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ON EXISTENCE OF OPTIMAL PROGRAMS OF CAPITAL ACCUMULATION WITH EXHAUSTIBLE RESOURCES

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I. *Introduction*

In the present paper we shall consider a model of capital accumulation with exhaustible resources and prove the existence of an optimal program of capital accumulation and resource depletion. The result in this paper is a generalization of the theorem proved by Mitra (1980, thm. 5.1) and is also an extension of the existence theorem of an overtaking optimal program proved by Brock and Haurie (1976, thm. 4.2) to the case of economies with exhaustible resources.

The problem on the existence of optimal resource depletion has been considered by Solow (1974) and Dasgupta and Heal (1979) in specific models, and by Mitra (1980) in a more general model. Their models include one capital good and one exhaustible resource, and they closely examined the substitutability between the capital good and the exhaustible resource, which implies the existence of an optimal program. In this paper we shall consider the problem in a multi-sector economy with many exhaustible resources and prove that an optimal program of resource depletion exists if the exhaustibility of resources can be overcome by capital accumulation. One of the most important assumptions for our existence theorem is that the technology is enough developed for capital accumulation to offset the effect of resource depletion, that is, the exhaustible resources are "inessential" in the sense of Dasgupta and Heal (1979).

The present paper is composed in the following way. In section II mathematical notation will be explained and a technical lemma will be presented, whose proof will be given in Appendix. In section III we shall present a stationary model of capital accumulation, which is pretty standard in growth theory, and show some basic properties of the model. In section IV we shall prove the existence of an overtaking optimal program of resource depletion and capital accumulation. Also we shall discuss the equivalence of our model with that of Mitra (1980). In section V the case of discounted utilities will be considered supplementarily and the existence of an optimal program will be proved in a simple manner. In section VI all the lemmas in sections III and IV will be proved.

II. *Mathematical Preliminaries*

Let N denote the set of all positive integers. For each $n \in N$, R^n denotes an n-dimensional Euclidean space. By $R^n{}_+$ we denote the nonnegative orthant of R^n. When $n=1$, we write R and R_+ instead of R^1 and $R^1{}_+$. To denote R_+, symbol $[0, \infty)$ is also used. Also, $R^n{}_-$ is used for $-R^n{}_+$. For any $x, y \in R^n$, $x \geqq y$ means $x - y \in R^n{}_+$. The inner product of vectors x and y is denoted by $x \cdot y$. The Euclidean norm of any $x \in R^n$ is denoted by $||x||$, i.e., $||x|| = \sqrt{x \cdot x}$.

For each $n \in N$, $L_1{}^n$ denotes the set of all integrable functions from $[0, \infty)$ to R^n, and $L_\infty{}^n$ the set of all essentially bounded measurable functions from $[0, \infty)$ to R^n. When $n=1$, we write L_1 and L_∞ instead of $L_1{}^1$ and $L_\infty{}^1$. The norm of any $f \in L_1{}^n$ (or $L_\infty{}^n$) is denoted by $||f||_1$ (or $||f||_\infty$).

For each $g \in L_1{}^n$, a linear map from $L_\infty{}^n$ to R can be defined by

$$f \to \int_0^\infty f(t) \cdot g(t) dt.$$

Therefore, by abusing the notation, $L_1{}^n$ can be regarded as a set of some linear maps from $L_\infty{}^n$ to R. There are some locally convex topologies for space $L_\infty{}^n$ in which $L_1{}^n$ is the set of *all* continuous linear maps from $L_\infty{}^n$ to R. Of such topologies, the weakest one is called "wear-star topology" [see Dunford and Schwartz (1957, p. 289, thm. 5, p. 421, thm. 9, and p. 462)].

Remarks 2.1. The relative weak-star topology for any bounded subset of $L_\infty{}^n$ is metrizable [see Dunford and Schwartz (1957, p. 426, thm. 1)].

Remarks 2.2. Any bounded subset of $L_\infty{}^n$ is relatively weak-star compact by Alaoglu's theorem [see Dunford and Schwartz (1957, p. 424, thm. 2)].

The following lemma is useful in proving Lemmas 3.2 and 4.3 later.

Lemma 2.1. Let $\{f_\nu\}_{\nu \in N}$ be a sequence in $L_\infty{}^n$ converging to a point $f_0 \in L_\infty{}^n$ in the weak-star topology. Then, there exists a sequence $\{\bar{f}_\nu\}_{\nu \in N}$ in $L_\infty{}^n$ such that each $\bar{f}_\nu$ is a convex combination of some elements in $\{f_i | i \geqq \nu\}$ and such that $\bar{f}_\nu$ converges to f_0 almost everywhere.

In this paper, any function f: $[0, \infty) \to R^n$ is said to be *absolutely continuous* if the restriction of f on any compact interval is absolutely continuous in the usual sense. Also, the derivative of f is denoted by $\dot{f}$.

III. *The Stationary Model*

We shall consider a model of economic growth, in which there are l kinds of exhaustible resources and m kinds of capital goods. The technology of the economy is described by a subset T of $R \times R^m \times R^l{}_- \times R^m$. The expression, $(\alpha, x, y, z) \in T$, where $\alpha \in R, x \in R^m, y \in R^l$, and $z \in R^m$, means that if amount x of capital goods exist and amount $-y$ of exhaustible resources are depleted, then level α of social satisfaction and level z of capital accumulation can be realized. Namely, T is the set of all possible combinations of social utility level, capital stock, resource depletion, and investment level. It has been already assumed that resources can not be produced and that utility level cannot be affected by quantity of existing resources, but by level of resource depletion.

Let w_0 and x_0 denote the initial endowments of resources and capital goods respectively. The economy continues from time 0 to ∞. To describe the state of the economy at each point in time, we shall use a triplet (u, r, k) of a measurable function u: $[0, \infty)\rightarrow R$, absolutely continuous functions r: $[0, \infty)\rightarrow R^l$ and k: $[0, \infty)\rightarrow R^m$. That is, $u(t)$ denotes level of social utility at time t, $r(t)$ resource stock, $-\dot{r}(t)$ resource depletion, $k(t)$ capital stock, and $\dot{k}(t)$ investment. Such a triplet (u, r, k) is said to be *a feasible program* if the following hold:

$r(0)=w_0$ and $k(0)=x_0$.

$(u(t), k(t), \dot{r}(t), \dot{k}(t))\in T$ for almost every $t\in[0, \infty)$.

$-\int_0^\infty \dot{r}(t)dt\leqq w_0$ (equivalently $r(t)\geqq 0$ for all $t\in[0, \infty)$).

And let $\boldsymbol{P}$ denote the set of all feasible programs.

Assumption 1. The technology T satisfies the following conditions:

(i) T is a closed convex subset of $R\times R^m\times R^l\times R^m$.

(ii) For all $\varepsilon>0$ there exists $\delta>0$ such that $(\alpha, x, y, z)\in T$ and $||x||\leqq\varepsilon$ imply $||(\alpha, y, z)||\leqq\delta$.

(iii) There exists a number $\bar{\beta}>0$ such that $(\alpha, x, y, z)\in T$ and $||x||\geqq\bar{\beta}$ imply $x\cdot z\leqq 0$.

The above assumption is pretty standard and is commonly used to insure the uniform boundedness of feasible programs.

Lemma 3.1. There exists a number $\bar{b}>0$ such that $||u||_\infty\leqq\bar{b}$, $||\dot{r}||_\infty\leqq\bar{b}$, $||\dot{k}||_\infty\leqq\bar{b}$, $||r||_\infty\leqq\bar{b}$, and $||k||\leqq\bar{b}$ for all $(u, r, k)\in\boldsymbol{P}$.

Define a set $\boldsymbol{F}$ by

$$\boldsymbol{F}=\{(u, \dot{r}, \dot{k}) \mid (u, r, k)\in\boldsymbol{P}\}.$$

Then, clearly, a natural map from $\boldsymbol{P}$ to $\boldsymbol{F}$ defined by

$$(u, r, k)\rightarrow(u, \dot{r}, \dot{k})$$

is one to one and onto. Therefore we can identify $\boldsymbol{P}$ with $\boldsymbol{F}$. Thus, since $\boldsymbol{F}$ is a bounded subset of $L_\infty\times L_\infty{}^l\times L_\infty{}^m$ by Lemma 3.1, we can regard $\boldsymbol{P}$ as a bounded subset of $L_\infty\times L_\infty{}^l\times L_\infty{}^m$. In addition, we can show a certain compactness of $\boldsymbol{P}$.

Lemma 3.2. $\boldsymbol{P}$ can be regarded as a convex and weak-star compact subset of $L_\infty\times L_\infty{}^l\times L_\infty{}^m$, i.e., $\boldsymbol{F}$ is a convex and weak-star compact subset of $L_\infty\times L_\infty{}^l\times L_\infty{}^m$.

IV. *Existence of Optimal Programs*

A feasible program (u, r, k) is said to be *overtaken* by another feasible program (u', r', k') if there exist $\varepsilon>0$ and $t_0\in[0, \infty)$ such that

$$\int_0^s u'(t)dt>\int_0^s u(t)dt+\varepsilon \text{ for all } s\in[t_0\ \infty).$$

A feasible program is said to be *an optimal program* if it is not overtaken by any other feasible program.

In order to prove the existence of optimal programs, we shall assume the existence of a unique optimal steady state dispensing with exhaustible resources:

Assumption 2. There exists $(\bar{\alpha}, \bar{x}, 0, 0) \in T$ satisfying the following:
(i) $\bar{\alpha} \geqq \alpha$ for all $(\alpha, x, y, 0) \in T$.
(ii) $(\bar{\alpha}, x, 0, 0) \in T$ implies $x = \bar{x}$.

The above assumption insures that utility level $\bar{\alpha}$ can be maintained forever without resources if we have capital stock $\bar{x}$. In other words, amount $\bar{x}$ of capital goods can be completely substituted for exhaustible resources. Condition (i) says that utility level greater than $\bar{\alpha}$ cannot be maintained forever even if we have any amount of exhaustible resources and capital goods. Condition (ii) implies the uniqueness of such capital stock $\bar{x}$.

A feasible program (u, r, k) is said to be *good* if

$$\liminf_{s \to +\infty} \int_0^s (u(t) - \bar{\alpha})dt > -\infty.$$

Assumption 3. There exists at least one good program, say (u_0, r_0, k_0).

This assumption says that the technology is good enough to keep utility level $u_0(t)$ close to $\bar{\alpha}$ at almost all point in time. Namely, capital accumulation offsets resource depletion. In this sense, the exhaustible resources are not essential in the economy.

Finally we assume a kind of richness of the technology.

Assumption 4. $0 \in \text{int } \{z \in R^m | (\alpha, x, y, z) \in T\}$.

We should compare our assumptions with those of Mitra (1980). First, in his model, the economy is unbounded, that is, the capital stock can grow indefinitely large. On the other hand, in this paper we assume by Assumption 1 that the economy is bounded. However, this difference is only a matter of the measurement of capital stock. By changing the way of capital measurement, his model can be transformed to a "bounded" model.

Second, in Mitra's model, there is no optimal steady state, while we assume the existence of the optimal steady state $\bar{x}$ in Assumption 2. In his model exhaustible resources are indispensable for production and they cannot be replaced by any finite amount of capital stock. But, if a huge amount of capital goods exist, only a small amount of exhaustible resources are necessary for production. Namely, intuitively speaking, an infinite amount of capital goods can be substituted for exhaustible resources. Thus, capital stock $\bar{x}$ in our model corresponds to the infinite amount of capital stock in his model. This is why there is no steady state in his model, which comes from the difference of mathematical formulation. By changing the way of capital measurement and by "compactifying" the production technology, his model can be reduced to our model.

Third, the exhaustible resources are indispensable for production in Mitra's model, while in our model amount $\bar{x}$ of capital stock can be completely substituted for exhaustible resources. However, we do not exclude the indispensability of exhaustible resources for production, for we do not assume the optimal steady state $\bar{x}$ can be reached from the initial capital stock x_0 in finite time. In Mitra's model capital stock cannot become infinite in finite time. Since the infinite amount of capital stock in his model corresponds to capital stock $\bar{x}$ in our model as was pointed out, there is no virtual difference between his model and our model concerning the indispensability of exhaustible resources for production. All we assume in this paper is Assumption 3, the existence of a good program, which corresponds to the inessentiality of exhaustible resources in the sense of Dasgupta and Heal (1979). In Mitra's paper, he showed an explicit condition on the curvitures of the production function and the utility function, from which the inessentiality of exhaustible resources can be derived.

We can prove the following lemmas and corollaries.

Lemma 4.1. If (u, r, k) is a good program, then

$$\lim_{s\to+\infty} \frac{1}{s}\int_0^s k(t)dt = \bar{x}.$$

Lemma 4.2. There exists $\bar{p}\in R^m$ such that $\bar{\alpha}\geqq\alpha+\bar{p}\cdot z$ for all $(\alpha, x, y, z)\in T$.

Lemma 4.3. There exists $(u_*, r_*, k_*)\in \boldsymbol{P}$ such that

$$\int_0^\infty(\bar{\alpha}-u(t)-\bar{p}\cdot\dot{k}(t))dt \geqq \int_0^\infty(\bar{\alpha}-u_*(t)-\bar{p}\cdot\dot{k}_*(t))dt$$

for all $(u, r, k)\in \boldsymbol{P}$.

Corollary 1. For all $(u, r, k)\in \boldsymbol{P}$ and $\varepsilon>0$, there exists $s_0\in[0, \infty)$ such that

$$\bar{p}\cdot(k_*(s)-k(s))+\varepsilon \geqq \int_0^s u(t)dt - \int_0^s u_*(t)dt \text{ for all } s\in[s_0, \infty).$$

Corollary 2. (u_*, r_*, k_*) is a good program, i.e.,

$$\liminf_{s\to+\infty} \int_0^s (u_*(t)-\bar{\alpha})dt > -\infty.$$

We are now ready to prove the existence of an optimal program.

Theorem 4.1. Under Assumptions 1, 2, 3, and 4, there exists an optimal program.

Proof: We shall show that program (u_*, r_*, k_*) in Lemma 4.3 is an optimal program.

Suppose that a feasible program (u, r, k) overtake program (u_*, r_*, k_*), i.e., there exist $\varepsilon_0>0$ and $t_0\in[0, \infty)$ such that

$$\int_0^s u(t)dt > \int_0^s u_*(t)dt + \varepsilon_0 \text{ for all } s\in[t_0, \infty).$$

Then, since (u_*, r_*, k_*) is a good program according to Corollary 2 of Lemma 4.3, (u, r, k) is also a good program. Moreover, by Corollary 1 of Lemma 4.3, for any fixed number $\varepsilon>0$ there exists $s_0\in[0, \infty)$ such that

$$\bar{p}\cdot(k_*(z)-k(s))+\varepsilon \geqq \int_0^s u(t)dt - \int_0^s u_*(t)dt \text{ for all } s\in[s_0, \infty).$$

Without loss of generality, we can assume that $s_0\geqq t_0$. Therefore, the above two inequalities imply that $\bar{p}\cdot(k_*(s)-k(s))+\varepsilon\geqq\varepsilon_0$ for all $s\in[s_0, \infty)$. Hence, by integration we have

$$\int_0^s \bar{p}\cdot(k_*(t)-k(t))\,dt-\int_0^{s_0}\bar{p}\cdot(k_*(t)-k(t))\,dt+(s-s_0)\varepsilon$$
$$\geqq (s-s_0)\varepsilon_0 \text{ for all } s\in[s_0,\ \infty), \text{ i.e.,}$$
$$\bar{p}\cdot\left(\frac{1}{s}\int_0^s k_*(t)dt-\frac{1}{s}\int_0^s k(t)dt\right)-\frac{1}{s}\int_0^{s_0}\bar{p}\cdot(k_*(t)-k(t))dt$$
$$+\frac{1}{s}(s-s_0)\varepsilon\geqq\frac{1}{s}(s-s)\varepsilon_0 \text{ for all } s\in[s_0,\ \infty).$$

Since $(u_*,\ r_*,\ k_*)$ and $(u,\ r,\ k)$ are good programs, by Lemma 4.1 we can conclude that $\varepsilon\geqq\varepsilon_0$. In addition, since ε is an arbitrary positive number, it follows that $\varepsilon_0=0$. This is a contradiction. Thus, no program overtakes program $(u_*,\ r_*,\ k_*)$. *Q.E.D.*

Of course, the above existence theorem includes the case in which there is no exhaustible resource in the economy, i.e., $w_0=0$. In such a case the above theorem corresponds, with minor differences, to the theorem proved by Brock and Haurie (1976), and also to the theorems by Gale (1967), McKenzie (1968), and Brock (1970) in descrete time models.

V. *The Discounted Utilities*

In this section we shall consider the case in which future utilities are discounted. Let δ be a positive number, which may be called a discount rate. By Lemma 3.1 we know that $\int_0^\infty u(t)e^{-\delta t}dt$ is finite for all $(u,\ r,\ k)\in \boldsymbol{P}$. Therefore we can define the following optimality criterion. A feasible program $(u,\ r,\ k)$ is said to be *δ-optimal* if $\int_0^\infty u(t)e^{-\delta t}dt\geqq\int_0^\infty u'(t)e^{-\delta t}dt$ for all feasible program $(u',\ r',\ k')$.

Theorem 5.1. Under Assumption 1, for any $\delta>0$ there exists a δ-optimal program, provided that $\boldsymbol{P}\neq\phi$.

Proof: Let $U=\{u \mid (u,\ r,\ k)\in\boldsymbol{P}\}$. Then, by Lemma 3.2, U is a weak-star compact subset of L_∞. Also, a map defined by $t\to e^{-\delta t}$ is an integrable function, i.e., an element of L_1. Hence, a map from U to R defined by

$$u\to\int_0^\infty u(t)e^{-\delta t}dt$$

is weak-star continuous. Therefore, the maximum of the function is attained at a point in U. Thus a δ-optimal program exists. *Q.E.D.*

We should note that the above theorem requires only Assumption 1, which is weak and is commonly used. The existence of exhaustible resources does not make any trouble in proving the existence of optimal programs if the utility is discounted. In fact, the above theorem is essentially the same as the usual existence theorem of optimal growth paths [see, for example, Takekuma (1980)].

VI. *Proofs of Lemmas*

Proof of Lemma 3.1. Let $(u, r, k)\in \boldsymbol{P}$. Then $r(0)=w_0$, $k(0)=x_0$, and $r(t)\geqq 0$, $(u(t), k(t), \dot{r}(t), \dot{k}(t))\in T$ for almost every $t\in[0, \infty)$. Since $\dot{r}(t)\leqq 0$ for almost every $t\in[0, \infty)$, $||r||_\infty\leqq ||w_0||$. Also, by Assumption 1 (iii), there exists a number $\bar{\beta}>0$ such that $k(t)\cdot\dot{k}(t)\leqq 0$ for almost every $t\in[0, \infty)$ with $||k(t)||\geqq\bar{\beta}$. Namely, $\frac{d}{dt}(||k(t)||)=\frac{1}{||k(t)||}(k(t)\cdot\dot{k}(t))\leqq 0$ for almost every $t\in[0, \infty)$ with $||k(t)||\geqq\bar{\beta}$. Therefore, since k is an absolutely continuous function, we can conclude that $||k(t)||\leqq\max\{\bar{\beta}, ||x_0||\}$ for all $t\in[0, \infty)$. Hence, by Assumption 1 (ii), there exists $\delta>0$ such that $||(u(t), \dot{r}(t), \dot{k}(t))||\leqq\delta$ for almost every $t\in[0, \infty)$. By putting $\bar{b}=\max\{||w_0||, \bar{\beta}, ||x_0||, \delta\}$, we complete the proof of the lemma. *Q.E.D.*

Proof of Lemma 3.2. From Lemma 3.1, it follows that $\boldsymbol{F}$ is a bounded subset of $L_\infty\times L_\infty{}^l\times L_\infty{}^m$. Therefore, $\boldsymbol{F}$ is relatively weak-star compact [see Remark 2.2]. Also, the convexity of $\boldsymbol{F}$ immediately follows from the convexity of technology T. Thus, it suffices only to prove that $\boldsymbol{F}$ is weak-star closed. In proving the closedness, we can use sequences, because the weak-star topology for $\boldsymbol{F}$ is metrizable [see Remark 2.1].

Let $\{(u_n, f_n, g_n)\}_{n\in N}$ be a sequence in $\boldsymbol{F}$ converging to a point $(u_0, f_0, g_0)\in L_\infty\times L_\infty{}^l\times L_\infty{}^m$ in the weak-star topology. We shall prove that $(u_0, f_0, g_0)\in\boldsymbol{F}$.

For each $n\in N$, or $n=0$, define functions r_n: $[0, \infty)\rightarrow R^l$ and k_n: $[0, \infty)\rightarrow R^m$ by

$$r_n(t)=w_0+\int_0^t f_n(s)ds \quad\text{and}\quad k_n(t)=x_0+\int_0^t g_n(s)ds.$$

Then, by definition of $\boldsymbol{F}$, for each $n\in N$

$(u_n(t), k_n(t), \dot{r}_n(t), \dot{k}_n(t))\in T$ for almost every $t\in[0, \infty)$ and $r_n(t)\geqq 0$ for all $t\in[0, \infty)$.

Apply Lemma 2.1 to the sequence $\{(u_n, f_n, g_n)\}_{n\in N}$, and we have a sequence $\{(\bar{u}_n, \bar{f}_n, \bar{g}_n)\}_{n\in N}$ in $L_\infty\times L_\infty{}^l\times L_\infty{}^m$ such that each $(\bar{u}_n, \bar{f}_n, \bar{g}_n)$ is a convex combination of some elements in $\{(u_i, f_i, g_i)\mid i\geqq n\}$ and such that $(\bar{u}_n, \bar{f}_n, \bar{g}_n)$ converges to (u_0, f_0, g_0) almost everywhere. For each $n\in N$, define functions e_n: $[0, \infty)\rightarrow R^l$ and h_n: $[0, \infty)\rightarrow R^m$ by

$$e_n(t)=w_0+\int_0^t \bar{f}_n(s)ds \quad\text{and}\quad h_n(t)=x_0+\int_0^t \bar{g}_n(s)ds.$$

Then, since each $(\bar{u}_n, e_n, h_n)$ is a convex combination of some elements in $\{(u_i, r_i, k_i)\mid i\geqq n\}$, it follows from Assumption 1 (i) that for each $n\in N$

$(u_n(t), h_n(t), \dot{e}_n(t), \dot{h}_n(t))\in T$ for almost every $t\in[0, \infty)$ and $e_n(t)\geqq 0$ for all $t\in[0, \infty)$.

Since $(\bar{u}_n, \bar{f}_n, \bar{g}_n)$ converges to (u_0, f_0, g_0) almost everywhere

$(\bar{u}_n(t), \dot{e}_n(t), \dot{h}_n(t))$ converges to $(u_0(t), \dot{r}_0(t), \dot{k}_0(t))$ for almost every $t\in[0, \infty)$.

Also, since (f_n, g_n) converges to (f_0, g_0) in the weak-star topology, $(r_n(t), k_n(t))$ converges to $(r_0(t), k_0(t))$ for each $t\in[0, \infty)$. Therefore, since (e_n, h_n) is a convex combination of some elements in $\{(r_i, k_i)\mid i\geqq n\}$,

$(e_n(t), h_n(t))$ converges to $(r_0(t), k_0(t))$ for each $t\in[0, \infty)$.

Thus, since T is closed, we can conclude that

$(u_0(t), k_0(t), \dot{r}_0(t), \dot{k}_0(t))\in T$ for almost every $t\in[0, \infty)$ and $r_0(t)\geqq 0$ for all $t\in[0, \infty)$.

This proves that $(u_0, r_0, k_0)\in\boldsymbol{P}$, i.e., $(u_0, f_0, g_0)\in\boldsymbol{F}$. *Q.E.D.*

Proof of Lemma 4.1. Let (u, r, k) be a good program. Then, by Lemma 3.1, $\left\|\frac{1}{s}\int_0^s u(t)dt\right\| \leqq \bar{b}$ and $\left\|\frac{1}{s}\int_0^s k(t)dt\right\| \leqq \bar{b}$ for all $s \in [0, \infty)$.

Let $\alpha_* \in R$, $x_* \in R^m$, and $\{s_n\}_{n \in N}$ be a sequence in $[0, \infty)$ with $\lim_{n\to\infty} s_n = +\infty$ such that

$$\lim_{n\to\infty}\frac{1}{s_n}\int_0^{s_n} u(t)dt = \alpha_* \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{s_n}\int_0^{s_n} k(t)dt = x_*.$$

Then, by Lemma 3.1,

$$\lim_{n\to\infty}\frac{1}{s_n}\int_0^{s_n} \dot{r}(t)dt = \lim_{n\to\infty}\frac{1}{s_n}(r(s_n)-r(0)) = 0 \quad \text{and}$$

$$\lim_{n\to\infty}\frac{1}{s_n}\int_0^{s_n} \dot{k}(t)dt = \lim_{n\to\infty}\frac{1}{s_n}(k(s_n)-k(0)) = 0.$$

Since technology T is convex and $(u(t), k(t), \dot{r}(t), \dot{k}(t)) \in T$ for almost every $t \in [0, \infty)$,

$$\left(\frac{1}{s_n}\int_0^{s_n} u(t)dt,\ \frac{1}{s_n}\int_0^{s_n} k(t)dt,\ \frac{1}{s_n}\int_0^{s_n} \dot{r}(t)dt,\ \frac{1}{s_n}\int_0^{s_n} \dot{k}(t)dt\right) \in T \text{ for all } n \in N.$$

Hence, since T is closed, $(\alpha_*, x_*, 0, 0) \in T$. In addition, since (u, r, k) is a good program, there exist a number G and $s_0 \in [0, \infty)$ such that

$$\int_0^s (u(t)-\bar{\alpha})dt \geqq G \text{ for all } s \in [s_0, \infty).$$

Therefore,

$$\alpha_* = \lim_{n\to\infty}\frac{1}{s_n}\int_0^{s_n} u(t)dt \geqq \lim_{n\to\infty}\frac{1}{s_n}\left(\int_0^{s_n} \bar{\alpha}dt + G\right) = \bar{\alpha}.$$

Thus, by Assumption 2, $\alpha_* = \bar{\alpha}$ and $x_* = \bar{x}$. This completes the proof of the lemma. *Q.E.D.*

Proof of Lemma 4.2. Define a set C by

$$C = \{(\alpha, z) \in R \times R^m \mid \alpha < \alpha' \text{ and } z = z' \text{ for some } (\alpha', x', y', z') \in T\}.$$

Then, C is convex and, by Assumption 2 (i), $(\bar{\alpha}, 0) \notin C$. Therefore, by a separation theorem, there exists $(\pi, p) \in R \times R^m$ with $(\pi, p) \neq 0$ such that $\pi\bar{\alpha} \geqq \pi\alpha + p\cdot z$ for all $(\alpha, z) \in C$. By construction of C, $\pi \geqq 0$ and

$$\pi\bar{\alpha} \geqq \pi\alpha + p\cdot z \text{ for all } (\alpha, x, y, z) \in T.$$

Suppose that $\pi = 0$. Then, $p\cdot z \leqq 0$ for all $(\alpha, x, y, z) \in T$. Therefore, by Assumption 4, $p = 0$. That is, $(\pi, p) = 0$, a contradiction. Hence, $\pi \neq 0$. Put $\bar{p} = \frac{1}{\pi}p$. This completes the proof. *Q.E.D.*

Proof of Lemma 4.3. Define

$$\mathrm{M} = \inf\left\{\int_0^\infty (\bar{\alpha} - u(t) - \bar{p}\cdot\dot{k}(t))dt \mid (u, r, k) \in \boldsymbol{P}\right\}.$$

Then, by Lemma 4.2, Assumption 3, and Lemma 3.1,

$$0 \leqq M \leqq \lim_{s\to\infty}\int_0^s (\bar{\alpha} - u_0(t) - \bar{p}\cdot\dot{k}_0(t))dt$$

$$= \lim_{s\to\infty}\left[\int_0^s (\bar{\alpha} - u_0(t))dt - \bar{p}\cdot(k_0(s) - k(0))\right] < +\infty.$$

Also, there exists a sequence $\{(u_n, r_n, k_n)\}_{n \in N}$ in $\boldsymbol{P}$ such that

$$\lim_{n\to\infty}\int_0^\infty(\bar{\alpha}-u_n(t)-p\cdot\dot{k}_n(t))dt=M.$$

Without loss of generality, by Lemma 3.2, we can assume that there exists $(u_*, r_*, k_*)\in\boldsymbol{P}$ and $(u_n, \dot{r}_n, \dot{k}_n)$ converges to $(u_*, \dot{r}_*, \dot{k}_*)$ in the weak-star topology. Apply Lemma 2.1 to the sequence $\{(u_n, \dot{r}_n, \dot{k}_n)\}_{n\in N}$, and we have a sequence $\{(\bar{u}_n, \bar{f}_n, \bar{g}_n)_{n\in N}$ in $L_\infty\times L_\infty{}^l\times L_\infty{}^m$ such that each $(\bar{u}_n, \bar{f}_n, \bar{g}_n)$ is a convex combination of some elements in $\{(u_i, \dot{r}_i, \dot{k}_i)\mid i\geqq n\}$ and such that $(\bar{u}_n, \bar{f}_n, \bar{g}_n)$ converges to $(u_*, \dot{r}_*, \dot{k}_*)$ almost everywhere. Therefore,

$$\lim_{n\to\infty}\int_0^\infty(\bar{\alpha}-\bar{u}_n(t)-\bar{p}\cdot\bar{g}_n(t))dt=M$$

and $(\bar{\alpha}-\bar{u}_n(t)-\bar{p}\cdot\bar{g}_n(t))$ converges to $(\bar{\alpha}-u_*(t)-\bar{p}\cdot\bar{g}(t))$ for almost every $t\in[0, \infty)$. Hence, by Fatou's lemma,

$$\int_0^\infty(\bar{\alpha}-u_*(t)-\bar{p}\cdot\dot{k}_*(t))dt\leqq M.$$

This implies the lemma. *Q.E.D.*

Proof of Corollary 1 of Lemma 4.3. Obvious from Lemma 4.3. *Q.E.D.*

Proof of Corollary 2 of Lemma 4.3. By Assumption 3 and Corollary 1 of Lemma 4.3, there exist $\varepsilon_0>0$ and $s_0\in[0, \infty)$ such that

$$\bar{p}\cdot(k_*(s)-k_0(s))+\varepsilon_0\geqq\int_0^s u_0(t)dt-\int_0^s u_*(t)dt \text{ for all } s\in[s_0, \infty).$$

Therefore,

$$\int_0^s(u_*(t)-\bar{\alpha})dt\geqq\int_0^s(u_0(t)-\bar{\alpha})dt-\varepsilon_0-\bar{p}\cdot(k_*(s)-k_0(s))$$

for all $s\in[s_0, \infty)$. Since (u_0, r_0, k_0) is a good program, together with Lemma 3.1 this clearly implies the corollary. *Q.E.D.*

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APPENDIX

Proof of Lemma 2.1. We shall use the following sublemma. For proof, see Takekuma (1980, lem. 2.1).

Sublemma. Let $\{f_\nu\}_{\nu\in N}$ be a sequence in $L_\infty{}^n$ converging to a point $f_0\in L_\infty{}^n$ in the weak-star topology. Then, for any $s\in[0, \infty)$ there exists a sequence $\{\bar{f}_\nu\}_{\nu\in N}$ in $L_\infty{}^n$ such that each $\bar{f}_\nu$ is a convex combination of some elements in $\{f_i \mid i\geqq\nu\}$ and such that $\bar{f}_\nu$ converges to f_0 almost everywhere in $[0, s]$.

Put $s=1$ in the sublemma. Then, we have a sequence $\{f_\nu{}^1\}_{\nu\in N}$ in $L_\infty{}^n$ such that each $f_\nu{}^1$ is a convex combination of some elements in $\{f_i \mid i\geqq\nu\}$ and such that $f_\nu{}^1$ converges to f_0 almost everywhere in $[0, 1]$. We can easily check that $f_\nu{}^1$ converges to f_0 in the weak-star topology. Therefore, again we can apply the sublemma to sequence $\{f_\nu{}^1\}_{\nu\in N}$. Put $s=2$ in the sublemma. Then, we have a sequence $\{f_\nu^2\}_{\nu\in N}$ in L_∞^n such that each f_ν^2 is a convex combination of some elements in $\{f_i^1 \mid i \geqq \nu\}$ and such that f_ν^2 converges to f_0 almost everywhere in [0, 2]. We can easily check that f_ν^2 converges to f_0 in the weak-star topology. Thus, again we can apply the sublemma to sequence $\{f_\nu^2\}_{\nu\in N}$.

Repeat this process. Then, we have sequences $\{f_\nu^s\}_{\nu\in N}$, where $s\in N$, such that each f_ν^s is a convex combination of some elements in $\{f_i \mid i \geqq \nu\}$ and such that f_ν^s converges to f_0 almost everywhere in [0, s]. Hence, by putting $\bar{f}_\nu = f_\nu^\nu$ for each $\nu\in N$, we have a sequence $\{\bar{f}_\nu\}_{\nu\in N}$ desired in Lemma 2.1. *Q.E.D.*

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