Hitotsubashi Journal of Economics 25 (1984) 93-104. © The Hitotsubashi Academy

A NEW CONCEPT OF EQUILIBRIUM FOR A PRIVATE OWNERSHIP ECONOMY WITH PROFIT-MAKING FIRMS

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I. *Introduction*

One of the important results in general equilibrium theory is the coincidence of the core and the set of competitive equilibria in large economies, which was first observed by F.Y. Edgeworth. The equivalence between the core and the competitive equilibrium has been proved in two types of economies. One is a pure exchange economy, and the other is an economy with production, which is a generalization of the pure exchange economy. However, the results in those two economies do not essentially differ from each other in that the definition of core is based only on the consumer's criterion of utility maximization. Namely, the producer's criterion of profit maximization is not appropriately incorporated in the theory of core.

In general equilibrium theory on economies containing producers, or firms, roughly speaking, there are three approaches. The first one is by L. Walras, whose economic model was presented in a general formulation by L.W. McKenzie [6]. Walras considered an economy where no profits arise in firms, while he took into account producers, or "enterpreneurs" in his terminology. In the economy, the law of "constant returns to scale" prevails, that is, the production possibility set is a cone with vertex at the origin, and every producer's profit is zero in equilibrium. In such a zero-profit economy, prodercers do not play any important role.

The second approach is of a private ownership economy with a fixed list of producers by K.J. Arrow and G. Debreu [1] and G. Debreu [2]. Unlike Walras' economy, producers' profits can be positive in the private ownership economy. Therefore, the behaviors of producers have significant effects on the economy through the distribution of their profits to consumers. As a result, the theory of core is not directly applicable to such an economy with producers.

In order to define a core of an economy with production, the third approach was taken by W. Hildenbrand [4], that is, a coalition production economy. He proved that an equilibrium existence theorem for the coalition production economy includes that for the private ownership economy. However, the coalition production economy corresponds to an economy where firms are personalised, i.e., each firm is the property of a single consumer who can fully control it. This kind of personalised firm is equivalent to the entrepreneur in

* I would like to thank Hiroaki Osana and Akira Yamazaki for their comments.

J.R. Hicks [3, p. 100]. In other words, production technologies are separately possessed by consumers. Thus, the core of the coalition production economy, like that of the pure exchange economy, can be defined on the basis of only consumers' preferences.

In the present paper, we shall introduce a new concept of equilibrium, which depends on producers' decisions as well as consumers'. Our equilibrium is totally different from the usual core in that the producer's criterion of profit maximization is incorporated. But, our equilibrium applies to the pure exchange economy, and, in that case, it belongs to the core. In this sense, our equilibrium concept is a generalization of the core. While various equilibrium concepts have been proposed in game theory, our equilibrium seems close to the strong Nash equilibrium (see T. Ichiishi [5] and D. Schmeidler [7]). In fact, we can easily show our equilibrium is a special case of the strong Nash equilibrium.

The purpose of this paper is to give a new interpretation of the competitive equilibrium in the economy where consumers and producers coexist. As a basic economy, we shall adopt an private ownership economy with a fixed list of firms, which is usually called "the Arrow-Debreu economy." A new concept of equilibrium for the economy is defined and is called "a stable contract" in this paper (see Def. 4). On the other hand, the usual Walrasian equilibrium is defined and is called "a competitive contract" in this paper (see Def. 5). First we shall prove that any competitive contract is a stable contract (Theorem 1). Namely, the Walrasian equilibrium is shown to be the equilibrium in our sense. Next we shall prove that, if the economy is large, any allocation realized under a stable contract is an allocation which is realized under a competitive contract (Theorem 2). In other words, our equilibrium is shown to be equivalent to the Walrasian equilibrium.

We shall use the following notation: R^k denotes a k-dimensional Euclidean space, where k is positive integer. $R^k{}_+$ is the non-negative orthant of R^k. When $k=1$, we write R and R_+ instead of R^1 and $R^1{}_+$. By 2^{R^k}, we denote the class of all the subsets of R^k. Subscripts attached to vectors will be used exclusively to denote coordinates. Following standard practice, for x and y in R^k we take $x>y$ to mean $x_i>y_i$ for all i; $x\geqq y$ to mean $x_i\geqq y_i$ for all i; and $x\geq y$ to mean $x\geqq y$ but not $x=y$. The inner product $\sum_{i=1}^k x_iy_i$ of two members x and y of R^k is denoted by $x\cdot y$. The Euclidean norm of any x in R^k is denoted by $||x||$, i.e., $||x||=\sqrt{x\cdot x}$. The integral of a vector function is to be taken as the vector of integrals of the components. The symbol $\sim$ will be used for set-theoretic subtraction, whereas the symbol — will be reserved for ordinary algebraic subtraction.

II. *A Model of Private Ownership Economy*

We shall consider a private ownership economy with a fixed list of consumers and producers. Let $(A, \mathscr{A}, \nu)$ be a measure space of economic agents, which is a complete probability space. The elements of set A are interpreted as economic agents, family $\mathscr{A}$ as a collection of sets of agents, and the number $\nu(C)$ for each $C\in\mathscr{A}$ as the size of set C relative to set A of all the agents. In the economy there are two kinds of economic agents, that is, consumers (households) and producers (firms). The sets of all the consumers and all the producers are denoted by S and T respectively, and it is assumed that S, $T\in\mathscr{A}$, $S\cup T=A$, and $S\cap T=\phi$. Let $(S, \mathscr{S}, \lambda)$ and $(T, \mathscr{T}, \mu)$ denote sub-spaces of $(A, \mathscr{A}, \nu)$. Also, let $(S\times T, \mathscr{S}\times\mathscr{T}, \lambda\times\mu)$ denote the product space of $(S, \mathscr{S}, \lambda)$ and $(T, \mathscr{T}, \mu)$.

There are finitely many different commodities in the economy. The number of commodities is m, which is a positive integer. Let $X: S \to 2^{R^m}$ be a measurable map, i.e., $\{(a, x) \in S \times R^m \mid x \in X(a)\}$ is a measurable subset of $S \times R^m$. For each $a \in S$, set $X(a)$ is interpreted as the net-consumption set of consumer a. Also, let $\succ: S \to 2^{R^m \times R^m}$ be a measurable map, i.e., $\{(a, x, x') \in S \times R^m \times R^m \mid (x, x') \in \succ_a\}$ is a measurable subset of $S \times R^m \times R^m$, such that $\succ_a \subset X(a) \times X(a)$ for all $a \in S$. For each $a \in S$, set $\succ_a$ is interpreted as the preference relation of consumer a defined on $X(a)$. Usually, $(x, x') \in \succ_a$ is written as $x \succ_a x'$, which means that consumer a prefers x to x'.

Let $Y: T \to 2^{R^m}$ be a measurable map, i.e., $\{(a, y) \in T \times R^m \mid y \in Y(\mathrm{a})\}$ is a measurable subset of $T \times R^m$. For each $a \in T$, set $Y(a)$ is interpreted as the production set of producer a. All the profits of producers are assumed to be distributed to consumers in a fixed way, i.e., there is a measurable function $\theta: S \times T \to R_+$ such that

$$\int_S \theta(a, a') d\lambda(a) = 1 \text{ for all } a' \in T,$$

with $\theta(a, a')$ standing for the share of consumer $a \in S$ in the profit of producer $a' \in T$. Namely, if producer a' earns profit $\pi(a')$, then consumer a can get dividend $\theta(a, a') \cdot \pi(a')$ from porducer a'.

Let $(A \times A, \mathscr{A} \times \mathscr{A}, \nu \times \nu)$ be a product space of $(A, \mathscr{A}, \nu)$. To describe the contracts among agents in the economy, we use a triplet (f, q, π) of measurable functions.

Definition 1. A triplet (f, q, π) of measurable functions, $f: A \times A \to R^m$, $q: A \times A \to R^m$, and $\pi: T \to R_+$ is called *a contract* if the following conditions are fulfilled.

(i) $f(a, a') = -f(a', a)$ for all $a, a' \in A$.
(ii) $q(a, a') = q(a', a)$ for all $a, a' \in A$.
(iii) Function f is integrable.
(iv) A map from $A \times A$ to R defined by
$(a, a') \to q(a, a') \cdot f(a, a')$
is integrable.
(v) Function π is integrable.

Contracts of commodity transactions between any two agents are described by functions f and q. Agents a and a' have made a contract such that agent a buy from agent a' net-amount $f(a, a')$ of commodities at prices $q(a, a')$. Namely, $f(a, a')$ and $q(a, a')$ denote an agreement of commodity transaction between agents a and a'. Thus, condition (i) and (ii) of Definition 1 are natural requirements on functions f and q.

Profit payments from producers to consumers are described by function π. Producer a' has made a promise to each consumer a such that producer a' will pay profit dividend $\theta(a, a') \cdot \pi(a')$ to consumer a. Namely, $\pi(a')$ denotes an announcement of profit payment by producer a'.

Conditions (iii), (iv), and (v) of Definition 1 are purely mathematical and they do not impose any economic restriction.

III. *Stable Contracts among Agents*

Somehow agents in the economy make contracts with each other, and a contract (f, q, π) will be made. However, we shall assume that such a contract is conditional on circumstances being favorable. Namely, it is assumed that every agent has a right to cancel the contract when it turns out to be unfavorable. In addition, it is assumed that every agent has a right to recontract in order to make a new better contract.

First of all, agents care whether contracts are feasible for them in the following sense.

Definition 2. A contract (f, q, π) is said to be *feasible* for agent $a\in A$ if, when agent a is a consumer, i.e., $a\in S$,

(i) $\int_A f(a, a')d\nu(a')\in X(a)$ and

(ii) $\int_A q(a, a')\cdot f(a, a')d\nu(a')\leqq\int_T \theta(a, a')\pi(a')d\mu(a')$,

or when agent a is a producer, i.e., $a\in T$,

(iii) $-\int_A f(a, a')d\nu(a')\in Y(a)$ and

(iv) $-\int_A q(a, a')\cdot f(a, a')d\nu(a')\geqq\pi(a)$.

In the above definition, when agent a is a consumer, condition (i) implies that total net-transaction of commodities, $\int_A f(a, a')d\nu(a')$, is acceptable to consumer a. Condition (ii) means that total net-expenditure in commodity transactions, $\int_A q(a, a')\cdot f(a, a')d\nu(a')$, is not greater than total dividend of profits, $\int_T \theta(a, a')\cdot\pi(a')d\mu(a')$, which consumer a will receive from producers. When agent a is a producer, condition (iii) says that total net-transaction of commodities, $-\int_A f(a, a')d\nu(a')$, is possible to producer a. Condition (iv) means that total net-revenue in commodity transactions, $-\int_A q(a, a')\cdot f(a, a')d\nu(a')$, is not smaller than profit $\pi(a)$, which producer a will pay to consumers.

Given a contract, say $(\hat{f}, \hat{q}, \hat{\pi})$, if it is not feasible for some agents, they will try to make another contract which is feasible for them. Moreover, if contract $(\hat{f}, \hat{q}, \hat{\pi})$ is feasible for them, they will try to make a better contract.

Definition 3. A contract $(\hat{f}, \hat{q}, \hat{\pi})$ can be *improved upon* by a set of agents, $C\in\mathscr{A}$, if there exists another contract (f, q, π) which is feasible for all agent in C and satisfies the following conditions.

(i) For all agent $a\in C$, if contract $(\hat{f}, \hat{q}, \hat{\pi})$ is feasible for agent a, then, when agent a is a consumer, i.e., $a\in C\cap S$,

$\int_A f(a, a')d\nu(a')\succ_a\int_A \hat{f}(a, a')d\nu(a')$,

or when agent a is a producer, i.e., $a\in C\cap T$,

$\pi(a)>\hat{\pi}(a)$.

(ii) $f(a, a')=\begin{cases}\hat{f}(a, a') & \text{for } (a, a')\in(A\sim C)\times(A\sim C)\\ \hat{f}(a, a'), \text{ or } 0 & \text{for } (a, a')\in C\times(A\sim C), \text{ or } (A\sim C)\times C.\end{cases}$

(iii) $q(a, a')=\hat{q}(a, a')$ for $(a, a')\in A\times(A\sim C)$, or $(A\sim C)\times A$.

(iv) $\pi(a)=\hat{\pi}(a)$ for $a\in T\sim C$.

The above definition says that, given a contract $(\hat{f}, \hat{q}, \hat{\pi})$ which may not be feasible for all the agents in the economy, by changing the contract, a set of some agents, C, can make another contract (f, q, π) which is better for them than the original contract $(\hat{f}, \hat{q}, \hat{\pi})$, or at least feasible for them. Conidtion (i) says that contract (f, q, π) is more favorable for all agent in C than contract $(\hat{f}, \hat{q}, \hat{\pi})$, i.e., all consumer in C can enjoy a preferable consumption of commodities and all producer in C can pay a greater profit. Of course, agents in C cannot change the whole of contract $(\hat{f}, \hat{q}, \hat{\pi})$, but only a part of it.

In commodity transactions, any two agents can change at will the contract between them if they agree. Thus, agents in C can determine their contracts, $f(a, a')$ and $q(a, a')$ for $(a, a')\in C\times C$, as they like. However, it should be assumed that a contract between any two agents cannot be affected by any other third agent. That is, condition (ii) and (iii) say that the contracts among agents in $A\sim C$, $\hat{f}(a, a')$ and $\hat{q}(a, a')$ for $(a, a')\in(A\sim C)\times(A\sim C)$, must remain unchanged.

Moreover, it seems natural to assume that one of any two agents can cancel the contract between them without agreement of the other. Namely, condition (ii) insures that each agent $a\in C$ can either keep or cancel the contract with agent $a'\in A\sim C$, i.e., can choose either $\hat{f}(a, a')$ or 0. Note that $q(a, a')=\hat{q}(a, a)$ for $(a, a')\in C\times(A\sim C)$ in condition (iii) even when agent a revokes the contract with agent a'. But, no generality is lost since prices have no importance in that case. In this sense, contracts are assumed not to be binding, but to be conditional.

In addition, the profit payment of each producer cannot be affected by other agents. Namely, condition (iv) insures that the profit payments by producers in $T\sim C$ must remain unchanged.

There may be some contracts that nobody can improve upon.

Definition 4. A contract $(\hat{f}, \hat{q}, \hat{\pi})$ is said to be *stable* if it cannot be improved upon by any set of agents, $C\in\mathscr{A}$, with $\nu(C)>0$.

Here we should note a simple property of stable contracts.

Lemma 3.1. Assume that $0\in X$(a) for almost every agent in S, and that $0\in Y(a)$ for almost every agent in T. If $(\hat{f}, \hat{q}, \hat{\pi})$ is a stable contract, then it is feasible for almost every agent in A.

Proof. Let $(\hat{f}, \hat{q}, \hat{\pi})$ be a stable contract, and define

$C=\{a\in A \mid (\hat{f}, \hat{q}, \hat{\pi}) \text{ is not feasible for agent } a\}.$

Then, we can easily check that $C\in\mathscr{A}$. Define a contract (f, q, π) by

$$f(a, a')=\begin{cases}\hat{f}(a, a'), & \text{for } (a, a')\in(A\sim C)\times(A\sim C)\\ 0 & \text{otherwise,}\end{cases}$$

$q(a, a')=\hat{q}(a, a')$ for all $(a, a')\in A\times A$,

and

$$\pi(a)=\begin{cases}\hat{\pi}(a) & \text{for } a\in T\sim C\\ 0 & \text{otherwise}\end{cases}$$

Then, the assumption of this lemma implies that (f, q, π) is feasible for almost every agent in C. Therefore, $(\hat{f}, \hat{q}, \hat{\pi})$ can be improved upon by C. Since $(\hat{f}, \hat{q}, \hat{\pi})$ is stable, $\nu(C)=0$.

Q.E.D.

IV. *Competitive Contracts*

Of contracts, there may be some contracts which can be characterized by a particular price vector.

Definition 5. A contract $(\hat{f}, \hat{q}, \hat{\pi})$ is said to be *competitive* if it is feasible for almost every agent in the economy and there exists a vector $p\in R^m$ that has the following properties.

(i) $\hat{q}(a, a')=p$ for all $(a, a')\in A\times A$.

(ii) For almost every consumer $a\in S$,

$$x\succ_a\int_A\hat{f}(a, a')d\nu(a') \text{ implies } p\cdot x>\int_T\theta(a, a')\hat{\pi}(a')d\mu(a').$$

(iii) For almost every producer $a\in T$,

$$\hat{\pi}(a)\geqq p\cdot y \text{ for all } y\in Y(a).$$

In Definition 5, since $(\hat{f}, \hat{q}, \hat{\pi})$ is feasible for almost every agent in the economy in the sense of Definition 2, for almost every consumer $a\in S$,

$$p\cdot\int_A\hat{f}(a, a')d\nu(a')\leqq\int_T\theta(a, a')\hat{\pi}(a')d\mu(a')$$

and for almost every producer $a\in T$,

$$\hat{\pi}(a)=-p\cdot\int_A\hat{f}(a, a')d\nu(a').$$

Therefore, under a competitive contract, commodities are exchanged at a single price system, every consumer is maximizing utility subject to budget constraint, and every producer is maximizing profit subject to technological constraint. Furthermore, by Definition 1 of contracts we can show that

$$\int_S\left[\int_A\hat{f}(a, a')d\nu(a')\right]d\nu(a)=\int_T\left[-\int_A\hat{f}(a, a')d\nu(a')\right]d\nu(a),$$

which means that demand and supply are balanced. Thus, a competitive contract exactly corresponds to the Walrasian equilibrium.

First we shall show a relation between a competitive contract and a stable contract.

Theorem 1. Any competitive contract is stable.

Proof. Let $(\hat{f}, \hat{q}, \hat{\pi})$ be a competitive contract, i.e., there exists a vector $p\in R^m$ satisfying all the conditions of Definition 5. To get a contradiction, suppose that $(\hat{f}, \hat{q}, \hat{\pi})$ were not stable. Then, it can be improved upon by a set of some agents, $C\in\mathscr{A}$ with $\nu(C)>0$. That is, there exists a contract (f, q, π), which is feasible for all agent in C in the sense of Definition 2, and which satisfies all the conditions of Definition 3.

Since $(\hat{f}, \hat{q}, \hat{\pi})$ and (f, q, π) are feasible for almost every agent in C, it follows from Definition 3 (i) and Definition 5 (ii) (iii) that

$$p\cdot\int_A f(a, a')d\nu(a') > \int_T \theta(a, a')\hat{\pi}(a')d\mu(a')$$

for almost every $a \in C \cap S$, and that

$\pi(a) > \hat{\pi}(a)$ for almost every $a \in C \cap T$.

Therefore, first,

$$p\cdot\int_{C\cap S}\left[\int_A f(a, a')d\nu(a')\right]d\nu(a) \geqq \int_{C\cap S}\left[\int_T \theta(a, a')\hat{\pi}(a')d\mu(a')\right]d\lambda(a),$$

where strict inequallty holds if $\nu(C\cap S)>0$. Second, since $\nu(C\cap S)=0$ implies $\nu(C\cap T)>0$, and since $\int_{C\cap S}\theta(a, a')d\lambda(a) \leqq 1$ for all $a' \in T$,

$$\int_{C\cap T}\left[1-\int_{C\cap S}\theta(a, a')d\lambda(a)\right]\pi(a')d\mu(a')$$
$$\geqq \int_{C\cap T}\left[1-\int_{C\cap S}\theta(a, a')d\lambda(a)\right]\hat{\pi}(a')d\mu(a'),$$

where strict inequality holds if $\nu(C\cap S)=0$. Hence,

$$p\cdot\int_{C\cap S}\left[\int_A f(a, a')d\nu(a')\right]d\nu(a) + \int_{C\cap T}\left[1-\int_{C\cap S}\theta(a, a')\,d\lambda(a)\right]\pi(a')d\mu(a')$$
$$> \int_{C\cap S}\left[\int_T \theta(a, a')\,\hat{\pi}(a')\,d\mu(a')\right]d\lambda(a) + \int_{C\cap T}\left[1-\int_{C\cap S}\theta(a, a')d\lambda(a)\right]\hat{\pi}(a')d\mu(a').$$

Thus, by arrangement,

$$p\cdot\int_{C\cap S}\left[\int_A f(a, a')d\nu(a')\right]d\nu(a) - \int_{C\cap T}\hat{\pi}(a)d\mu(a)$$
$$> \int_{C\cap S}\left[\int_{T\sim C}\theta(a, a')\hat{\pi}(a')d\mu(a')\right]d\lambda(a)$$
$$+ \int_{C\cap S}\left[\int_{C\cap T}\theta(a, a')\,\pi(a')\,d\mu(a')\right]d\lambda(a) - \int_{C\cap T}\pi(a)d\mu(a).$$

Here, note that $\hat{\pi}(a) \geqq -p\cdot\int_A f(a, a')d\nu(a')$ for almost every $a \in C\cap T$ by Definition 5 (iii), and that $\pi(a)=\hat{\pi}(a)$ for all $a \in T\sim C$ by Definition 3 (iv). Therefore, the above inequality implies the following.

$$(4.1)\quad p\cdot\int_C\left[\int_A f(a, a')d\nu(a')\right]d\nu(a)$$
$$> \int_{C\cap S}\left[\int_T \theta(a, a')\,\pi(a')\,d\mu(a')\right]d\lambda(a) - \int_{C\cap T}\pi(a)d\mu(a).$$

On the other hand, since (f, q, π) is feasible for all agent in C, by Definition 2 (ii) (iv) we have

$$(4.2)\quad \int_{C\cap S}\left[\int_T \theta(a, a')\,\pi(a')\,d\mu(a')\right]d\lambda(a) - \int_{C\cap T}\pi(a)d\mu(a)$$
$$\geqq \int_C\left[\int_A q(a, a')\cdot f(a, a')d\nu(a')\right]d\nu(a).$$

From (4.1) and (4.2), it follows that

$$p\cdot\int_C\left[\int_A f(a, a')d\nu(a')\right]d\nu(a) > \int_C\left[\int_A q(a, a')\cdot f(a, a')d\nu(a')\right]d\nu(a).$$

Since $q(a, a')=p$ for all $(a, a') \in (A\sim C)\times C$ by Definition 3 (iii) and Definition 5 (i), we can conclude that

$$p\cdot\int_C\left[\int_C f(a, a')d\nu(a')\right]d\nu(a) > \int_C\left[\int_C q(a, a')\cdot f(a, a')d\nu(a')\right]d\nu(a).$$

However, this is a contradiction, because both sides of the above inequality are zero by Definition 1 of contracts. *Q.E.D.*

V. *Identity of Stable Contracts and Competitive Contracts*

In order to establish the converse of Theorem 1, we need some additional assumptions. First, the economy is assumed to be so large that every agent is negligibly small compared with the whole economy, and that he cannot have any effect on the economy by himself. In other words, the economy is perfectly competitive.

Assumption 1. The measure space $(A, \mathscr{A}, \nu)$ is non-atomic.

For consumers, we assume the following.

Assumption 2. For almost every consumer $a \in S$, the following hold:
(i) preference relation $\succ_a$ is irreflexive and transitive.
(ii) for all $x_0 \in X(a)$, $\{x \mid x \succ_a x_0\}$ is open in $X(a)$.
(iii) $X(a)$ is a convex subset of R^m.
(iv) $0 \in \text{int } X(a)$.
(v) (monotonicity) $x \in X(a)$ and $x' > x$ imply $x' \succ_a x$.

Finally, for producers, we assume the following.

Assumption 3. For almost every producer $a \in T$, the following hold:
(i) $0 \in Y(a)$.
(ii) (free disposability) $y \in Y(a)$ and $y' \leqq y$ imply $y' \in Y(a)$.

The converse of Theorem 1 can be proved under these assumption. Namely, we can show that stable contracts are equivalent to competitive contracts.

Theorem 2. Under Assumptions 1, 2, and 3, if a contract $(\hat{f}, \hat{q}, \hat{\pi})$ is stable, then there exist measurable functions $\bar{q}: A \times A \to R^m$ and $\bar{\pi}: T \to R_+$ such that $(\hat{f}, \bar{q}, \bar{\pi})$ is a competitive contract.

Of course, the above theorem is not exactly the converse of Theorem 1, because $(\bar{q}, \bar{\pi})$ is not equal to $(\hat{q}, \hat{\pi})$ in general. In fact, even if the economy is large, a unique price system does not necessarily hold under some stable contracts. However, stable contracts are equivalent to competitive contracts in that any stable contract always gives a rise to an allocation of commodities which is realized under a competitive contract.

VI. *Proof of Theorem 2*

To prove Theorem 2, let $(\hat{f}, \hat{q}, \hat{\pi})$ be a stable contract. Then, by Assumptions 2 (iv), 3 (i), and Lemma 3.1, we know that

(6.1) $\int_A \hat{f}(a, a')d\nu(a') \in X(a)$ for almost every $a \in S$ and

$-\int_A \hat{f}(a, a')d\nu(a') \in Y(a)$ for almost every $a \in T$.

Define a map F: $A \to 2^{R^{m+1}}$ by

$$F(a)=\begin{cases}\left\{(x, -\alpha)\in R^m\times R \mid x\succ_a\int_A \hat{f}(a, a')d\nu(a') \quad\text{and}\quad \alpha\leqq\int_T\theta(a, a')\hat{\pi}(a')d\mu(a')\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } a\in S \\ \{(-y, \beta)\in R^m\times R \mid y\in Y(a) \text{ and } \beta>\hat{\pi}(a)\} \quad\text{for } a\in T.\end{cases}$$

Moreover, define

$L=\{(h, \sigma) \mid h\colon A\to R^m$ and $\sigma\colon A\to R$ are integrable functions such that $(h(a), \sigma(a))\in F(a)\cup\{0\}$ for almost every $a\in A\}$

and

$$Z=\left\{\left(\int_A h(a)d\nu(a), \int_A\sigma(a)d\nu(a)\right)\in R^m\times R \mid (h, \sigma)\in L.\right\}$$

The following is a key lemma in this proof, which will be proved in the next section.

Lemma 6.1. Z is a non-empty convex subset of R^{m+1} such that $Z\cap R^{m+1}_{--}=\phi$, where R^{m+1}_{--} denotes the negative orthant of R^{m+1}.

By virtue of this lemma, we can apply a separation theorem to set Z, and we have a vector $(p, \delta)\in {R_+}^m\times R_+$ with $(p, \delta)\neq 0$ such that

$$p\cdot\int_A h d\nu+\delta\int_A\sigma d\nu\geqq 0 \text{ for all } (h, \sigma)\in L.$$

Also, we know that (see Hildenbrand [4, Prop. 6, p. 63]) that

$$\inf\left\{p\cdot\int_A h d\nu+\delta\int_A\sigma d\nu \mid (h, \sigma)\in L\right\}$$
$$=\int_A\inf\{p\cdot w+\delta\gamma \mid (w, \gamma)\in F(a)\cup\{0\}\}\,d\nu(a).$$

Therefore, $\inf\{p\cdot w+\delta\gamma \mid (w, \gamma)\in F(a)\cup\{0\}\}=0$ for almost every $a\in A$. Thus, we can conclude that

(6.2) $\inf\{p\cdot w+\delta\gamma \mid (w, \gamma)\in F(a)\}\geqq 0$ for almost every $a\in A$.

Define maps $\bar{q}\colon A\times A\to R^m$ by

$\bar{q}(a, a')=p$ for each $(a, a')\in A\times A$

and $\bar{\pi}\colon T\to R_+$ by

$\bar{\pi}(a)=\delta\hat{\pi}(a)$ for each $a\in T$.

Then, all we have to do is to show that $(\hat{f}, \bar{q}, \bar{\pi})$ is a competitive contract.

From (6.2), it follows that for almost every $a\in S$,

$$p\cdot x\geqq\delta\int_T\theta(a, a')\hat{\pi}(a')d\mu(a')=\int_T\theta(a, a')\bar{\pi}(a')d\mu(a') \quad\text{for all}\quad x\succ_a\int_A\hat{f}(a, a')d\nu(a').$$

Therefore, by Assumption 2 (ii), (iii), (iv), we can prove in a standard manner that for almost every $a\in S$,

(6.3) $x\succ_a\int_A\hat{f}(a, a')d\nu(a')$ implies $p\cdot x>\int_T\theta(a, a')\bar{\pi}(a')d\mu(a')$.

Also, from (6.2), it follows that for almost every $a\in T$,

$\delta\beta\geqq p\cdot y$ for all $y\in Y(a)$ and $\beta>\hat{\pi}(a)$.

Therefore, for almost every $a\in T$,

(6.4) $\bar{\pi}(a)\geqq p\cdot y$ for all $y\in Y(a)$.

Finally we shall prove that contract $(\hat{f}, \bar{q}, \bar{\pi})$ is feasible. Under Assumption 2 (ii), (6.3) and (6.1) imply that, for almost every $a\in S$,

$$p\cdot\int_A\hat{f}(a, a')d\nu(a')-\int_T\theta(a, a')\bar{\pi}(a')d\mu(a')\geqq 0.$$

Also, (6.4) and (6.1) imply that, for almost every $a\in T$,

$$p\cdot\int_A \hat{f}(a, a')d\nu(a')+\bar{\pi}(a)\geqq 0.$$

On the other hand, by definition (see Definition 1), we know that

$$\int_S\left[p\cdot\int_A \hat{f}(a, a')d\nu(a')-\int_T\theta(a, a')\bar{\pi}(a')d\mu(a')\right]d\lambda(a)$$
$$+\int_T\left[p\cdot\int_A \hat{f}(a, a')d\nu(a')+\bar{\pi}(a)\right]d\mu(a)$$
$$=p\cdot\int_A\left[\int_A \hat{f}(a, a')d\nu(a')\right]d\nu(a)$$
$$=0.$$

Therefore, we can conclude that

$$p\cdot\int_A \hat{f}(a, a')d\nu(a')=\int_T\theta(a, a')\bar{\pi}(a')d\mu(a') \text{ for almost every } a\in S$$

and

$$-p\cdot\int_A \hat{f}(a, a')d\nu(a')=\bar{\pi}(a) \text{ for almost every } a\in T.$$

Together with (6.1), this shows that contract $(\hat{f}, \bar{q}, \bar{\pi})$ is feasible.

Thus, it has been shown that contract $(\hat{f}, \bar{q}, \bar{\pi})$ is competitive, and Theorem 2 has been proved.

VII. *Proof of Lemma 6.1*

By definition of L, obviously $0\in Z$, and $Z\neq\phi$. The convexity of Z follows from Assumption 1 (see Hildenbrand [4, Thm. 3, p. 62]).

Now, to get a contradiction, suppose that $Z\cap R^{m+1}_{--}\neq\phi$, i.e., there is $(h, \sigma)\in L$ such that $\int_A hd\nu<0$ and $\int_A\sigma d\nu<0$. Let $C=\{a\in A \mid (h(a), \sigma(a))\neq 0\}$ Then, $\nu(C)>0$, and, since $(h, \sigma)\in L$,

(7.1) $(h(a), \sigma(a))\in F(a)$ for almost every $a\in C$.

Also, we have

(7.2) $\dfrac{1}{\nu(C)}\int_C hd\nu\equiv w<0$ and

(7.3) $-\int_{C\cap S}\left[\int_T\theta(a, a')\hat{\pi}(a')d\mu(a')\right]d\lambda(a)+\int_{C\cap T}\sigma(a)d\mu(a)\leqq\int_C\sigma d\nu<0.$

To complete the proof, we must drive a contradictory fact that contract $(\hat{f}, \hat{q}, \hat{\pi})$ can be improved upon by agents in C, i.e., there exists a contract, say (f, q, π), which satisfies all the conditions of Definition 3. In what follows, we shall show how to construct such (f, q, π).

Let $E=\{(a, a')\in C\times C \mid h(a)=h(a')$ and $(a, a')\in S\times T$, or $(a, a')\in T\times S\}$ and define f: $A\times A\to R^m$ by

$$f(a, a')=\begin{cases}\hat{f}(a, a') & \text{for } (a, a')\in(A\sim C)\times(A\sim C)\\ \dfrac{h(a)-h(a')}{\nu(C)} & \text{for } (a, a')\in(C\times C)\sim E\\ \dfrac{h(a)-h(a')+w}{\nu(C)} & \text{for } (a, a')\in E\cap(S\times T)\end{cases}$$

$$\begin{cases} \dfrac{h(a)-h(a')-w}{\nu(C)} & \text{for} \quad (a, a')\in E\cap(T\times S) \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (7.2) and Assumption 2 (v), 3 (ii), we can show that conditions (i) and (iii) in Definition 2 hold for all agent in C. Also, by (7.1) and Assumption 2 (i), (v), we can show that condition (i) in Definition 3 holds for all agent in $C\cap S$. Condition (ii) in Definition 3 is implied by definition of f.

Define π: $T\rightarrow R_+$ by

$$\pi(a)=\begin{cases} \hat{\pi}(a) \text{ for } a\in T\sim C \\ \sigma(a) \text{ for } a\in C\cap T. \end{cases}$$

Then, by (7.1), we can show that condition (i) in Definition 3 holds for all agent in $C\cap T$. Condition (iv) in Definition 3 is implied by definition of π.

In defining q, there are two cases.

Case 1: When $\nu(C\cap T)=0$, define q: $A\times A\rightarrow R^m$ by

$$q(a, a')=\begin{cases} \hat{q}(a, a') \text{ for } (a, a')\in A\times(A\sim C), \text{ or } (A\sim C)\times A \\ 0 \quad \text{otherwise.} \end{cases}$$

Case 2: When $\nu(C\cap T)>0$, define q: $A\times A\rightarrow R^m$ by

$$q(a, a')=\begin{cases} \hat{q}(a, a') & \text{for } (a, a')\in A\times(A\sim C), \text{ or } (A\sim C)\times A \\ \dfrac{\pi(a')\int_T\theta(a, t)\pi(t)d\mu(t)}{\int_{C\cap T}\pi(t)d\mu(t)}\cdot\dfrac{f(a, a')}{||f(a, a')||} & \\ & \text{for } (a, a')\in(C\times C)\cap(S\times T) \\ \dfrac{\pi(a)\int_T\theta(a', t)\pi(t)d\mu(t)}{\int_{C\cap T}\pi(t)d\mu(t)}\cdot\dfrac{-f(a, a')}{||f(a, a')||} & \\ & \text{for } (a, a')\in(C\times C)\cap(T\times S) \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (7.3), we can show that conditions (ii) and (iv) in Definition 2 hold for all agent in C. Condition (iii) is implied by definition of q.

Thus, we have shown that contract $(\hat{f}, \hat{q}, \hat{\pi})$ can be improved upon by agents in C, which is a contradiction. This completes the proof of the lemma.

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REFERENCES

[1] Arrow, K.J. and G. Debreu, 1954, Existence of an Equilibrium for a Competitive Economy, *Econometrica* 22, 265–290.
[2] Debreu, G., 1959, *Theory of Value*, Wiley, New York.
[3] Hicks, J.R., 1946, *Value and Capital*, second edition, Oxford.
[4] Hildenbrand, W., 1974, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey.

[5] Ichiishi, T., 1981, A Social Coalitional Equilibrium Existence Lemma, *Econometrica* 46, 369–377.
[6] McKenzie, L.W., 1959, On the Existence of General Equilibrium for a Competitive Market, *Econometrica* 27, 54–71.
[7] Schmeidler, D., 1980, Walrasian Analysis via Strategic Outcome Functions, *Econometrica* 48, 1585–1593.