiable with respect to \( p \) for all \( I \), the assumption of convexity is clearly equivalent to the condition that the second partial derivative with respect to \( p \) is nonnegative. Using this condition, we can easily show that \( w(x, L) = x^\alpha L^\beta \), where \( \alpha > 0, \beta \geq 0, \) and \( w(x, L) = \log x \) satisfy the condition of convexity.

If the condition of convexity is not assumed in the above proposition, we cannot necessarily say that increasing risk lowers the reservation price. To see this consider the following situation.

\[
w(x, L) = \begin{cases} x + 40, & \text{if } x \geq 2/3, \\ 122x - 122/3, & \text{if } 0 \leq x \leq 2/3, \end{cases}
\]

\( P = \{0.1, 1.4, 1.6, 2.9\} \),
\[
\{f(p_{i})\}_{i=1}^{4} = \{0, 1/2, 1/2, 0\},
\]
\[
\{f'(p_{i})\}_{i=1}^{4} = \{1/2, 0, 0, 1/2\}.
\]

This situation satisfies condition (5) through (7). Suppose \( n = 2, I = 1.01, \) and that \( c = 0.01 \). Clearly we have
\[
u(p, I) = \begin{cases} 1/p + 40, & \text{if } p \leq 3I/2, \\ 122I/p - 122/3, & \text{if } p \geq 3I/2. \end{cases}
\]

We can easily show that this indirect utility function does not satisfy the condition of convexity.

By simple calculation we have
\[
38.14 < \sum_{i=1}^{4} u(p_{i}, 1)f(p_{i}) < 38.15, \quad 25.70 < \sum_{i=1}^{4} u(p_{i}, 1)f(p_{i}) < 25.71,
\]
\[
u(1.4, 1.01) > 40.72, \quad 36.35 < \nu(1.6, 1.01) < 36.36, \quad \nu(2.9, 1.01) < 1.82.
\]

Let \( \xi \) and \( \xi \) be the reservation prices corresponding to \( f \) and \( f' \) respectively. From the above information we have \( \xi = 1.4 \) and \( \xi = 1.6 \). So in this case increasing risk raises the reserva-
The reason is that the increase in utility due to the probability change in the lowest two prices is smaller than the decrease in utility due to the probability change in the highest two prices and this causes the second term in the maximization of (1) to decrease when the search becomes riskier. But when the condition of convexity is satisfied, the former is always larger than or equal to the latter.

B. Search with Recall

We now consider the case in which recall is allowed. When recall is allowed in search with a finite time horizon, as Proposition 8 below will show, the reservation price is generally lower than that when recall is not allowed. In expected-cost-minimizing search with an infinite time horizon possibility of recall does not change the situation at all. If a searcher can observe prices only a finite number of times, which is obviously the case when income is finite, possibility of recall dramatically changes the situation. Since recall is very common in most price searches, it is important to study how possibility of recall changes the situation.

We note that possibility of recall changes the searcher's actual probability distribution of the prices after a search. To see this let

\[ \{f(p_1), \ldots, f(p), \ldots, f(p_m)\} \]  

be the original distribution of the prices. Suppose the researcher has observed price \( p \) in his first search. Then his actual probability distribution of the prices in his second search will be

\[ \{f(p_1), \ldots, f(p) + (1 - F(p)), 0, \ldots, 0\}, \]  

where \( F(p) \) is the probability that he observes a price lower than or equal to \( p \).

To calculate the optimal stopping rule let \( \bar{V}_n(p, I) \) be the maximal expected utility for the searcher who can search \( n \) times now with recall, has currently price \( p \) as his best price (i.e., \( p \) is the lowest of all the prices he has observed so far including the price he has just observed), and has current income \( I \) after the cost for the observation he has just made has been paid (\( n \) includes the observation he has just made). Then

\[ \bar{V}_1(p, I) = u(p, I) \]

and

\[ \bar{V}_n(p, I) = \max \{u(p, I), \bar{V}_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_{i < p}} \bar{V}_{n-1}(p_i, I - c)f(p_i)\} \quad (n \geq 2). \]

We note that we have used the idea in the above paragraph. The first term in the maximization of (10) is the maximal utility he can attain by ceasing search, while the second term is the maximal expected utility he can attain by continuing search. We define

\[ z_n(p, I) = \bar{V}_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_{i < p}} \bar{V}_{n-1}(p_i, I - c)f(p_i). \]

Thus the optimal stopping policy for \( n \geq 2 \) is to

\[ \text{stop searching if } u(p, I) \geq z_n(p, I) \]

and

\[ \text{continue searching if } u(p, I) < z_n(p, I). \]

If \( n = 1 \), the searcher must stop searching no matter which price he has observed.

When recall is allowed, we have to change slightly the definition of the reservation
price property discussed in the introduction. We now say that this search strategy exhibits the reservation price property if there is a critical price such that when the searcher's best price is lower than or equal to that price, he stops searching, and when the searcher's best price is higher than that price, he continues searching. This critical price is called the reservation price in our present model.

Let
\[ \xi^1 = p_m \]
and
\[ \xi^n = \max \{ p \in P : u(p, I) \geq z_n(p, I) \} \quad (n \geq 2). \]

We would like to prove that the above stopping rule, (11), exhibits the reservation price property and that the reservation price is \( \xi^n \). In other words we want to show that for \( n \geq 2 \)
\[ \{ p \in P : u(p, I) \geq z_n(p, I) \} = \{ p \in P : p \leq \xi^n \}. \]

If \( n = 1 \), it is obvious that the above stopping rule exhibits the reservation price property.

Unfortunately this stopping rule does not in general exhibit the reservation price property. The main reason for this is that the second term in the maximization of (10) contains \( p \). We have to assume a certain condition about the searcher's utility function to guarantee the existence of the reservation price property. For the proof we first obtain a very useful relation in Lemma 5 and next show in Lemma 6 that \( \xi^n \) is well-defined. In Proposition 5 we prove that the above stopping rule exhibits the reservation price property when a certain condition is assumed.

**Lemma 5:** For an arbitrary pair of adjacent prices \( p^1 \) and \( p^2 \) such that \( p^1 < p^2 \), i.e., for an arbitrary pair of prices \( p^1 \) and \( p^2 \) such that \( p^1 = p_i \) implies \( p^2 = p_{i+1} \)
\[ z_n(p^1, I) - z_n(p^2, I) = \{ V_n-1(p_i, I - c) - V_n-1(p_{i+1}, I - c) \} (1 - F(p^1)). \]

**Proof:**
\[ z_n(p^1, I) - z_n(p^2, I) = \sum_{p_i \leq p^1} V_n-1(p_i, I - c) f(p_i) - \sum_{p_i \leq p^1} V_n-1(p_{i+1}, I - c) f(p_{i+1}) \]
\[ = \{ V_n-1(p^1, I - c) - V_n-1(p^2, I - c) \} (1 - F(p^1)). \]

**Lemma 6:** \( \xi^n \) is well defined, i.e., \( \xi^n \) exists for all \( n \).

**Proof:** Since \( \xi^1 = p_m \), we want to prove that
\[ \{ p \in P : u(p, I) \geq z_n(p, I) \} \]
is non-empty for all \( n \geq 2 \). Define
\[ M_n(p) = u(p, I) - z_n(p, I) \quad (n \geq 2). \]

To prove that the above set is non-empty, we have only to show that \( M_n(p_i) \geq 0 \) for \( n \geq 2 \). We use induction for the proof.

(i) \( n = 2 \).
\[ M_2(p_i) = u(p_i, I) - u(p_i, I - c)(1 - F(p_i)) - \sum_{p_i \leq p_i} u(p_i, I - c) f(p_i) \]
\[ \geq u(p_i, I) - u(p_i, I)(1 - f(p_i)) - u(p_i, I) f(p_i) \]
\[ = 0. \]
(ii) Suppose $M_{n-1}(p_1) \geq 0$, i.e.,

$$u(p_1, I) \geq \tilde{V}_{n-2}(p_1, I - c).$$

Then

$$M_n(p_1) = u(p_1, I) - \tilde{V}_{n-2}(p_1, I - c)$$

$$= u(p_1, I) - \max \{u(p_1, I - c), \tilde{V}_{n-2}(p_1, I - 2c)\}$$

$$= u(p_1, I) - u(p_1, I - c) \text{ by the induction hypothesis}$$

$$\geq 0.$$

$Q.E.D.$

**Proposition 5**: Suppose

$$u(p_j, I) - u(p_k, I) \geq u(p_j, I - c) - u(p_k, I - c)$$

(*)

for all $p_j < p_k$ and $c < I$. Then the above stopping rule, (11), exhibits the reservation price property and the reservation price is $\xi_n$ for all $n$.

**Proof**: By Lemma 6 $\xi_n$ exists for all $n$. It is obvious that $\xi_1$ is the reservation price for $n = 1$. So we consider the case in which $n \geq 2$. Since $\xi_n$ is defined in (12) as the highest price such that $u(p, I) \geq z_n(p, I)$, we have only to show that for each $n \geq 2$ and $I$

$$M_n(p) \equiv u(p, I) - z_n(p, I)$$

is decreasing in $p$. We show this by induction. Choose an arbitrary pair of adjacent prices $p^1$ and $p^2$ such that $p^1 < p^2$.

(i) $n = 2$.

$$M_2(p^1) - M_2(p^2)$$

$$\geq 0$$

by (*)

(ii) Suppose $M_{n-1}(p^1) - M_{n-1}(p^2) \geq 0$, i.e.,

$$u(p_1, I) - u(p_2, I) - (\tilde{V}_{n-2}(p_1, I - c) - \tilde{V}_{n-2}(p_2, I - c)) (1 - F(p')) \geq 0.$$

We have again used Lemma 5 to get the second inequality.

Let $G = M_n(p_1) - M_n(p_2)$. Then

$$G = u(p_1, I) - u(p_2, I) - (\tilde{V}_{n-2}(p_1, I - c) - \tilde{V}_{n-2}(p_2, I - c)) (1 - F(p'))$$

Let $H = \tilde{V}_{n-2}(p_1, I - c) - \tilde{V}_{n-2}(p_2, I - c)$. Then by (10)

$$H = \max \{A^1, B^1\} - \max \{A^2, B^2\},$$

where

$$A^i = u(p_i, I - c), \quad j = 1, 2$$

and

$$B^i = \tilde{V}_{n-2}(p_i, I - 2c)(1 - F(p')) + \sum_{i \leq p'} \tilde{V}_{n-2}(p_i, I - 2c)f(p_i), \quad j = 1, 2.$$

We consider four cases.

(a) $A^1 \geq B^1$ and $A^2 \geq B^2$.

$$G = u(p_1, I) - u(p_2, I) - (A^1 - B^1) (1 - F(p')) \geq 0 \text{ by (*)}.$$

(b) $A^1 \geq B^1$ and $A^2 < B^2$.

$$G = u(p_1, I) - u(p_2, I) - (A^1 - A^2) (1 - F(p'))$$

$$\geq 0 \text{ by (*)}.$$
(c) $A^1 < B^1$ and $A^2 \geq B^2$

$G = (u(p^1, I) - u(p^2, I)) - (B^1 - A^2)(1 - F(p^3))$
$\geq (u(p^3, I) - u(p^2, I)) - (B^1 - B^2)(1 - F(p^3))$
$= (u(p^1, I) - u(p^2, I)) - \{\bar{V}_{n-2}(p^1, I-2c) - \bar{V}_{n-2}(p^2, I-2c)\}(1 - F(p^3))^2$
$\geq (u(p^1, I) - u(p^2, I)) - (u(p^1, I-c) - u(p^2, I-c))(1 - F(p^3))$
$\geq 0,$

where the second inequality is due to the induction hypothesis and the third to (*).

(d) $A^1 < B^1$ and $A^2 < B^2$.

$G = (u(p^1, I) - u(p^2, I)) - (B^1 - B^2)(1 - F(p^3)) \geq 0$ by (c).

Q.E.D.

Thus (*) was used as a sufficient condition to guarantee the existence of the reservation price property. It was originally used in Harstad-Postlewaite [1] to prove that expected-utility-maximizing search without recall and with imperfect information about the distribution of prices exhibits the reservation price property if (*) is assumed. But even if a searcher has perfect information about the distribution, we need the condition to prove the existence of the reservation price property when recall is allowed.

The meaning of condition (*) is that the decrease in utility due to an increase in price is large when income is large. Rewriting (*) as

$u(p_j, I) - u(p_j, I - c) \geq u(p_k, I) - u(p_k, I - c),$

we can get another interpretation of (*): the decrease in utility due to a decrease in income is large when price is low. Because we used condition (*) in the proof for only adjacent prices, one may think that we used a condition weaker than (*). But it is quite easy to show that the two are equivalent.

If $u(p, I)$ is differentiable, we can get a very simple condition equivalent to (*). The remark below gives this condition.

Remark: Suppose

$$\frac{\partial^2 u(p, I)}{\partial p \partial I}$$

exists for all $p$ and $I$. Then

$u(p_j, I) - u(p_j, I - c) \geq u(p_k, I) - u(p_k, I - c)$

for all $p_j < p_k$ and $c < I$ if and only if

$$\frac{\partial^2 u(p, I)}{\partial p \partial I} \leq 0$$

for all $p$ and $I$.

Since the proof is straightforward, it is omitted.

When $u(p, I)$ is differentiable, the above remark enables us to see easily if a given utility function satisfies condition (*). For example, we can easily show that $w(x, L) = \log x$ satisfies (*). Harstad-Postlewaite [1] showed (Proposition 3) that utility functions homogeneous of degrees $\alpha \geq 0$ also satisfy condition (*). Thus we can expect the class of utility functions which satisfy condition (*) is relatively large.

If condition (*) does not hold, we may not have the reservation price property. As a counter-example consider the following situation.

$$w(x, L) = \begin{cases} x + 40, & \text{if } x \geq 2/3, \\ 122x - 122/3, & \text{if } 0 \leq x \leq 2/3, \end{cases}$$
\[ P = \{1/320, 1/4, 3\} \]
\[ \{f(p_i)\}_{i=1}^{n} = \{1/20, 1/20, 18/20\} \]

Search cost, \( c \), is 1, \( n \) is 2, and income after the first search is 2.

It can easily be shown that this utility function does not satisfy condition (\(*\)). Consider the following case which will be used later again.

\[ u(1/4, 2) - u(3, 2) = 7 + 1/3, \]
\[ u(1/4, 1) - u(3, 1) = 44. \]

Thus we have
\[ u(1/4, 2) - u(3, 2) < u(1/4, 1) - u(3, 1), \]
which contradicts (\(*\)).

First suppose that the searcher has observed \( p = 3 \) in his first search. If he stops searching, he can attain \( w(2/3, 0) = 40 + 2/3 \). If he continues searching, then his expected utility is
\[ \frac{1}{20} w(320, 0) + \frac{1}{20} w(4, 0) + \frac{18}{20} w(1/3, 0) = 20.2. \]

So he stops searching. Now suppose that he has observed \( p = 1/4 \) in his first search. Then if he stops searching, he can attain \( w(8, 0) = 48 \). But if he continues searching, his expected utility is
\[ \frac{1}{20} w(320, 0) + \frac{19}{20} w(4, 0) = 59.8. \]

So he continues searching. Since he stops searching when he observes a higher price and continues searching when he observes a lower price, we do not have the reservation price property here.

The reason is given by the fact that the above utility function does not satisfy condition (\(*\)). Here we have
\[ u(p_j, I) - u(p_k, I) < u(p_j, I - c) - u(p_k, I - c) \]
for some \( p_j < p_k \) and \( c < I \). Rewriting (17), we have
\[ u(p_j, I) - u(p_j, I - c) < u(p_k, I) - u(p_k, I - c) \]
for some \( p_j < p_k \) and \( c < I \). If we set \( p_j = 1/4 \) and \( p_k = 3 \), (18) is true by (14).

(18) implies that cost of search in terms of foregone utility when he observes a low price in his first and second searches is smaller than that when he observes a high price in his first and second searches. Though (15) is not exactly the same as \( u(p_k, I - c) \) or \( u(3, 1) \), it is close since the probability that he observes \( p = 3 \) is very high. Similarly (16) is close to \( u(p_j, I - c) \) or \( u(1/4, 1) \). Thus difference in costs in terms of expected foregone utility explains this counter-example.

We now would like to study a few properties of the reservation price. To do so we need the following lemmas.

**Lemma 7:** \( \bar{V}_{n+1}(p, I) \geq \bar{V}_n(p, I) \) for all \( n, p, \) and \( I \).

**Lemma 8:** \( \bar{V}_n(p, I) \) is decreasing in \( p \).

**Lemma 9:** \( \bar{V}_n(p, I) \) is increasing in \( I \).

**Proposition 6:** Assume (\(*\)) in Proposition 5. Then if search cost increases, the reservation
price increases.

**Proof:** Since $\xi^1$ remains unchanged, we have only to show that $u(p, I) - z_n(p, I)$, which is a decreasing function of $p$ by (3), increases as $c$ increases for $n \geq 2$. If $c$ increases, $z_n(p, I)$ decreases by Lemma 9. Then obviously $u(p, I) - z_n(p, I)$ increases. Q.E.D.

**Proposition 7:** Assume (5) in Proposition 5. Then the reservation price is a decreasing function of $n (\xi^n \leq \xi^{n-1})$.

**Proof:** It is obvious that $\xi^1$ is larger than or equal to any reservation price. By Lemma 7 $z_n(p, I)$ is increasing in $n$. Thus $u(p, I) - z_n(p, I)$ is decreasing in $n$, which implies the reservation price is decreasing in $n$. Q.E.D.

**Lemma 10:** $V_n(p, I) \geq V_n(p, I)$ for all $n, p, I$.

**Proof:** We show this by induction.

(i) $n = 1$.

$V_n(p, I) - V_1(p, I) = u(p, I) - u(p, I) = 0$.

(ii) Suppose $V_{n-1}(p, I) \geq V_1(p, I)$. Then

$V_n(p, I) - V_{n-1}(p, I) = \max \{u(p, I), V_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_i \leq p} V_{n-1}(p_i, I - c)f(p_i)\} - \max \{u(p, I), V_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_i \leq p} V_{n-1}(p_i, I - c)f(p_i)\}$

$\geq \max \{u(p, I), V_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_i \leq p} V_{n-1}(p_i, I - c)f(p_i)\} - \max \{u(p, I), V_{n-1}(p, I - c)(1 - F(p)) + \sum_{p_i \leq p} V_{n-1}(p_i, I - c)f(p_i)\}$

$\geq 0$.

The first inequality follows from Lemma 1 and the last from the induction hypothesis. Q.E.D.

**Proposition 8:** Assume (5) in Proposition 5. Then the reservation price with recall is not higher than that without recall, i.e., $\xi^m \leq \xi^n$.

**Proof:** Since $\xi^1 = \xi^2 = p_m$, the above statement is true for $n = 1$. So we consider the case in which $n \geq 2$.

$u(\xi, I) \geq z_n(\xi^m, I) = V_{n-1}(\xi^m, I - c)(1 - F(\xi^m)) + \sum_{p_i \leq \xi^m} V_{n-1}(p_i, I - c)f(p_i)$

$\geq \sum_i V_{n-1}(p_i, I - c)f(p_i) \geq \sum_i V_{n-1}(p_i, I - c)f(p_i)$

where the second inequality follows from Lemma 8 and the last from Lemma 10. But by (3) $\xi^m$ is the highest price such that $u(p, I) \geq \sum V_{n-1}(p_i, I - c)f(p_i)$. Therefore we have $\xi^m \leq \xi^n$. Q.E.D.

Though Proposition 8 does not guarantee that the reservation price with recall is strictly lower than that without recall, in many cases strict inequality holds. To see this consider the following example. The direct utility function is $w(x, L) = x L$, search cost $c$ is 0.1, $n$ is 2, and income after the first search is 1.1. The price vector is given by

$p = \{3, 4, 5, 6\}$
and the probability distribution of the prices is given by
\[ \{f(p_i)\}_{i=1}^{4} = \{0.3, 0.1, 0.2, 0.4\} . \]

Then it is easy to see \( u(p, I) = \frac{1.21}{4p} \). Since \( u(p, 1.1) = \frac{1.21}{4p} \), \( u(5, 1.1) > 0.06, 0.058 > \sum_{i=1}^{4} u(p_i, 1)f(p_i) > 0.057, \) and \( u(6, 1.1) < 0.051, \) \( \xi^2 \) is equal to 5. On the other hand, the function defined by
\[
M_2(p) = \frac{1.21}{4p} - \frac{1}{4p}(1 - F(p)) - \sum_{p \leq p_i} \frac{f(p_i)}{4p_i}
\]
is decreasing in \( p \), since \( u(p, I) \) satisfies condition (*) in Proposition 5, and
\[
M_2(5) = -0.00075 < 0, \quad M_2(4) = 0.006875 > 0.
\]
Therefore, we have \( \xi^2 = 4 \), which is strictly lower than \( \xi^2 \).

### III. Search with an Unknown Distribution of Prices

In this section we consider search problems in which a searcher has imperfect information about the distribution of the prices. In this situation observation of a price conveys information about the true distribution of the prices. The searcher updates his beliefs with this information in Bayesian fashion and recalculates the reservation price. We can build completely different models by assuming he accepts or rejects prices either before or after he recalculates the reservation price.

To see the difference consider the following example, which can be found both in Rothschild [7] and Kohn-Shavell [2]. There are three prices,
\[ p = \{1, 2, 3\} \]
An expected-cost-minimizing searcher has sufficiently large income and cost of search is very small, say 0.01. His prior beliefs admit only two possible price distributions
\[ \{0, 0, 1\}, \quad \{0.99, 0.01, 0\} . \]

Suppose he accepts or rejects prices before he recalculates the reservation price. If he observes \( p = 3 \), then he understands that the true distribution is the former, and stops searching. But if he observes \( p = 2 \), he understands that the true distribution is the latter, and continues searching because the probability that he finds \( p = 1 \), is very high and search cost is very small. If he observes a higher price, he stops searching and if he observes a lower price, he continues searching. So we do not have the reservation price property here. But if he recalculates the reservation price after he has observed prices and decides to accept or reject them, then we may have the reservation price property (the reservation price in the former distribution must be 3 and that in the latter must be 1).

In Rothschild [7] the searcher accepts or rejects prices before he recalculates the reservation price, while in Kohn-Shavell [2] and Lippman-McCall [3] he accepts or rejects prices after he recalculates the reservation price.

**A. Prices Accepted or Rejected before the Reservation Price Is Recalculated**

Rothschild [7] showed that if an expected-cost-minimizing searcher has an infinite time
horizon and Dirichlet priors, then the optimal search without recall has the reservation price property even if he accepts or rejects prices before he recalculates the reservation price. He also showed that increasing risk lowers the reservation price in this situation. But his proof is incomplete. Harstad-Postlewaite [1] showed that if an expected-utility-maximizing searcher in this situation has a utility function with the property

\[ u(p_j, I) - u(p_k, I) \geq u(p_j, I - c) - u(p_k, I - c) \tag{\ast} \]

for all \( p_j < p_k \) and \( c < I \), then the optimal search exhibits the reservation price property.

Here we will see that if an expected-utility-maximizing searcher has a convex indirect utility function, then under the same assumptions as above increasing risk lowers the reservation price. Since the proof is very long, we will see only the outline of the model and the proposition we want to prove. The proof is written in the appendix.

Let \( N \equiv (N_1, \ldots, N_m) \) be the number of times each price has been observed and let \( \rho = \Sigma N_i, \mu_i = \rho N_i \) \((i = 1, \ldots, m)\), and \( \mu = (\mu_1, \ldots, \mu_m) \). Denote by \( \lambda(\mu, \rho) \) the current prior over \( P \) for a searcher with history \((\mu, \rho)\). Rothschild [8] showed that if a searcher has a Dirichlet prior, then \( \lambda(\mu, \rho) = \mu_i \) for all \((\mu, \rho)\).

Let

\[ V_1(\mu, \rho, I) = \sum_i \lambda(\mu, \rho) u(p_i, I). \]

\( V_1(\mu, \rho, I) \) is the expected utility for the searcher allowed to search only once, assuming history \((\mu, \rho)\), and holding current income \( I \) after the cost of the search has been paid.

Let

\[ V_n(\mu, \rho, I) = \sum_i \lambda(\mu, \rho) \max \{u(p_i, I), V_{n-1}[h_i(\mu, \rho), I - c]\}, \tag{19} \]

where

\[ h_i(\mu, \rho) = \left[ \frac{\mu_1}{1 + \rho}, \ldots, \frac{\mu_{i-1}}{1 + \rho}, \frac{\mu_i + \rho}{1 + \rho}, \frac{\mu_{i+1}}{1 + \rho}, \ldots, \frac{\mu_m}{1 + \rho}, \frac{\rho}{1 + \rho} \right]. \]

\( V_n(\mu, \rho, I) \) is the maximal expected utility for the searcher who can search \( n \) times, has prior experience \((\mu, \rho)\), and income \( I \) after the cost of the first search has been paid.

Since

\[ V_{n-1}(\mu, \rho, I) \leq V_n(\mu, \rho, I) \leq u(p_i, I), \]

\[ V(\mu, \rho, I) = \lim_{n \to \infty} V_n(\mu, \rho, I) \]

is finite. Taking the limit of (19), we have

\[ V(\mu, \rho, I) = \sum_i \lambda(\mu, \rho) \max \{u(p_i, I), V[h_i(\mu, \rho), I - c]\}. \]

Thus the optimal stopping policy is to cease search, upon eliciting price \( p_i \) with experience \((\mu, \rho)\) and income \( I \), if \( u(p_i, I) \geq V[h_i(\mu, \rho), I - c] \), otherwise to search again.

The reservation price property here is that

\[ V[h_k(\mu, \rho), I - c] \leq u(p_k, I) \leq u(p_j, I) \]

implies

\[ V[h_j(\mu, \rho), I - c] \leq u(p_j, I). \]

Harstad-Postlewaite [1] showed that if condition (\ast) above is satisfied in this situation, the above stopping policy exhibits the reservation price property. It is obvious from the above
statement of the reservation price property that the searcher with history \((\mu, \rho)\) sets the reservation price for his next search before he actually observes a price.

Rothschild [7] proved that the optimal strategy for an expected-cost-minimizing searcher implies a finite number of searches. His argument also holds in the present model, though finite income trivially implies finite search. This fact shows that to prove propositions about \(V(\mu, \rho, I)\) it is sufficient to prove them for all \(V_n(\mu, \rho, I)\).

**Proposition 9:** Suppose \(u(p, I)\) is a convex function of \(p\) for all \(I\) and that
\[
u(p_i, I) - u(p, I) \geq u(p, I - c) - u(p, I - c)
\]
for all \(p_i < p\) and \(c < I\). If the searcher’s prior is Dirichlet, then increasing risk in the sense of Rothschild-Stiglitz lowers the reservation price.

B. Prices Accepted or Rejected after the Reservation Price Is Recalculated

Lippman-McCall [3] showed that an expected-cost-minimizing search without recall and with a finite time horizon exhibits the reservation price property if prices are accepted or rejected after the searcher has recalculated the reservation price. We extend his model to an expected-utility-maximizing search.

The expected-utility-maximizing searcher is assumed to have imperfect information about the \(k\) parameters, \(\gamma = (\gamma_1, \ldots, \gamma_k)\), of the true price distribution \(f(p)\). He has a subjective distribution \(h(\gamma \mid \theta)\) over the unknown parameters, where \(\theta\) is a vector representing the parameters of his subjective distribution. This subjective distribution shows all the information he has about moments of the true price distribution. Each time he observes a price, \(\theta\) is revised in Bayesian fashion and a new value is calculated by
\[
\theta' = T(\theta, p),
\]
where \(T\) is a transformation illustrating the dependence of \(\theta'\) on \(\theta\) and price \(p\). After each observation his subjective distribution is revised, and he decides either to accept the price or to continue searching.

Let \(V_n(p, \theta, I)\) be the searcher’s maximal expected utility when he can observe \(n\) times without recall, has just observed \(p, \theta\) represents the parameters of his subjective distribution of \(\gamma \mid \theta\) \((\theta\) includes any information contained in price \(p\)), and he has current income \(I\) after the cost of the search for \(p\) has been paid (as before \(n\) includes the observation he has just made). Then by the definition of \(u(p, I)\)
\[
V_1(p, \theta, I) = u(p, I)
\]
and
\[
V_n(p, \theta, I) = \max \{u(p, I), \sum_j \sum_{i=1}^{n-1} V_{n-i}(p, T(\theta, p_i), I - c)f(p_i \mid \gamma_i)h(\gamma_i \mid \theta)\}.
\]
The first term in the maximization of (20) is the maximal utility the searcher can attain by stopping search, while the second term is the maximal expected utility he can attain by continuing search. It is obvious that he uses the information contained in \(p\) to decide whether to stop or continue searching, because \(\theta\) in the second term includes the information.

Denote by \(z_n(\theta, I)\) the second term in the maximization. Then as before the optimal stopping policy for \(n \geq 2\) is to

stop searching if \(u(p, I) \geq z_n(\theta, I)\)

and

\[
V_n(p, \theta, I) \geq z_n(\theta, I)
\]

(21)
continue searching if $u(p, I) < z_n(\theta, I)$.

If $n = 1$, the searcher must stop searching no matter which price he observes. Let

$\xi^1(\theta) = p_m$

and

$\xi^n(\theta) = \max \{ p \in P : u(p, I) \geq z_n(\theta, I) \} \quad (n \geq 2)$.

We would like to prove that the above stopping rule exhibits the reservation price property and that the reservation price is $\xi^n(\theta)$. We need the following lemmas to prove this and study properties of $\xi^n(\theta)$.

**Lemma 11:** $\xi^n(\theta)$ is well-defined, i.e., $\xi^n(\theta)$ exists for all $n$.

**Proof:** Since $\xi^1(\theta) = p_m$, we want to prove that

$\{ p \in P : u(p, I) \geq z_n(\theta, I) \}$

is non-empty for all $n \geq 2$. To prove that the above set is non-empty we have only to show that $u(p_i, I) \geq z_n(\theta, I)$ for $n \geq 2$. We show this by induction.

(i) $n = 2$.

$\sum_j \sum_i V_j(p_i, T(\theta, p_i), I-\gamma) h(\gamma^j|\theta).
= \sum_j \sum_i u(p_i, I-\gamma) f(p_i|\gamma^j) h(\gamma^j|\theta) \leq u(p_i, I) \leq u(p_i, I).

(ii) Suppose that for any $\theta$ and $c > I$

$u(p, I) \geq \sum_j \sum_i V_{n-2}(p_i, T(\theta, p_i), I-2c) f(p_i|\gamma^j) h(\gamma^j|\theta).

Then

$\sum_j \sum_i V_{n-1}(p_i, T(\theta, p_i), I-2c) f(p_i|\gamma^j) h(\gamma^j|\theta)
= \sum_j \sum_i \max \{ u(p_i, I-\gamma) \}
\sum_k \sum_a V_{n-2}(p_a, T(\theta, p_a), I-2c)
\times f(p_a|\gamma^j) h(\gamma^j|T(\theta, p_a)) f(p_a|\gamma^j) h(\gamma^j|\theta)
\leq \sum_j \sum_i \max \{ u(p_i, I-\gamma) \}
\sum_k \sum_a V_{n-2}(p_a, T(\theta, p_a), I-2c)
\times f(p_a|\gamma^j) h(\gamma^j|T(\theta, p_a)) f(p_a|\gamma^j) h(\gamma^j|\theta)
= \sum_j \sum_i u(p_i, I-\gamma) f(p_i|\gamma^j) h(\gamma^j|\theta)
= u(p_i, I-\gamma).

The second equality follows from the induction hypothesis. Q.E.D.

**Lemma 12:** $V_n(p, \theta, I)$ is increasing in $I$ for each $n$.

**Proposition 10:** The above stopping rule, (21), exhibits the reservation price property and the reservation price is $\xi^n(\theta)$.

**Proof:** By Lemma 11 $\xi^n(\theta)$ exists for all $n$. It is obvious that $\xi^1(\theta)$ is the reservation price for $n = 1$. So we consider the case in which $n \geq 2$. Since $\xi^n(\theta)$ is defined to be the highest price such that $u(p_i, I) \geq z_n(\theta, I)$ and $u(p, I)$ is decreasing in $p$, it is clear that $\xi^n(\theta)$ is the reservation price. Q.E.D.

**Proposition 11:** If search cost increases, the reservation price increases.

**Proof:** This is obvious by Lemma 12. Q.E.D.
Proposition 12: The reservation price is a decreasing function of the number of times the 
searcher can observe prices \( \xi^n(\theta) \leq \xi^{n-1}(\theta) \).
(The proof is similar to that for Proposition 3.)

We note that to prove the above three propositions we did not use condition \((\ast)\), which 
was used to guarantee the existence of the reservation price property when the searcher has 
imperfect information about the distribution and accepts or rejects prices before he recal-
culates the reservation price.

IV. Concluding Remarks

There are several problems that were not studied in this paper. Some of them are 
not so interesting and others are just complicated. For example, we did not study the search 
problem in which an expected-utility-maximizing searcher knows the true distribution of 
prices and is allowed to search infinitely many times. As far as the searcher has finite leisure 
endowment this problem trivially reduces to the finite time horizon problem we studied in 
this paper. Another problem we did not discuss in this paper is about how increasing risk 
affects the reservation price when recall is allowed. Though we conjecture that it lowers 
the reservation price if convexity of the indirect utility function is assumed, for the proof 
we have to consider more than twenty cases depending on how high are the reservation 
price and the searcher’s best price in relation to the four prices to which we apply a mean 
preserving spread.

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Appendix

To prove that increasing risk lowers the reservation price, we need the following lemmas. 
Lemma 13 and Lemma 14 are proved in Harstad-Postlewaite [1]. Lemma 15 follows from 
the convexity of the indirect utility function, but the proof is much longer and harder than 
that for Proposition 9. In these lemmas we repeatedly use the following notation.

\[
\hat{h}_t(\mu, \rho) = \left[ \frac{\mu_1}{1+s\rho}, \ldots, \frac{\mu_4+s\rho}{1+s\rho}, \ldots, \frac{\mu_m}{1+s\rho}, \frac{\rho}{1+s\rho} \right].
\]

\( \hat{h}_t(\mu, \rho) \) is \((\mu, \rho)\) updated after \(s\) observations of \(p_t\). Thus for example 
\( \hat{h}_t^3(\mu, \rho) = h_t(\mu, \rho) \) and \( \hat{h}_t^3(\mu, \rho) = h_t[\hat{h}_t(\mu, \rho)] \). In the proofs we use the following relation 
\( h_k \hat{h}_t(\mu, \rho) = h_k[\hat{h}_t(\mu, \rho)] = h_t^k h_k(\mu, \rho) \).

In most cases \((\mu, \rho)\) is omitted to simplify notations.

Lemma 13 (Harstad-Postlewaite): Suppose 
\[
u(p_t, I) - u(p_k, I) \geq u(p_t, I - c) - u(p_k, I - c)
\]
for all \(p_t < p_k\) and \(c < I\) and that the searcher’s prior is Dirichlet.
Then for $p_j < p_k$
\[ V_n[h_j^*(\mu, \rho), I-c] \geq V_n[h_k^*(\mu, \rho), I-c]. \]

**Lemma 14** (Harstad-Postlewaite): Suppose
\[ u(p_j, I) - u(p_k, I) \geq u(p_j, I-c) - u(p_k, I-c) \]
for all $p_j < p_k$ and $c < I$ and that the searcher’s prior is Dirichlet. Then for $p_j < p_k$
\[ u(p_j, I) - u(p_k, I) \geq V_n[h_j^*(\mu, \rho), I-c] - V_n[h_k^*(\mu, \rho), I-c]. \]

**Lemma 15:** Suppose $u(p, I)$ is a convex function of $p$ for all $I$ and that
\[ u(p_j, I) - u(p_k, I) \geq u(p_j, I-c) - u(p_k, I-c) \]
for all $p_j < p_k$ and $c < I$. If the searcher’s prior is Dirichlet, then for all $n, s$, and $p_j' < p_k'$
\[ p_s' < p_l' \]
\[ W = V_n[h_{l'}^*, I] - V_n[h_{k'}^*, I] \geq 0. \]

**Proof:** We use induction to prove the above inequality.
(i) $n = 1$.
\[ W = \frac{sp}{1+sp} \left( \frac{u(p_{j'}, I) - u(p_{j''}, I)}{p_{j'} - p_{j''}} - \frac{u(p_{k'}, I) - u(p_{k''}, I)}{p_{k'} - p_{k''}} \right) \geq 0 \]
by the convexity.
(ii) Suppose that for any $s$
\[ \frac{V_n[h_{j'}^*, I] - V_n[h_{j''}, I]}{p_{j'} - p_{j''}} \geq 0. \]
Let $r_{k'}$ satisfy
\[ V_n[h_{k'}^*, I] = \sum_{i=1}^{r_{k'}} v_{j'} u(p_{j}, I) + \sum_{r_{k'}+1}^{m} v_{j'} V_n[h_{j'}^*, I-c], \]
where
\[ v_{j'} = \begin{cases} \frac{p_j + s}{1 + sp}, & \text{if } j = k' \\ \frac{p_j}{1 + sp}, & \text{if } j \neq k'. \end{cases} \]
The reservation price property guarantees the existence of such an $r_{k'}$. We note that $h_{k'}^*$ is $h_{j'}^*$ updated after an observation of $p_j$. We use later a relation $h_{k'}^* = h_{j'}^*$, where the right hand side is $h_j$ updated after $s$ observations of $p_{k'}$. These somewhat confusing notations are used for simplicity. It follows from Lemma 13 that $r_{k'} \leq r_{k''} \leq r_{s'} \leq r_{s''}$. We define
\[ w_{i'k'} = V_n[h_{i'}^*, I] - V_n[h_{k'}^*, I] \quad (i' < k'). \]
\[ w_{i'k'} \quad \text{and} \quad w_{s'k'} \]
are the numerators of $W$. To prove $W \geq 0$ we study $w_{i'k'}$ and $w_{s'k'}$ separately. We study each of the two in four different cases. Since all cases can be treated similarly, we show the detail of the first case in each of the two and give only the results in the other cases.

We first study $w_{i'k'}$. We consider four cases depending on the relationship of $r_{k'}$ and $r_{s'}$ to 1' and 2'. These four cover all the cases possible.
(a) $1' \leq r_{k'}$ and $2' \leq r_{s'}$. 

\[ w_{1^*} \geq \frac{s_p}{1 + s_p} [u(p_{1^*}, I) - u(p_{2^*}, I)] + \sum_{r_1^*+1}^{r_2^*} \frac{\mu_{1^*}}{1 + s_p} [V_{n-1}[h_{1^*}^s, I - c] - u(p_{1^*}, I)] + \sum_{r_{2^*}+1}^{m} \frac{\mu_{2^*}}{1 + s_p} [V_{n-1}[h_{2^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]]. \]

The inequality follows from the fact that \( V_{n-1}[h_{1^*}^s, I - c] \geq u(p_{1^*}, I) \) for \( j \geq r_1^* + 1 \) and \( V_{n-1}[h_{2^*}^s, I - c] \geq V_{n-1}[h_{2^*}^s, I - c] \) by Lemma 13.

(2) \( r_1^* < 1' < 2' < r_3^* \):

\[ w_{1^*} \geq \frac{s_p}{1 + s_p} [u(p_{1^*}, I) - u(p_{2^*}, I)] + \sum_{r_1^*+1}^{n} \frac{\mu_{1^*}}{1 + s_p} [V_{n-1}[h_{1^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]]. \]

(3) \( 1' \leq r_1^* \) and \( r_3^* < 2' \):

\[ w_{1^*} \geq \frac{s_p}{1 + s_p} [V_{n-1}[h_{1^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]] + \sum_{r_{2^*}+1}^{m} \frac{\mu_{2^*}}{1 + s_p} [V_{n-1}[h_{2^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]]. \]

(4) \( r_1^* < 1' \) and \( r_3^* < 2' \):

\[ w_{1^*} \geq \frac{s_p}{1 + s_p} [V_{n-1}[h_{1^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]] + \sum_{r_{2^*}+1}^{m} \frac{\mu_{2^*}}{1 + s_p} [V_{n-1}[h_{2^*}^s, I - c] - V_{n-1}[h_{2^*}^s, I - c]]. \]

We next study \( w_{3^*} \). We consider four cases depending on the relationship of \( r_3^* \) and \( r_4^* \) to 3' and 4'. These four cover all the cases possible.

(a') \( 3' \leq r_3^* \) and \( 4' \leq r_4^* \):

\[ w_{3^*} \geq \frac{s_p}{1 + s_p} [u(p_{3^*}, I) - u(p_{4^*}, I)] + \sum_{r_{3^*}+1}^{r_4^*} \frac{\mu_{3^*}}{1 + s_p} [V_{n-1}[h_{3^*}^s, I - c] - u(p_{3^*}, I)] + \sum_{r_{4^*}+1}^{m} \frac{\mu_{4^*}}{1 + s_p} [V_{n-1}[h_{4^*}^s, I - c] - V_{n-1}[h_{4^*}^s, I - c]]. \]

The inequality follows from the fact that \( u(p_{3^*}, I) \geq V_{n-1}[h_{3^*}^s, I - c] \) for \( j \leq r_4^* \).

(b') \( 3' \leq r_3^* \) and \( r_4^* < 4' \):

\[ w_{3^*} \geq \frac{s_p}{1 + s_p} [u(p_{3^*}, I) - u(p_{4^*}, I)] + \sum_{r_{4^*}+1}^{m} \frac{\mu_{4^*}}{1 + s_p} [V_{n-1}[h_{4^*}^s, I - c] - V_{n-1}[h_{4^*}^s, I - c]]. \]

(c') \( r_3^* < 3' \leq 4' \leq r_4^* \):

\[ w_{3^*} \geq \frac{s_p}{1 + s_p} [V_{n-1}[h_{3^*}^s, I - c] - V_{n-1}[h_{4^*}^s, I - c]]. \]
Suppose \( r_d' < 3' \) and \( r_d'' < 4' \).

\[
W_{r_d'} \leq \frac{sp}{1 + sp} \left[ V_{n-1}[h_{d'}', I-c] - V_{n-1}[h_{d''}^s, I-c] \right]
\]

\[
+ \sum_{r_{d''} + 1}^{n} \frac{\mu_{j'}}{1 + sp} \left[ V_{n-1}[h_{d''}^t, I-c] - V_{n-1}[h_{d''}^t, I-c] \right].
\]

In the eight cases considered above each of the four pairs, i.e., \([\alpha), (\beta)\], \([\gamma), (\delta)\], \([\alpha'), (\beta')\], and \([\gamma'), (\delta')\], reduces to the same inequality relation. So in order to prove the lemma we seem to have to consider four combinations of these pairs. But \([\gamma), (\delta)\] and \([\alpha'), (\beta')\] do not occur simultaneously. This can be seen from Lemma 14.

Suppose \( 3' \leq r_{d'} \) as in \([\alpha'), (\beta')\). Then

\[
u(p_{d'}) \geq V_{n-1}[h_{d'}^s, I-c] = V_{n-1}[h_{d'}^{s+1}, I-c].
\]

Using this inequality and Lemma 14, we have

\[
u(p_{d'}) - V_{n-1}[h_{d'}^{s+1}, I-c] \geq u(p_{d'}, I) - V_{n-1}[h_{d'}^{s+1}, I-c] \geq 0.
\]

Thus \( u(p_{d'}, I) \geq V_{n-1}[h_{d'}^s, I-c] \). So we have \( 2' \leq r_{d'} \), which contradicts \([\gamma), (\delta)\].

Thus we have only to consider three combinations.

(a) \([\alpha) or (\beta)\) and \([\alpha') or (\beta')\).

\[
W \geq \frac{s}{1 + s} \left\{ \frac{u(p_{d'}, I) - u(p_{d'}, I)}{p_{d'} - p_{d'}} - \frac{u(p_{d'}, I) - u(p_{d'}, I)}{p_{d'} - p_{d'}} \right\}
+ \sum_{r_{d''} + 1}^{n} \frac{\mu_{j'}}{1 + sp} \left[ \frac{V_{n-1}[h_{d''}^t, I-c] - V_{n-1}[h_{d''}^t, I-c]}{p_{d'} - p_{d'}} - \frac{V_{n-1}[h_{d''}^t, I-c] - V_{n-1}[h_{d''}^t, I-c]}{p_{d'} - p_{d'}} \right] \geq 0
\]

by the convexity and the induction hypothesis.

(b) \([\alpha) or (\beta)\) and \([\gamma') or (\delta')\).

Using Lemma 14 in \([\alpha) or (\beta)\], we have

\[
W \geq \frac{sp}{1 + sp} \left\{ \frac{V_{n-1}[h_{d'}^{s+1}, I-c] - V_{n-1}[h_{d'}^{s+1}, I-c]}{p_{d'} - p_{d'}} - \frac{V_{n-1}[h_{d'}^{s+1}, I-c] - V_{n-1}[h_{d'}^{s+1}, I-c]}{p_{d'} - p_{d'}} \right\}
+ \sum_{r_{d''} + 1}^{n} \frac{\mu_{j'}}{1 + sp} \left[ \frac{V_{n-1}[h_{d''}^t, I-c] - V_{n-1}[h_{d''}^t, I-c]}{p_{d'} - p_{d'}} - \frac{V_{n-1}[h_{d''}^t, I-c] - V_{n-1}[h_{d''}^t, I-c]}{p_{d'} - p_{d'}} \right] \geq 0
\]

by the induction hypothesis.

(c) \([\gamma) or (\delta)\) and \([\gamma') or (\delta')\). In this case we get the same inequality as that in (b).

Q.E.D.

**Proposition 9:** Suppose \( u(p, I) \) is a convex function of \( p \) for all \( I \) and that

\[
u(p_{d'}, I) - u(p_{d'}, I) \geq u(p_{d'}, I-c) - u(p_{d'}, I-c)
\]

for all \( p_j < p_{d'} \) and \( c < I \). If the searcher's prior is Dirichlet, then increasing risk in the sense of Rothschild-Stiglitz lowers the reservation price.
Proof: Suppose \( \tilde{\mu} \) is riskier than \( \mu \). We have only to show
\[
V_n(\tilde{\mu}, \rho, I) \geq V_n(\mu, \rho, I)
\]
for all \( n \).

Like the proof for Proposition 4 it will suffice to prove this proposition when \( \tilde{\mu} \) and \( \mu \) differ by a single mean preserving spread, so we shall consider four prices \( p_1' < p_2' < p_3' < p_4' \) such that \( \tilde{\mu}_i = \mu_i \) for \( i \in \{1', 2', 3', 4'\} \) and
\[
\tilde{\mu}_{i'} - \mu_{i'} = \tilde{\mu}_4' - \mu_4' = Q > 0,
\]
and
\[
\mu_{i'} - \mu_{i'}' = \tilde{\mu}_d' - \mu_d' = R > 0,
\]
where
\[
Q(p_2' - p_1') + R(p_3' - p_4') = 0. \tag{23}
\]
We show (22) by induction.

(i) \( n = 1 \).

By an argument similar to (i) of the proof for Proposition 4
\[
V_1(\tilde{\mu}, \rho, I) - V_1(\mu, \rho, I) = Q \{u(p_1', I) - u(p_{2'}, I)\} - R \{u(p_{3'}, I) - u(p_{4'}, I)\} \geq 0.
\]

(ii) Suppose \( V_{n-1}(\tilde{\mu}, \rho, I) \geq V_{n-1}(\mu, \rho, I) \).

For simplicity let \( \tilde{h}_j = h_j(\tilde{\mu}, \rho) \). Let \( r \) satisfy
\[
V_n(\mu, \rho, I) = \sum \tilde{\mu}_j u(p_j, I) + \sum \tilde{\mu}_j V_{n-1}(h_j, I - c).
\]
The reservation price property guarantees the existence of such an \( r \). Note that
\[
V_n(\tilde{\mu}, \rho, I) = \sum \tilde{\mu}_j u(p_j, I) + \sum \tilde{\mu}_j V_{n-1}[h_j, I - c] \geq \sum \tilde{\mu}_j u(p_j, I) + \sum \tilde{\mu}_j V_{n-1}[h_j, I - c].
\]
The last inequality follows from the induction hypothesis. Let
\[
Y(\tilde{\mu}, \mu) = \sum (\tilde{\mu}_j - \mu_j)u(p_j, I) + \sum (\tilde{\mu}_j - \mu_j)V_{n-1}[h_j, I - c].
\]
Then we have only to show that \( Y(\tilde{\mu}, \mu) \geq 0 \).

There are five cases to consider, depending on the relationship of \( \tilde{\mu} \) to \( p_1', p_2', p_3', \) and \( p_4' \):

(a) \( p_r \geq p_{4'} \).

\[
Y(\tilde{\mu}, \mu) = Q \{u(p_1', I) - u(p_{2'}, I)\} - R \{u(p_{3'}, I) - u(p_{4'}, I)\} \geq 0.
\]
This inequality is the same as that in (i).

(b) \( p_{4'} > p_r \geq p_{3'} \).

\[
Y(\tilde{\mu}, \mu) = Q \{u(p_1', I) - u(p_{2'}, I)\} - R \{u(p_{3'}, I) - V_{n-1}[h_{4'}, I - c]\} \geq 0.
\]

(c) \( p_d' > p_r \geq p_f' \).
\[
Y(\bar{p}, \mu) = Q\left( u(p_f', I) - u(p_d', I) \right) - R \left[ V_{n-1}[h_3', I-c] - V_{n-1}[h_4', I-c] \right]
\geq Q\left( u(p_f', I) - u(p_d', I) \right) - R \left( u(p_d', I) - u(p_f', I) \right)
\geq 0.
\]
The first inequality follows from Lemma 14.

(d) \( p_d' > p_f \geq p_{f'} \).
\[
Y(\bar{p}, \mu)
= Q\left( u(p_{f'}, I) - V_{n-1}[h_3', I-c] \right) - R \left( V_{n-1}[h_3', I-c] - V_{n-1}[h_4', I-c] \right)
\geq Q\left( V_{n-1}[h_1', I-c] - V_{n-1}[h_2', I-c] \right) - R \left( V_{n-1}[h_3', I-c] - V_{n-1}[h_4', I-c] \right)
= Q\left( V_{n-1}[h_1', I-c] - V_{n-1}[h_2', I-c] \right) - \frac{Q(p_d'-p_{f'})}{p_d'-p_{f'}} \left( V_{n-1}[h_3', I-c] - V_{n-1}[h_4', I-c] \right)
\geq 0.
\]
The second equality follows from (23) and the last inequality follows from Lemma 15.

(e) \( p_{f'} > p_r \).
In this case \( Y(\bar{p}, \mu) \) is just equal to the expression after the first inequality in (d), so the same argument as above applies.

\[Q.E.D.\]

\textbf{REFERENCES}


