Hitotsubashi Journal of Economics 24 (1983) 143-148. © The Hitotsubashi Academy

A COMPARISON OF CONVERGENCE SPEED OF OLD AND NEW ITERATIVE PROCESSES FOR AN INPUT-OUTPUT SYSTEM

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I. *Introduction*

It has been pointed out in numerous articles [e.g., Montias (1959)] that the old, standard iterative process employing power series expansion for solving the standard linear input-output system has the disadvantage of generally slow convergence speed. Recently, to speed up this process, a new iterative process using step-wise aggregation has been developed by Vakhutinsky *et al.* (1979), Manove and Weitzman (1978) and Kuboniwa (1982). While it was verified that numerical experiments guarantee the speedier convergence of this new process, the theoretical setting for the comparison of convergence speed was confined to the very special cases in these articles.

This note makes a comparison of convergence speed of the two processes in a more general setting. Two sufficient conditions, using upper bounds of non-maximal eigenvalues of a stochastic matrix, will be presented to state that the new process might converge more speedily than the old, standard process. This note will suggest some implications of these conditions, which may appear to be somewhat paradoxical in that the new process may not be as effective in the adjustment of relative output ratios as in that of output scale. This note limits investigation to the step-wise aggregation process in the simplest case, and deals exclusively with the standard input-output model in value terms.

II. *The Old and New Iterative Processes*

We begin by putting down the standard static input-output model that is assumed to be the economy-wide planning problem. Let there be n fully disaggregated commodities ($i, j = 1, \ldots, n$), and let $x=[x_i]$, $y=[y_i]$ and $A=(a_{ij})$ denote respectively, fully disaggregated output n-column vector, final demand n-column vector, and input-output n by n matrix appropriate to the given economy. For convenience we assume in the sequel that the nonnegative matrix A is productive and indecomposable, and that y is semi-positive. The model is written as

$$x = Ax + y, \tag{1}$$

and the unique positive solution to this equation is given by

$$x^* = (I-A)^{-1}y. \tag{2}$$

The old iterative method for finding a well-balanced output near x^* is described as follows: at round t the target is specified to be

$$x_t = Ax_{t-1} + y. \quad (t=1, 2, \ldots) \tag{3}$$

In view of the assumptions of the input-output matrix the process (3) should converge for any initial output vector x_0 with nonnegative elements. It can also be stated that its convergence speed is generally determined by the magnitude of the Frobenius eigenvalue λ^* of the matrix A, and of the distance between x_0 and x^* [see Manove and Weitzman (1978), and Montias (1959)].

These statements may be clarified as follows.

Define $E_t = x_t - x^*$, and it follows from (1) and (3) that

$$E_t = A^t E_0. \quad (t=1, 2, \ldots) \tag{4}$$

Since A is productive, we have $A^t \to 0$ $(t \to \infty)$, and $E_t \to 0$ $(t \to \infty)$ for any E_0; $x_t \to x^*$ $(t \to \infty)$ for any x_0. Further, let P and Λ be the matrix of row eigenvectors of A and the diagonal matrix of the corresponding eigenvalues; let P_i and $\bar{P}_i$ denote the ith row of P and the ith column of P^{-1}, and let λ_i be the ith diagonal of $\Lambda(\lambda_1 = \lambda^*)$. If an additional assumption is made that A is diagonalizable, we then have $A = P^{-1}\Lambda P$. Hence we can rewrite (4) as

$$E_t = P^{-1}\Lambda^t P E_0 = \sum_{i=1}^{n} \lambda_i^t (P_i E_0) \bar{P}_i. \tag{5}$$

$\bar{P}_1$ is strictly positive since A is indecomposable. Suppose that $P_1 E_0 \neq 0$, and the absolute value of the first term of (5) is also strictly positive. Because the magnitude of the eigenvalues λ's other than λ^* is less than λ^*, the geometrical decline factor of the slowest damping term is λ^*. Accordingly, we can evaluate the rate of convergence of (3) by the unique, positive Frobenius eigenvalue of A, λ^*.

We next turn to the new iterative process for solving (1). The simplest version of this process can be written as

$$\begin{aligned} x_t &= Az_t x_{t-1} + y, \\ z_t &= \frac{py}{p(x_{t-1} - Ax_{t-1})} = \frac{px_t}{px_{t-1}}. \quad (t=1, 2, \ldots) \end{aligned} \tag{6}$$

where p denotes the n-row vector of aggregation weights. As (1) is defined in value terms, p may be specified as the n-row vector e whose elements are all unity. The advantage of (6) may lie in that the convergence will be speedier if we go beyond the old method correction by using the single parameter z: if $z > 1$ we are 'overcorrecting'; if $z < 1$ we are 'undercorrecting'; if $z = 1$ the new process mirrors the old process (3). Hence, the new process may be more effective when the initial output x_0 is far from the solution x^*; $x_0 \gg x^*$ or $x_0 \ll x^*$. It should be noted that a similar idea is employed in the successive overrelaxation (SOR) method [see Berman (1979, ch. 7)].

We now proceed to the aspect of convergence speed of (6). Define

$$w_t = x_t / ex_t,$$

and

$$M = A + (ey)^{-1} y \cdot (e - eA).$$

Then the process (6) can be written in the form

$$w_t = Mw_{t-1} = \ldots = M^t w_0. \tag{7}$$

As A is nonnegative, $e \geqq eA$. Considering the semipositiveness of y and $eM = e$, we find that M is a column stochastic matrix. As A is indecomposable, M is also indecomposable. Therefore $w_t \to w^*$ $(t \to \infty)$ for any positive w_0, and $x_t \to x^*$ for any positive x_0. Since the Frobenius eigenvalue of M equals unity, we should evaluate the convergence speed of (6), or (7) by the second largest magnitude of eigenvalues of M [see Howard (1960, ch. 1)]. It should be noted that we are assuming that the speed of convergence of w_t in the new process is proportional to that of x_t in the new process. This assumption seems to be plausible, for it is not easy to find examples where x_t converges slowly and w_t converges very fast in the same process. For instance, if x_t is a scalar, both x_t and w_t in the new process converge to x^* and w^*, respectively, on its first round.

III. *A Comparison of Convergence Speed of Two Processes*

For a nonnegative matrix A we have the inclusion [Nikaido (1968, Theorem 7.5)]:

$$\min_j \sum_{i=1}^{n} a_{ij} \leqq \lambda^* \leqq \max_j \sum_{i=1}^{n} a_{ij} \ (j=1, \ldots, n). \tag{8}$$

It should be noted that if we define the norm of A as l_1-norm we have $||A|| = \max_j \sum_{i=1}^{n} a_{ij}$. On the other hand, we have at least two upper bounds for the magnitude of an eigenvalue $\rho \neq 1$ of a column stochastic matrix $M=(m_{ij})$ [see Seneta (1981, Theorem 2.10)] and [Berman (1979, Theorem 5.10)]:

$$|\rho| \leqq \frac{1}{2} \max_{i,j} \sum_{s=1}^{n} |m_{si} - m_{sj}|; \tag{9}$$

$$|\rho| \leqq \min \{1 - \sum_{i=1}^{n} \min_j (m_{ij}), \sum_{i=1}^{n} \max_j (m_{ij}) - 1\}. \tag{10}$$

Letting the right sides of (9) and (10) denote B_1 and B_2, we have $|\rho| \leqq B_k (k=1, 2)$. Hence, considering (8), it is sufficient for the speedier convergence of the new process (6) to state that $B_k < \min_j \sum_{i=1}^{n} a_{ij}$ for either $k=1$ or 2. If we employ (9) and (10), in view of the definition of M, we will be able to show that sufficient conditions for the faster convergence of (6) can be presented only by a_{ij}'s, excluding y_i's. The results are summarized in the following:

Proposition

The new process (6) *converges more speedily than the old process* (3) *if one of the following conditions is fulfilled:*

[a] $\max_j \sum_{s=1}^{n} a_{sj} < 3 \min_i \sum_{s=1}^{n} a_{si} - \max_{i,j} \sum_{s=1}^{n} |a_{si} - a_{sj}|$;

[b] $\min \{\max_j \sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} \min_j a_{ij}, \sum_{i=1}^{n} \max_j a_{ij} - \min_j \sum_{i=1}^{n} a_{ij}\} < \min_j \sum_{i=1}^{n} a_{ij}$.

Proof Noting that in view of the definition of M

$$m_{ij} = a_{ij} + (\sum_k y_k)^{-1} y_i \cdot (1 - \sum_{i=1}^{n} a_{ij}),$$

we then have:

$$B_1=\frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|m_{si}-m_{sj}|$$

$$=\frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|a_{si}-a_{sj}+(\sum_k y_k)^{-1}\cdot y_s(\sum_s a_{sj}-\sum_s a_{si})|$$

$$\leqq\frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|a_{si}-a_{sj}|+\frac{1}{2}\max_{i,j}(\sum_s a_{sj}-\sum_s a_{si})$$

$$=\frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|a_{si}-a_{sj}|+\frac{1}{2}(\max_j \sum_s a_{sj}-\min_i \sum_s a_{si})$$

$$<\frac{1}{2}(3\min_i \sum_s a_{si}-\max_j \sum_s a_{sj})+\frac{1}{2}(\max_j \sum_s a_{sj}-\min_i \sum_s a_{si}) \qquad \text{(using [}a\text{])}$$

$$<\min_i \sum_{s=1}^{n} a_{si}.$$

Hence, $|\rho|\leqq B_1<\lambda^*$ $(\rho\neq 1)$ if the condition [*a*] holds. Let us next see the case of condition [*b*]. As is easily verified, we can have

$$1-\sum_{i=1}^{n}\min_j(m_{ij})=1-\sum_i \min_j \{a_{ij}+(\sum_k y_k)^{-1}y_i(1-\sum_i a_{ij})\}$$

$$=1-\sum_i \min_j a_{ij}-(\sum_k y_k)^{-1}\cdot\sum_i y_i\cdot\min(1-\sum_i a_{ij})$$

$$=\max\sum_{i=1}^{n}a_{ij}-\sum_{i=1}^{n}\min a_{ij},$$

and in an analogous manner

$$\sum_{i=1}^{n}\max(m_{ij})-1=\sum_{i=1}^{n}\max_j a_{ij}-\min_j\sum_{i=1}^{n}a_{ij}.$$

This shows that B_2 is equal to the lefthand side of condition [*b*]; $B_2<\min_j\sum_{i=1}^{n}a_{ij}$. Proposition follows. Q.E.D.

We may give clear economic meanings to these conditions. Let us begin by condition [*b*]. Basically every row of an input-output matrix has at least one very small component so that $\min_j a_{ij}$ is basically equal to zero, hence $\max_j \sum_{i=1}^{n} a_{ij}-\sum_{i=1}^{n}\min_j a_{ij}=\max_j \sum_{i=1}^{n} a_{ij}$, and is never smaller than $\min_j \sum_{i=1}^{n} a_{ij}$. The condition therefore reduces to

$$\sum_{i=1}^{n}\max_j a_{ij}<2\min_j\sum_{i=1}^{n}a_{ij}.$$

This, seems a very strong condition.

Similarly, condition [*a*] is stronger than $\max_j \sum_{s=1}^{n} a_{sj} < 3 \min_i \sum_{s=1}^{n} a_{si}$, which implies that the value of the inputs of the industry who uses the most inputs is smaller than three times the value of the inputs to that industry which uses the smallest amount. A very strong condition indeed!

Speed of convergence of planning procedures consists of two factors: adjustment speed of output scale and that of relative output ratios. Concerning scale adjustment, the new process is very effective. This was already shown in my article [Kuboniwa (1982)]. Further, if we employ the von Neumann aggregation weights, the new process is effective in both the scale and relative ratios adjustment [Kuboniwa (1982), Manove and Weitzman (1978)]. However, as the above conditions suggest, in a general setting of theoretical frame work, the new process may not be as effective in the adjustment of relative output ratios as in that of output scale.

IV. *Concluding Remarks*

We obtained two sufficient conditions for the faster convergence of the new iterative process which we call the step-wise aggregation process. We clarified that these conditions are very strong from economic view points. This economic interpretation suggests that the new process may not be as efficient in the adjustment of relative output ratios as in that of output scale. Let us conclude this note with the following additional remarks. First, unlike the SOR method, we can assign clear economic meaning to the step-wise aggregation process presented here [see Kuboniwa (1982)]. Secondly, this note shows that, unlike modern textbooks on nonnegative matrices [see e.g., Seneta (1981, 2.2, 2.5 and 7.5)], an iterative method for the solution of a linear system is closely related to the 'probability algorithm' and the theory of eigenvalues of a stochastic matrix.

ACKNOWLEDGMENT

This work was supported by the Nippon Keizai-Kenkyu Shorei Zaidan (Japan Foundation for Encouraging Economic Research) Grant in 1982.

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