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COMPETITIVE FIRM STRUCTURES AND EQUILIBRIA IN A COALITION PRODUCTION ECONOMY†

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I. *Introduction*

R.J. Aumann (1964) introduced a continuum of traders in order to make precise the concept of perfect competition in a general equilibrium analysis of exchange economy. Since then W. Hildenbrand (1968, 1970), C. Oddou (1972), D. Sondermann (1974), and T. Ichiishi (1977) have introduced production in an Aumann economy. The idea, originated in a work by Hildenbrand (1968), is to consider a "coalition production economy." In such an economy a production set $Y(C)$ is assigned to each coalition C of consumers in such a way that if C is a null set of the given measure space of economic agents, then $Y(C)$ consists of the zero vector only, that is, positive productions are not possible for a null coalition of consumers.

The concept of a coalition production economy arose from the necessity of specifying production possibilities for every coalition of consumers in defining the core of a production economy. In the context of a market economy, however, there seem to be some difficulties in the interpretation of production set correspondence. Its suggested interpretation is that $Y(C)$ represents the production set of a coalition C as a production unit. It raises two difficulties.

First, the behavior of each coalition C as a production unit is not fully justified. On one hand, only the coalitions of positive measure are given their *raison d'être*, and on the other, every coalition of positive measure takes prices as given although it can affect the total outcome of economy. Second, unless the production set correspondence is additive or superadditive, the meaning of a production set $Y(C)$ is ambiguous. For example, if there are decreasing returns to coalition scale, then the production set assigned to the coalition of all consumers A as a production unit, $Y(A)$, may be strictly contained in the production set $Y(C)$ of a smaller coalition of consumers C. However, as a coalition of consumers, the coalition A should be able to achieve whatever its subcoalition can achieve. Hence, the set $Y(A)$ cannot be taken to represent what the coalition A can produce. Hildenbrand (1968, 1970) assumed the production set correspondence to be additive. In that case any measurable partition of A can be regarded as a set of production units. That is, it is essentially devoid of any "coalition structures." In this particular sense it has a Walrasian flavor. We should note, however, that the additivity of production set correspondence is different from constant returns to scale. It means constant returns to "coalition scale." Sondermann (1974) assumed the production set correspondence to be (strongly) superad-

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ditive. In this case the only effective production unit of the economy is the coalition of all consumers A.

The consideration of the second difficulty noted above suggests that we should distinguish what a coalition of consumers as a production unit can produce from what it can produce by organizing various production units among its members. We are thus led to introduce the concept of mean production set correspondence which assigns to each coalition C of consumers what it can produce by organizing various production units among them. The original datum of the production sector of an economy is the assignment of production sets to various coalitions of consumers as a production unit. By allowing variation in "production structures" we derive the mean production set correspondence of the economy. Intuitively one expects the mean production set correspondence to have superadditivity. This, in turn, suggests a way to provide a new interpretation of Sondermann's production economy (1974).

In order to avoid the first difficulty, one can either allow strategic behavior of producers, or introduce competitive firms. Considering difficulties of incorporating monopolistic or oligopolistic behaviors of firms in a general equilibrium model, we shall adopt the latter approach.

One basic problem of a large production economy becomes apparent when we consider competitive firms. We need to introduce a measure space of competitive firms explicitly. When describing the production sector on the basis of individual firms with an atomless measure space of consumers, one must face the difficulty of attaching appropriate weights to various firms. If one deals with only a finite number of firms as in a "private ownership economy" in Hildenbrand (1970), or with a countably many firms as in Ichiishi (1977), one may assume away this difficulty implicitly. One must note that in an economy with a continuum of agents aggregated amounts of commodities in the economy are measured in units per capita. This means that one is not allowed to introduce an arbitrary measure space of firms which is independent of the measure space of consumers; aggregated amounts of commodities in the production sector should also be measured in units per capita of consumers, not in units per capita of producers.

Formally, this paper is concerned with the problem of how to introduce competitive firms and competitive firms structures into a general equilibrium model with a measure space of economic agents in such a way that at a competitive eqilibrium the formation of particular firm structure is explained. The "production set correspondence" in a conventional "coalition production economy" can be given an interpretation as the mean production set correspondence in the framework of a production economy with competitive firm structures.

II. *Description of a Production Economy with Firm Structures*

1. *Economic Agents in a Production Economy—Producing Agents or Firms as Secondary Economic Agents.*

Primary economic agents are the foundation of all economic organizations in an economy. On one hand, each individual primary economic agent plays the role of a *consumer*. Hence, information concerning these agents is given by a specification of their individual needs, tastes, and initial commodity endowments. On the other hand, any other economic organizations or agents are considered to be formed by primary economic agents. In this sense economic agents other than consumers are secondary agents. Among secondary economic agents our sole interest here is in *producing agents* called *producers* or *firms*. In particular, our main effort lies in giving a microeconomic description of secondary economic agents and then formalizing production structures generated by these secondary economic agents in a competitive market economy.

We take a modified view of Hicksian production economy [see Hicks (1946, Chapters VI and VIII)]. In an exchange economy primary economic agents come to the market as trading individuals with supplies of certain commodities or services and they obtain other commodities in one way only—by exchange. In a production economy we need to "take into account the fact that they can sometimes obtain new commodities in another way—by technical transformation, or production. Clearly they will not adopt this method unless it is more advantageous than simple exchange; that means that it will only be advantageous to convert one set of exchangeable goods into another set, by production, if the set acquired has a higher market value than the set given up. Therefore *under different market conditions, different opportunities for production will become profitable; and these different opportunities may be open to different people.* In this way, the class of persons who acquire goods by technical transformation rather than by simple sale of their services (the class of entrepreneurs) may change" [Hicks (1946, p. 78); italics are mine].

Now, every primary economic agent "possesses supplies of one or both of two sorts of resources—(1) factors of production which can be disposed of on the market, (2) entrepreneurial resources which cannot be disposed of in that way, but which can be used, in combination with the other sort of factors, to produce disposable products. Given a set of market prices, for factors and products, any one who possesses entrepreneurial resources will be able to determine whether the utilization of those resources in production will yield a positive surplus. If it will do so, he becomes an entrepreneur. ... his demand for factors and supply of products (on business account) is determined; consequently the amount of his surplus is determined. This surplus now becomes part of his income on private account ..." [Hicks (1946, p. 100)]. Hicks restricted the class of entrepreneurs to single primary economic agents. Thus the problem of profit distribution did not arise. Namely, all the profits of a firm go to the single primary economic agent that functions as the entrepreneur.

In the description of a Hicksian production economy above, private and business accounts of primary economic agents are clearly distinguished. A private account is relevant to a single primary agent as it is the account of his consumption activity and of his individual income. There is no *a priori* need for restricting a business account to a single primary agent. Under a given market condition a single primary economic agent may not find it advantageous to engage in a productive activity as an entrepreneur. Nevertheless, it is possible that the same primary economic agent finds it profitable to engage in such an activity together with some of the other primary agents as an entrepreneur. Hence, in a more general description of a Hicksian production economy we do not limit the class of entrepreneurs to single primary agents but allow any "coalition" of primary economic agents to be a potential entrepreneur which can organize a production unit. In such a model how the surplus on a business account should be divided among the primary agents functioning as an entrepreneur is no longer a trivial problem. V. Boehm (1972), C. Oddou (1972), and D. Sondermann (1974) provided an answer to the problem of determining a meaningful profit distribution among primary economic agents. We adopt for our model their idea of determining an individual profit distribution.

We regard an entrepreneurial resource possessed by a primary economic agent to be indivisible. This implies that a single primary agent can join only one coalition which functions as an entrepreneur. A coalition of primary economic agents identified as an entrepreneur is a producing agent or a firm. Entrepreneurial resources and other nonmarketed factors are not introduced explicitly into our model, and we assume that the only way we can detect these resources is through their influences on the shapes of production sets that individual producing agents are controlling.

Thus we start from a model that formalizes the range of technological alternatives open to a producing organization identified with a coalition of primary economic agents. Let $Y(F)$ be the production set assigned to a coalition F of primary economic agents as a producer. $y \in Y(F)$ means that by pooling all their entrepreneurial and nonmarket factors F can transform an input vector $y^- \in R^l_+$ of marketed commodities into an output vector $y^+ \in R^l_+$ such that $y = y^+ - y^-$, independently of the actions of the other primary economic agents. Note that if a primary economic agent a, or a coalition F of primary economic agents does not possess entrepreneurial factors, $Y(\{a\}) = \{0\}$, or $Y(F) = \{0\}$ respectively.

2. *Production Structure or Firm Structure as a List of Entrepreneurs*

In our extended Hicksian production economy a producing agent is determined by an agreement among a group of primary economic agents to function as an entrepreneur whenever they find it to their benefit to engage in a productive activity under a given market condition. Since a primary economic agent belongs to only one firm in which he shares the functions of an entrepreneur, a partition of all the primary economic agents in the economy can be regarded as a list of such agreements among members of various groups of primary economic agents. We may consider such a list of agreements to represent a "syndicate" structure among the primary economic agents in the sense of Gabszewicz and Drèze (1971). The purpose of forming syndicates in a production economy is not to gain bargaining power in allocation of fixed resources, but rather to engage in a more profitable production activity. Therefore we call a partition of the primary economic agents a *production structure* or a *firm structure.* It can be regarded as a list of entrepreneurs or a list of producing agents in the economy.

It is possible that a different production structure gives rise to a different set of production alternatives of the economy; if the batches of entrepreneurial resources corresponding to a given production structure are broken up and reorganized, there may be possible gains and losses in relative efficiencies of producing commodities. Hence, we explicitly introduce various production structures into the model. A competitive equilibrium shall be defined so that possible gains in efficiency through reallocation of nonmarketed resources among producing agents are captured within the market mechanism.

3. *Other Models of Production Economy*

Hicks (1946) and Arrow and Debreu (1954) presented a model of production economy with a fixed finite number of firms each owning an initial endowment of entrepreneurial resources. There is a well known difficulty for this type of formulation of a production economy [see Koopmans (1957, pp. 64–66 and pp. 68–71)]. It arises from the specification of a given number of producers. Koopmans noted that "... the creation or dissolution of a productive unit is by its very nature an economic act. ... It follows that a postulate assigning production sets with decreasing returns to scale to a number of producers given in advance is tantamount to prescribing and freezing the assignment to various production processes of a certain number of indivisible commodities. Since these commodities are not introduced explicitly, but only implicitly through their influence on the shapes of the production sets, such a model cannot be used to explore possible gains in efficiency through reshuffling of these indivisible resources among producers ..." [Koopmans (1957, p.65)].

In the Walras' model of a production economy [Walras (1954, Part IV)] all the entrepreneurs earn zero profits at an equilibrium. The economic basis for zero profits is given by the free entry of producers taken to be implied by the concept of perfect competition among producers. McKenzie (1955, 1959) formalized a Walrasian model introducing the aggregate production set of the economy which is a cone, in a commodity space, with vertex at the origin. Clearly, a Walrasian model abstracts from the production structures of the economy.

We already noted that a coalition production economy with an additive production set correspondence may be considered as abstracting from production structures; the total production possibilities of the economy are independent of which partition of primary economic agents being regarded as a production structure. But it appears that one cannot defend the additivity of the correspondence on economic grounds. For this reason in a later section we shall give an alternative interpretation of a coalition production economy with an additive production set correspondence. We shall interpret it as a Hicksian production economy with a fixed (infinite) set of producers.

Perhaps, as an approximation, the concept of free entry may be accepted in a finite production economy. However, we have placed ourselves in an "ideal" competitive economy with an atomless measure space of economic agents. This means that a concept such as free entry which purports to approximate perfect competition should not be employed in our model. Instead, this concept itself may have to be explained within the model. We find it difficult to clarify the concept in our model. It seems to us that the free entry assumption requires that the potential production capability of an economy must be "infinitely greater" than the aggregate consumption of the economy. Clearly this property need not hold even in a model which incorporates a continuum of agents.

The preceding remarks may explain the particular way we have chosen in building a model of production economy with an atomless measure space of primary economic agents. We describe the production sector starting from individual competitive firms each of which is assigned a production set. If a list of entrepreneurs or a production structure is fixed, the economy is nothing but a Hicksian economy. However, we allow variations in production structures. The "aggregate" or the mean production set of the economy is derived by considering possible reallocations of entrepreneurial resources, possessed by individual primary economic agents, among producing units, firms.

III. *Mathematical Model of a Production Economy with Competitive Firm Structures*

1. *Measure Space of Primary Economic Agents and Consumption Sector*

Let $(A, \mathscr{A}, \nu)$ be an atomless measure space, i.e., A is a set, $\mathscr{A}$ denotes a σ-algebra of subsets in A, and ν is a countably additive positive measure on $\mathscr{A}$ with $\nu(A)=1$. We assume $\{a\}\in\mathscr{A}$ for every $a\in A$. The economic meaning of this space is that A is the set of all primary economic agents, $\mathscr{A}$ is the set of all coalitions of primary economic agents, and $\nu(C)$ is the fraction of the totality of primary economic agents belonging to the coalition C for each $C\in\mathscr{A}$.

There are l commodities whose combinations are represented as vectors in l-dimensional Euclidean space R^l. All of these commodities are marketed so that their prices are well defined.

To each primary economic agent $a\in A$, as a consumer, are assigned his *consumption set* $X(a)$, his *preference relation* $\precsim_a$ over $X(a)$, and his *initial endowment* $e(a)$. $X(a)$ is the set of vectors in the commodity space R^l corresponding to the feasible consumptions of agent a. The preference relation $\precsim_a$ means that $x\precsim_a y$ if and only if y is preferred to or indifferent with x for the agent a. His initial endowment $e(a)$ is a vector in the commodity space R^l. Given a price vector $p\in R^l$, $p\cdot e$ (a) is a part of his income.

Given a price vector $p\in R^l$ and an income $M\in R$ a primary economic agent a as a consumer maximizes his preference level within his budget constraint. Define for each $(p, M)\in R^{l+1}$,

$$\begin{aligned} B(a, p, M) &= \{x\in X(a) \mid p\cdot x\leq M\}, \\ \phi(a, p, M) &= \{x\in B(a, p, M) \mid z\precsim_a x \text{ for every } z\in B(a, p, M)\}, \\ \psi(a, p, M) &= \phi(a, p, M) \qquad \text{if } M>\min p\cdot X(a) \\ &= B(a, p, M) \qquad \text{if } M\leq\min p\cdot X(a). \end{aligned}$$

$B(a, p, M)$ is the *budget set*, $\phi(a, p, M)$ the *demand set*, and $\psi(a, p, M)$ the *quasi-demand set* of the primary agent a when his income is given by M under the market price vector p.

Let $\mathscr{P}$ denote the set of all continuous complete preorderings defined on a closed, convex subset of R^l. Since a binary relation $\precsim$ on a subset X of R^l is defined by its graph $\{(x, y)\in X\times X \mid x\precsim y\}$, the set $\mathscr{P}$ is a collection of closed subsets in R^l, namely, of those subsets which are the graph of a continuous complete preordering. Now the consumption sector of a production economy is described by an *exchange economy* $\mathscr{E}$ which is a measurable mapping of the measure space $(A, \mathscr{A}, \nu)$ into $\mathscr{P}\times R^l$. That is, for each agent a in A, $\precsim_a=pr_1\circ\mathscr{E}(a)$ is the graph of his preference relation on his consumption set $X(a)$, and we have $X(a)=\{x\in R^l \mid (x, x)\in\precsim_a\}$ and $x\precsim_a y$ if and only if $(x, y)\in\precsim_a$. $e(a)=pr_2\circ\mathscr{E}(a)$ is his initial endowment of marketed commodities. (Here pr_1 and pr_2 denote the projections of $\mathscr{P}\times R^l$ onto $\mathscr{P}$ and R^l respectively.) We will always assume that the mapping $\mathscr{E}$ has the following properties:

(X.1) (Local nonsatiation of $\precsim_a$) For every $x \in X(a)$ and every neighborhood U of x there is a vector $z \in X(a) \cap U$ with $z \succ_a x$, ν-a.e. in A.
(X.2) (Boundedness from below for the correspondence $X: A \to R^l$) There exists a ν-integrable function $g: A \to R^l$ such that $g(a) \leq X(a)$, ν-a.e. in A.
(X.3) (Finiteness of the mean endowment) The mapping $e: A \to R^l$ is integrable.

Given a measure space (S, Σ, μ) and a correspondence Φ from S into R^l, we denote by $\mathscr{L}_\Phi$ the set of all μ-integrable selections from the correspondence Φ, i.e., $\mathscr{L}_\Phi = \{f: S \to R^l \mid f$ is μ-integrable and $f(s) \in \Phi(s)$ μ-a.e. in $S\}$.

Given an exchange economy $\mathscr{E}: A \to \mathscr{P} \times R^l$, a function $f: A \to R^l$ in $\mathscr{L}_X$ is called a *consumption plan* or an *allocation* for $\mathscr{E}$. The integral $\int_A f d\nu$ of a function f on A will be denoted by $\int_A f$ whenever the underlying measure is clearly understood from the context.

2. *Measure Spaces of Competitive Firms*

Since we are concerned with competitive production economies, the set of all possible firms consists of competitive, i.e., small (more precisely, infinitesimal) firms. A justification for restricting our attention to competitive firms may be given later. We define the set of all possible (competitive) firms, $\mathscr{N}$, by $\mathscr{N} = \{F \in \mathscr{A} \mid \nu(F) = 0\}$.

A firm structure or *a production structure* is a measurable partition $\mathscr{F}$ of A, i.e., $\cup \mathscr{F}$(=union of the sets belonging to $\mathscr{F}$) $= A$, $[E, F \in \mathscr{F}$ and $E \neq F]$ implies $E \cap F = \phi$, and every $F \in \mathscr{F}$ belongs to $\mathscr{A}$. If $\mathscr{F} \subset \mathscr{N}$, $\mathscr{F}$ is called a competitive firm structure. We only consider competitive firm structures. Hence, let us define $\mathscr{S} = \{\mathscr{F} \subset \mathscr{N} \mid \mathscr{F}$ is a measurable partition of $A\}$. $\mathscr{S}$ is the set of all competitive firm structures.

In order to carry on our analysis of competitive firm structures, we need a measure theoretical structure on each competitive firm structure. There is a natural way to introduce measure structure on a given firm structure $\mathscr{F}$. Let $\mathscr{F} \in \mathscr{S}$ be given. Since $\mathscr{F}$ is a partition of the set A, consider the canonical map from A onto $\mathscr{F}$, i.e., $Pr_{\mathscr{F}}: A \to \mathscr{F}$ defined by $a \mapsto Pr_{\mathscr{F}}(a)$ where $a \in Pr_{\mathscr{F}}(a)$. That is, the canonical map assigns to each primary economic agent the firm to which he belongs. Now define the set $\sum_{\mathscr{F}}$ consisting of subsets of $\mathscr{F}$ by: $\mathfrak{S} \in \sum_{\mathscr{F}}$ if and only if $Pr_{\mathscr{F}}^{-1}(\mathfrak{S}) \in \mathscr{A}$. It is easy to check that $\sum_{\mathscr{F}}$ is a σ-algebra defined on $\mathscr{F}$. Moreover $\sum_{\mathscr{F}}$ is the largest σ-algebra making the canonical map $Pr_{\mathscr{F}}$ measurable. $\sum_{\mathscr{F}}$ is interpreted to represent the set of coalitions of producing agents under the firm structure $\mathscr{F}$. By defininition of $\sum_{\mathscr{F}}$, a group of firms $\mathfrak{S}$ is a feasible coalition, i.e., $\mathfrak{S} \in \sum_{\mathscr{F}}$, if and only if the coalition of all primary agents belonging to some firm in $\mathfrak{S}$ is a feasible coalition of primary agents, i.e., $\cup \mathfrak{S} \in \mathscr{A}$.

There is also a natural measure induced on the measurable space $(\mathscr{F}, \sum_{\mathscr{F}})$. Define $\tau_{\mathscr{F}} = \nu \circ Pr_{\mathscr{F}}^{-1}$. Then, $\tau_{\mathscr{F}}$ is a measure defined on $(\mathscr{F}, \sum_{\mathscr{F}})$. Given a coalition of firms $\mathfrak{S}$ in the firm structure $\mathscr{F}$, $\tau_{\mathscr{F}}(\mathfrak{S})$ is interpreted as the weight of the coalition of firms $\mathfrak{S}$. It is given by the fraction of the totality of primary economic agents belonging to some firm in the coalition.

Thus each competitive firm structure $\mathscr{F} \in \mathscr{S}$ induces a measure space $(\mathscr{F}, \sum_{\mathscr{F}}, \tau_{\mathscr{F}})$ of competitive firms belonging to $\mathscr{F}$. Corresponding to a firm structure $\mathscr{F} \in \mathscr{S}$ and the set of coalitions of firms in $\mathscr{F}$, $\sum_{\mathscr{F}}$, it may be convenient to consider the set of coalitions of primary economic agents who belong to a firm in a coalition in $\sum_{\mathscr{F}}$. More specifically, define the set $\mathscr{A}_{\mathscr{F}}$ of subsets of A in $\mathscr{A}$ by $\mathscr{A}_{\mathscr{F}}=\{C\in\mathscr{A}\mid C=Pr_{\mathscr{F}}^{-1}(\mathfrak{S})$ for some $\mathfrak{S}\in\sum_{\mathscr{F}}\}$. Since $\sum_{\mathscr{F}}$ is a σ-algebra, $\mathscr{A}_{\mathscr{F}}$ is a sub σ-algebra of $\mathscr{A}$. We denote the restriction of measure ν to $\mathscr{A}_{\mathscr{F}}$ by $\nu_{\mathscr{F}}$, i.e., $\nu_{\mathscr{F}}=\nu\mid\mathscr{A}_{\mathscr{F}}$.

Let us now find the basic properties of firm structures in $\mathscr{S}$. We define a subset $\mathscr{S}_C$ of $\mathscr{S}$ by $\mathscr{S}_C=\{\mathscr{F}\in\mathscr{S}\mid$ there exists $\mathfrak{S}\in\sum_{\mathscr{F}}$ with $\cup\mathfrak{S}=C\}$, for $C\in\mathscr{A}$. Note that $\mathscr{S}_C\neq\phi$ for every $C\in\mathscr{A}$ as one has $\mathscr{F}=\{\{a\}\mid a\in A\}\in\mathscr{S}_C$. $\mathscr{S}_C$ is the set of all competitive firm structures under which there is a coalition of firms whose members are exactly those belonging to the coalition C of primary economic agents. We will define a relation $\sim_C$ on $\mathscr{S}_C$ by

> $\mathscr{F}_1\sim_C\mathscr{F}_2$ if and only if there exists $\mathfrak{S}$ belonging to $\sum_{\mathscr{F}_1}$ and $\sum_{\mathscr{F}_2}$ such that $\cup\mathfrak{S}=C$.

It is to be noted that if $\mathscr{F}\in\mathscr{S}_C$ and $\mathfrak{S}_1$, $\mathfrak{S}_2\in\sum_{\mathscr{F}}$ with $\cup\mathfrak{S}_1=\cup\mathfrak{S}_2=C$, then $\mathfrak{S}_1=\mathfrak{S}_2$. This is because $\mathscr{F}$ is a partition of A. Hence, if $\mathscr{F}_1\sim_C\mathscr{F}_2$, then the coalition of primary economic agents C forms the same set of firms under the firm structure $\mathscr{F}_1$ and $\mathscr{F}_2$. Also, it can be easily checked that the relation $\sim_C$ is an equivalence relation on $\mathscr{S}_C$. Let $[\mathscr{F}]_C$ be an equivalence class of the relation $\sim_C$ on $\mathscr{S}_C$. For the coalition C of primary economic agents, the firm structures in an equivalence class $[\mathscr{F}]_C$ are indifferent in the sense that the firms formed by C under one firm structure are identical with those formed under another. Some of the basic properties of firm structures are shown in the following:

Proposition 2.1: (1) *Let* $\mathscr{F}_1$, $\mathscr{F}_2\in\mathscr{S}$. *If* $\mathfrak{S}\in\sum_{\mathscr{F}_1}$ *and* $\mathfrak{S}\subset\mathscr{F}_2$, *then* $\mathfrak{S}\in\sum_{\mathscr{F}_2}$.

(2) Let C_1, $C_2\in\mathscr{A}$ *and* $C_1\subset C_2$. *Then, for any* $\mathscr{F}_1\in\mathscr{S}_{C_1}$, *there exists* $\mathscr{F}_2\in\mathscr{S}_{C_2}$ *such that* $\mathscr{F}_2\in[\mathscr{F}_1]_{C_1}$.

(3) *Let* C_1, $C_2\in\mathscr{A}$ *and* $C_1\cap C_2=\phi$. *Let* $\mathfrak{S}_1\in\sum_{\mathscr{F}_1}$, $\mathscr{F}_1\in\mathscr{S}_{C_1}$, $\mathfrak{S}_2\in\sum_{\mathscr{F}_2}$ *and* $\mathscr{F}_2\in\mathscr{S}_{C_2}$ *be such that* $C_1=\cup\mathfrak{S}_1$ *and* $C_2=\cup\mathfrak{S}_2$. *Then there exists* $\mathscr{F}\in\mathscr{S}_{C_1\cup C_2}$ *such that one has* $\mathfrak{S}_1\in\sum_{\mathscr{F}}$ *and* $\mathfrak{S}_2\in\sum_{\mathscr{F}}$, *in particular* $\mathfrak{S}_1\cup\mathfrak{S}_2\in\sum_{\mathscr{F}}$.

Intuitively properties (1)–(3) are obvious. (1) says that if a group of firms, $\mathfrak{S}$, whose members belong to both firm structures $\mathscr{F}_1$ and $\mathscr{F}_2$, is a feasible coalition of firms under the firm structure $\mathscr{F}_1$, then it is also a feasible coalition of firms under the firm structure $\mathscr{F}_2$. (2) simply says that a group of firms formed by a smaller coalition of primary economic agents can be formed by a larger coalition of primary economic agents. (3) says that if C_1 and C_2 are disjoint coaltions of primary economic agents, and that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are coalitions of firms formed by C_1 and C_2 respectively under the firm structures $\mathscr{F}_1$ and $\mathscr{F}_2$, then there is a firm structure $\mathscr{F}$ such that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are feasible coalitions of firms under $\mathscr{F}$.

The proposition 2.1 justifies the way in which we introduced measure structures on the sets of competitive firms. If a σ-algebra $\sum$ defined on $\mathscr{F}\in\mathscr{S}$ is smaller than the σ-algebra $\sum_{\mathscr{F}}$, then the intuitively obvious properties (1)–(3) of this proposition may not hold.

3. *Production Sets of Individual Firms and the Mean Production Set Correspondence*

The range of technological alternatives open to a firm F in $\mathscr{N}$ is given by the production set $Y(F)$ assigned to F. Thus for each firm structure $\mathscr{F}$ in $\mathscr{S}$ one has the correspondence $Y_{\mathscr{F}}: \mathscr{F} \to R^l$ with the property that $Y_{\mathscr{F}}(F) = Y_{\mathscr{F}'}(F)$ for any $\mathscr{F}$, $\mathscr{F}'$ in $\mathscr{S}$. We assume

(Y.1) $0 \in Y_{\mathscr{F}}(F)$ for every $F \in \mathscr{F}$ and $\mathscr{F} \in \mathscr{S}$.
(Y.2) The correspondence $Y_{\mathscr{F}}: \mathscr{F} \to R^l$ is measurable for each $\mathscr{F} \in \mathscr{S}$.

(Y.1) says that each producer can produce nothing. (Y.2) is a technical assumption. Since we are only considering competitive firm structures, it could be said that we are placing implicitly a basic economic assumption that at the point where firm sizes become nonnegligible firms face decreasing returns to coalition scale. Note, however, that increasing returns are compatible within our model. That is, we can allow increasing returns in general as long as firms are sufficiently small.

A function y from $\mathscr{F}$ into R^l is called a *production plan* or a *production assignment* if $y \in \mathscr{L}_{Y_{\mathscr{F}}}$. We will denote the integral $\int_{\mathscr{F}} y d\tau_{\mathscr{F}}$ of a function y on $\mathscr{F}$ by $\int_{\mathscr{F}} y$ whenever the context makes it clear as to the measure space in which the integral is defined.

Each $\mathscr{F}$ in $\mathscr{S}$ gives rise to a mean production set correspondence on the set of coalitions of primary economic agents $\mathscr{A}_{\mathscr{F}}$, $\boldsymbol{Y}_{\mathscr{F}}: \mathscr{A}_{\mathscr{F}} \to R^l$, which is defined by $\boldsymbol{Y}_{\mathscr{F}}(C) = \int_{\mathfrak{S}} Y_{\mathscr{F}}$, where $\mathfrak{S}$ belongs to $\sum_{\mathscr{F}}$ and $\cup \mathfrak{S} = C$, for every C in $\mathscr{A}_{\mathscr{F}}$. $\boldsymbol{Y}_{\mathscr{F}}$ is countably additive and $\nu_{\mathscr{F}}$-continuous as one has $\boldsymbol{Y}_{\mathscr{F}}(C) = \{0\}$ if $\nu_{\mathscr{F}}(C) = \nu_{\mathscr{F}} \circ Pr_{\mathscr{F}}^{-1}(\mathfrak{S}) = \tau_{\mathscr{F}}(\mathfrak{S}) = 0$ where $C = \cup \mathfrak{S}$ and $\mathfrak{S} \in \sum_{\mathscr{F}}$. Let us call the correspondence $\boldsymbol{Y}_{\mathscr{F}}$ as the *mean production set correspondence under the firm structure* $\mathscr{F}$. Given the firm structure $\mathscr{F}$, possible coalitions of primary economic agents are those which belong to $\mathscr{A}_{\mathscr{F}}$. $\boldsymbol{Y}_{\mathscr{F}}(C)$ represents the set of possible productions, measured in units per capita, of the coalition of primary agents C in $\mathscr{A}_{\mathscr{F}}$ when the firm structurre of the economy is given by $\mathscr{F}$ in $\mathscr{S}$.

Our next task is to show how the mean production set correspondence of the economy is defined using the basic information on the individual production sets of producing agents, and to investigate its properties.

We define the *mean production set correspondence of the economy* $\boldsymbol{Y}: \mathscr{A} \to R^l$ by

$$\boldsymbol{Y}(C) = \bigcup_{\mathscr{F} \in \mathscr{S}_C} \int_{Pr_{\mathscr{F}}(C)} Y_{\mathscr{F}} d\tau_{\mathscr{F}} \text{ where } \mathscr{S}_C = \mathscr{S} \text{ when } C = A.$$

Let us note the difference between the assignment of production sets to coalitions of primary economic agents as production units, and the mean production set correspondence of the economy $\boldsymbol{Y}: \mathscr{A} \to R^l$. The latter assigns to each coalition of primary economic agents the possible productions, measured in units per capita, of the coalition as the sum of possible productions of the firms formed by members of the coalition.

Some of the basic properties of the mean production set correspondence $\boldsymbol{Y}: \mathscr{A} \to R$ are given in the lemma 3.2 below.

Lemma 3.1: *Let* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *belong to* $\mathscr{S}$, *and* $\mathfrak{S}$ *to* $\sum_{\mathscr{F}_1}$ *and* $\sum_{\mathscr{F}_2}$. *Then, one has* $\int_{\mathfrak{S}} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1} = \int_{\mathfrak{S}} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$.

Lemma 3.2: *Let* $\boldsymbol{Y}: \mathscr{A} \to R^l$ *be defined as above.*

(1) $0 \in \boldsymbol{Y}(C)$ *for every* $C \in \mathscr{A}$, *and* $\boldsymbol{Y}(C) = \{0\}$ *if* $\nu(C) = 0$.
(2) $\boldsymbol{Y}: \mathscr{A} \to R^l$ *is monotone, i.e., if* C_1, $C_2 \in \mathscr{A}$ *and* $C_1 \subset C_2$, *then* $\boldsymbol{Y}(C_1) \subset \boldsymbol{Y}(C_2)$.

The results of the lemma 3.2 are intuitive. Since each producing agent can produce nothing, any coalition of firms can produce nothing. (2) says that since a larger coalition of primary economic agents can organize all the firms which can be formed by a smaller coalition of primary economic agents, the mean production set of the larger coalition should contain that of the smaller one. It is also intuitively clear that the mean production set correspondence of the economy should be superadditive regardless of the nature of production sets assigned to producing agents, because the sum of what could be produced by two disjoint coalitions of primary economic agents under various production structures should be in the mean production set of the union of these two coalitions as it is possible for the unified coalition to divide its members into smaller coalitions to achieve the results obtained by two smaller coalitions. The following proposition justifies this intuition and strengthens (2) of the lemma 3.2.

Proposition 3.3: The mean production set correspondence of the economy $Y: \mathscr{A} \to R^l$ *is superadditive, i.e.,* $Y(C_1)+Y(C_2) \subset Y(C_1 \cup C_2)$ *for any disjoint coalitions* C_1, C_2 *in* $\mathscr{A}$.

This result provides a way to reinterpret D. Sondermann's model of a production economy [see Sondermann (1974)]. In his model the production set correspondence $Y: \mathscr{A} \to R^l$ is assumed to satisfy strong superadditivity, i.e., $Y(C_1)+Y(C_2) \subset Y(C_1 \cup C_2)+Y(C_1 \cap C_2)$ for any C_1, C_2 in $\mathscr{A}$. If we are to interpret his production set correspondence literally, only one giant firm will result in an equilibrium and yet it takes prices as given. Of course such an assumption on the production set correspondence will not be accepted without difficulty from a realistic point of view. However, according to the proposition 3.3, the mean production set correspondence of the economy is superadditive (although it need not be strongly superadditive). Therefore, we may justify Sondermann's model of a production economy by reinterpreting his production set correspondence as the mean production set correspondence of the economy generated under competitive firm structures. The difference between Sondermann's model and ours is that his model describes only the aggregate behavior of the production sector whereas our model gives microeconomic descriptions of the production sector.

So far we have not placed any restrictions on the firm structures nor on the way in which the production sets are assigned to various producing agents. However, some requirements on the firm structures and the assignments of production sets to various firms seem to be unavoidable. In particular, we assume

(Y.3) If $\nu(C)=1$, then for any $\mathscr{F} \in \mathscr{S}$ there exists $\mathscr{F}' \in \mathscr{S}_C$ such that
$$\int_{\mathscr{F}} Y_{\mathscr{F}} = \int_{\mathscr{F}'} Y_{\mathscr{F}'}.$$

(Y.4) Let $\mathscr{F} \in \mathscr{S}$. Given an increasing sequence $(C_n)_n$ of subsets of A in $\mathscr{A}$, there exists a firm structure $\mathscr{F}'$ in $\mathscr{S}$ such that for any n there is a subset A_k of A in $\mathscr{A}_{\mathscr{F}'}$ having properties that $A_k \subset C_k$, $k \geq n$, and $\nu(C_k \backslash A_k)=0$, and that $\int_{\mathscr{F}} Y_{\mathscr{F}} = \int_{\mathscr{F}'} Y_{\mathscr{F}'}$.

Let $C_0 \in \mathscr{A}$ be a null set of primary economic agents, and put $C=A \backslash C_0$. Take a firm structure $\mathscr{F} \in \mathscr{S}_C$. C_0 cannot affect the production capability of the economy only in the following sense. Let $\mathfrak{S} \in \sum_{\mathscr{F}}$ be such that $C=Pr_{\mathscr{F}}^{-1}(\mathfrak{S})$. If the coalition of primary economic agents C forms the set of firms $\mathfrak{S}$ then regardless of the actions taken by C_0, that is for any firm structure $\mathscr{F}' \in [\mathscr{F}]_C$, one has $\int_{\mathscr{S}} Y_{\mathscr{S}} d\tau_{\mathscr{S}} = \int_{\mathfrak{S}} Y_{\mathscr{S}} d\tau_{\mathscr{S}} = \int_{\mathscr{S}'} Y_{\mathscr{S}'} d\tau_{\mathscr{S}'}$ since one has $\tau_{\mathscr{S}}(\mathfrak{S}) = \nu \circ Pr_{\mathscr{S}}^{-1}(\mathfrak{S}) = \nu(C) = 1$ and similarly $\tau_{\mathscr{S}'}(\mathfrak{S}) = 1$.

This does not, however, imply that a null set of primary agents cannot affect the essential production capability of the economy. For example, if we take a firm structure $\mathscr{F}' \in \mathscr{S} \backslash \mathscr{S}_C$, we may have $\int_{\mathscr{S}'} Y_{\mathscr{S}'} \supset \int_{\mathscr{S}} Y_{\mathscr{S}}$, and $\int_{\mathscr{S}'} Y_{\mathscr{S}'} \neq \int_{\mathscr{S}} Y_{\mathscr{S}}$ for all $\mathscr{F} \in \mathscr{S}_C$. This means that C_0 can influence the production capability of the economy. Obviously (Y.3) excludes such a possibility. Hence we may take (Y.3) to be saying that a null set of primary economic agents cannot affect the production capability of the economy. It is the sprit of models of large economies to exclude above mentioned possibility, and we consider (Y.3) to be a very reasonable requirement.

We say that a firm structure $\mathscr{F} \in \mathscr{S}$ is *regular* if, given an increasing sequence $(C_n)_n$ in $\mathscr{A}$ with $\cup_n C_n = A$ and a positive integer n, there exist an integer $k \geq n$ and $A_k \in \mathscr{A}_{\mathscr{F}}$ such that $A_k \subset C_k$ and $\nu(C_k \backslash A_k) = 0$. We may restrict ourselves to the set $\mathscr{S}_r \subset \mathscr{S}$ of competitive firm structures which are regular. But depending upon the σ-algebra $\mathscr{A}$, $\mathscr{S}_r$ could range from $\mathscr{S}_r = \mathscr{S}$ to the other extreme, that is, $\mathscr{S}_r = \{\mathscr{F}\}$ with $\mathscr{F} = \{\{a\} \mid a \in A\}$. Nevertheless, one may choose to work with $\mathscr{S}_r$ in which case the set $\mathscr{S}_r$ may be "small." Or, one may choose a particular σ-algebra $\mathscr{A}$ so that one has $\mathscr{S} = \mathscr{S}_r$. Here we choose to work under a weaker condition (Y.4) which restricts jointly the σ-algebra $\mathscr{A}$ (implicitly) and the manner in which production sets are assigned to various firms.

We assume

(Y.5) $Y(A)$ is closed and convex.

The first half of this assumption says, that if vectors arbitrarily close to $\bar{y}$ are in the mean production set of the economy $Y(A)$, then so is $\bar{y}$. In our present context this means that if vectors arbitrarily close to $\bar{y}$ represent possible aggregate productions of the economy, measured in units per capita, then there is a firm structure $\mathscr{F} \in \mathscr{S}$ under which $\bar{y}$ itself represents a possible aggregate production of the economy. The remaining half of the assumption implies nonincreasing returns to scale with respect to the marketed inputs.

The correspondence $Y: \mathscr{A} \to R^l$ can be shown to have additional properties stated below:

Lemma 3.4: (*1*) *Let C be a set in $\mathscr{A}$ with $\nu(C) = 1$. Then, $Y(C) = Y(A)$.*
(2) *If $(C_n)_n$ is an increasing sequence in $\mathscr{A}$ such that $\cup_n C_n = A$, then* $\lim_n \sup Y(C_n) = Y(A)$.

Here if $(S_n)_n$ is a sequence of subsets of R^l, then $\lim_n \sup S_n$ is the set of all points $\bar{y}$ in R^l such that every neighborhood of $\bar{y}$ intersects infinitely many S_n.

Now, for each price vector $p \in R^l$, define $S(p) = \{\bar{y} \in Y(A) \mid p \cdot \bar{y} = \max p \cdot Y(A)\}$. $S(p)$ represents the supply set of the economy under the market price vector p. Since we are only interested in those price vectors for which the total profit is finite, we can restrict ourselves to the set $P = \{p \in R^l \mid \sum_{i=1}^l p_i = 1\} \cap (\boldsymbol{ac}\, Y(A))^\circ$, where $\boldsymbol{ac}\, H$ denotes the asymptotic cone of H and $(H)^\circ$ the polar of H, considering normalization of price vectors.

Then we define $P_Y=\{p\in P \mid S(p)\neq\phi\}$, $P_Y{}^+=\{p\in P_Y \mid \max\ p\cdot Y(A)>0\}$, and $P^+=P_Y{}^+\cup$ int P.

We assume

(Y.6) For every p in $P_Y{}^+$, the set $S(p)$ is bounded.

(Y.7) If $(\lambda_n, C_n)_n$ is a finite sequence in $R_+\times\mathscr{A}$ such that $\sum_n\lambda_n\chi_{C_n}\leq\chi_A$, where $\chi_C(a)=1$ if $a\in C$, and $=0$ if $a\notin C$; then $\sum_n\lambda_n Y(C_n)\subset Y(A)$.

Our balancedness assumption (Y.7) is consistent with the properties of $Y:\mathscr{A}\rightarrow R^l$ obtained in the lemma 3.2 and the proposition 3.3. It is known that the superadditivity of Y is short of guaranteeing the balancedness even if $Y(A)$ is convex. If we define $v_p(C)=\sup p\cdot Y(C)$ for each C in $\mathscr{A}$ and p in P_Y, we can replace (Y.7) by a weaker assumption:

(Y.7′) If $(\lambda_n, C_n)_n$ is a finite sequence in $R_+\times\mathscr{A}$ such that $\sum_n\lambda_n\chi_{C_n}\leq\chi_A$, and if p is in P_Y, then $\sum_n\lambda_n v_p(C_n)\leq\max\ p\cdot Y(A)$.

Assumption (Y.6) is a technical one. It is satisfied, for example, if $Y(A)$ is a cone or strictly convex. Economic meanings of the balancedness assumption such as (Y.7) have not been well understood in economics. However, (Y.7′) is the minimal requirement for the model to have a "reasonable" profit distribution.

4. *Formation of Firm Structure and Profit Distribution Among Primary Economic Agents*

In this section we are going to see which firm structure is realized and how profits are distributed among the primary economic agents under the given market price vector p in P_Y. Perhaps at this point we should call to our attention that it is not appropriate to say, in the strict sense of the words, that this type of model explains the formation of firms in a market economy. For, even under a fixed firm structure $\mathscr{F}$, it may be the case that some firms are producing nothing, which is traditionally interpreted as saying that some firms are out of existence.

Let $p\in P_Y$ and $\bar{y}\in S(p)$. Take $\mathscr{F}^*\in\mathscr{S}$ and $y\in\mathscr{L}_{Y_{\mathscr{F}^*}}$ such that $\int_{\mathscr{F}^*}y=\bar{y}$. Then, we have $p\cdot y(F)=\sup p\cdot Y_{\mathscr{F}^*}(F)$ for $\tau_{\mathscr{F}^*}$-a.e. F in $\mathscr{F}^*$. Indeed, one has $\max\ p\cdot Y(A)=p\cdot\int_{\mathscr{F}^*}y=\int_{\mathscr{F}^*}p\cdot y$. It follows that $\int_{\mathscr{F}^*}p\cdot y\geq\sup p\cdot\int_{\mathscr{F}^*}Y_{\mathscr{F}^*}=\sup\int_{\mathscr{F}^*}p\cdot Y_{\mathscr{F}^*}=\int\sup p\cdot Y_{\mathscr{F}^*}$ [see Hildenbrand (1974, p.63)]. On the other hand, since y is in $\mathscr{L}_{Y_{\mathscr{F}^*}}$, we have $\sup p\cdot Y_{\mathscr{F}^*}(F)\geq p\cdot y(F)$ for $\tau_{\mathscr{F}^*}$ -a.e. F in $\mathscr{F}^*$. It follows that $\int\sup\ p\cdot Y_{\mathscr{F}^*}=\int_{\mathscr{F}^*}p\cdot y$, and $\sup p\cdot Y_{\mathscr{F}^*}(F)=p\cdot y(F)$ for $\tau_{\mathscr{F}^*}$-a.e. F in $\mathscr{F}^*$.

We also have $\int_{\mathfrak{S}}p\cdot y d\tau_{\mathscr{F}^*}=\sup\ p\cdot Y(C)$ for every C in $\mathscr{A}_{\mathscr{F}^*}$ where $\mathfrak{S}\in\sum_{\mathscr{F}^*}$ is such that $C=\cup\mathfrak{S}$. Suppose we had $\int_{\mathfrak{S}}p\cdot y d\tau_{\mathscr{F}^*}<\sup\ p\cdot Y(C)$ for some C in $\mathscr{A}_{\mathscr{F}^*}$. Then by definition of $Y(C)$ there exists $\mathscr{F}_1$ in $\mathscr{S}_C$ such that $\int_{\mathfrak{S}}p\cdot y d\tau_{\mathscr{F}^*}<\sup p\cdot\int\mathfrak{S}_1 d\tau_{\mathscr{F}_1}$, where $\mathfrak{S}_1$ is in $\sum_{\mathscr{F}_1}$ and $\cup\mathfrak{S}_1=C$. Define a firm structure $\mathscr{F}$ by $\mathscr{F}=\{F\in N \mid F\in\mathfrak{S}_1$ or $F\in\mathscr{F}^*\cap(A\backslash C)\}$ where $\mathscr{F}^*\cap(A\backslash C)=\{F\in\mathscr{F}^* \mid F\subset A\backslash C\}$. Then by (1) of the proposition

2.1, $\mathfrak{S}_1$ is in $\sum_{\mathscr{F}}$, and we have, by lemma 3.1, $\int_{\mathfrak{S}_1} Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1}$, and $\int_{\mathfrak{S}_2} Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \int_{\mathfrak{S}_2} Y_{\mathscr{F}^*} d\tau_{\mathscr{F}^*}$ where $\mathfrak{S}_2 = Pr_{\mathscr{F}}(A \backslash C) = Pr_{\mathscr{F}^*}(A \backslash C)$ belongs to both $\sum_{\mathscr{F}}$ and $\sum_{\mathscr{F}^*}$. Now, $\int_{\mathfrak{S}_2} p \cdot y d\tau_{\mathscr{F}^*} = p \cdot \int_{\mathfrak{S}_2} y d\tau_{\mathscr{F}^*} = \sup p \cdot \int_{\mathfrak{S}_2} Y_{\mathscr{F}^*} d\tau_{\mathscr{F}^*} = \sup p \cdot \int_{\mathfrak{S}_2} Y_{\mathscr{F}} d\tau_{\mathscr{F}}$, using $p \cdot y(F) = \sup p \cdot Y_{\mathscr{F}^*}(F)$ $\tau_{\mathscr{F}^*}$-a.e. F in $\mathscr{F}^*$. Hence we must have $\sup p \cdot \int_{\mathscr{F}} Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \sup \int_{\mathscr{F}} p \cdot Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \sup \int_{\mathfrak{S}_2} p \cdot Y_{\mathscr{F}} d\tau_{\mathscr{F}} + \sup \int_{\mathfrak{S}_1} p \cdot Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \sup p \cdot \int_{\mathfrak{S}_2} Y_{\mathscr{F}} d\tau_{\mathscr{F}} + \sup p \cdot \int_{\mathfrak{S}_1} Y_{\mathscr{F}} d\tau_{\mathscr{F}} = \sup p \cdot \int_{\mathfrak{S}_2} Y_{\mathscr{F}^*} d\tau_{\mathscr{F}^*} + \sup p \cdot \int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1} > \sup p \cdot \int_{\mathscr{F}^*} Y_{\mathscr{F}^*} d\tau_{\mathscr{F}^*} = p \cdot \int_{\mathscr{F}^*} y d\tau_{\mathscr{F}^*}$, contradicting the way y and $\mathscr{F}^*$ have been chosen.

Given a price vector $p \in P_Y$, the coalition C of primary economic agents can realize at most the profit $v_p(C) = \sup p \cdot Y(C)$, by forming the suitable set of firms among themselves. Thus, a sidepayment game $(A, \mathscr{A}, v_p)$ is defined for any given $p \in P_Y$. By the proposition 3.3 the correspondence Y is superadditive. It follows that $v_p : \mathscr{A} \to R_+$ is a superadditive. Hence it makes sense to talk about the core of the game $(A, \mathscr{A}, v_p)$. Let us define $\mathscr{C}(p) = \{\mu \in ba(A, \mathscr{A}) \mid \mu(A) = v_p(A)$ and $\mu(C) \geq v_p(C)$ for all C in $\mathscr{A}\}$ for p in P_Y. Here $ba(A, \mathscr{A})$ denotes the space of all bounded finitely additive scalar functions defined on $\mathscr{A}$. We denote by $ca(A, \mathscr{A})$ the space of all countably additive scalar functions defined on $\mathscr{A}$ [see Dunford and Schwartz (1958, pp. 160–161)].

$\mathscr{C}(p)$ denotes the core of the side-payment game $(A, \mathscr{A}, v_p)$. It has the properties shown below:

Lemma 4.1 Let p be in P_Y and y, $\mathscr{F}^*$ be chosen as above.

(1) *$\mathscr{C}(p)$ is nonempty.*

(2) *$\mathscr{C}(p) \subset ca(A, \mathscr{A})$.*

(3) *Each member of $\mathscr{C}(p)$ is absolutely continuous with respect to the measure ν.*

(4) *If μ is a member of $\mathscr{C}(p)$, then $\mu(C) = \int_{\mathfrak{S}} p \cdot y d\tau_{\mathscr{F}^*}$ for C in $\mathscr{A}_{\mathscr{F}^*}$ with $C = \cup \mathfrak{S}$ and $\mathfrak{S} \in \sum_{\mathscr{F}^*}$.*

By (2) and (3) of this lemma every member μ of $\mathscr{C}(p)$ is countably additive and absolutely continuous with respect to the measure ν. Therefore, μ adimts the Radon-Nikodym derivative [see Dunford and Schwartz (1954, Theorem III. 10.2, p. 176)] $\pi \in L_1(A, \mathscr{A}, \nu)$, i.e., $\mu(C) = \int_C \pi d\nu$ for every C in $\mathscr{A}$. Thus $\mathscr{C}(p)$ can be regarded as a subset of $L_1(A, \mathscr{A}, \nu)$. Here, $L_1(A, \mathscr{A}, \nu)$ denotes the set of equivalence classes of ν-integrable real-valued functions defined on A. We shall use the notation $L_1(A, \mathscr{A}, \nu; R^l)$ to denote the set of equivalence classes of ν-integrable functions from A into R^l [see Dunford and Schwartz (1945, pp. 119–125)].

The following proposition summarizes our remarks above:

Proposition 4.2 Let p be in P_Y. Under(Y.1)–(Y.7) there exist $\mathscr{F}^*$ in $\mathscr{S}$, y in $\mathscr{L}_{Y_{\mathscr{F}^*}}$, and a nonempty subset $\mathscr{C}(p)$ of $L_1(A, \mathscr{A}, \nu)$ such that for every π in $\mathscr{C}(p)$ we have

(1) *$p \cdot y(F) = \sup p \cdot Y_{\mathscr{F}^*}(F)$ for $\tau_{\mathscr{F}^*}$-a.e. F in $\mathscr{F}^*$,*

(2) $\int_{\mathscr{F}^*} p \cdot y d\tau_{\mathscr{F}^*} = \max p \cdot Y(A)$,

(3) $\int_C \pi d\nu = \int_{\mathfrak{S}} p \cdot y d\tau_{\mathscr{F}^*}$ *for every* C *in* $\mathscr{A}_{\mathscr{F}^*}$ *with* $\mathfrak{S}$ *in* $\sum_{\mathscr{F}^*}$ *such that* $C = \cup \mathfrak{S}$.

This proposition can be given the following economic interpretation. Given a vector of market prices $p \in P_Y$, for inputs and outputs, any coalition of primary economic agents that possesses entrepreneurial resources will be able to determine whether the utilization of those resources in production will yield a positive surplus. If it will do so, it may become an active firm. As such, it decides what arrangement of production makes its surplus, or profits, a maximum. This most profitable arrangement is determined by the production set which in turn depends upon entrepreneurial factors provided by the primary economic agents organizing the firm; consequently its demand for factors and supply of products are determined and so is the amount of surplus or profits of the agents. Under these circumstances it is natural to interpret that a firm structure $\mathscr{F}^*$ is *formed* among primary economic agents and that productions of producing agents under this firm structure are given by a production plan y. Indeed, almost every producer in $\mathscr{F}^*$ maximizes its profits subject to its production set under the production plan y, i.e., $p \cdot y(F) = \sup p \cdot Y_{\mathscr{F}^*}(F)$ for $\tau_{\mathscr{F}^*}$-a.e F in $\mathscr{F}^*$; the distribution of profits among primary agents as returns to their entrepreneurial resources is given by a function $\pi : A \to R_+$ in $\mathscr{C}(p)$. The profit distribution π is compatible with the production plan y. And there is no incentive on any part of primary economic agents to alter this distribution by forming a new set of firms among themselves.

5. *Competitive Equilibrium*

An exchange economy $\mathscr{E} : (A, \mathscr{A}, \nu) \to \mathscr{P} \times R^l$ together with a family of production set correspondences of $\mathscr{F}$ into R^l for each firm structure $\mathscr{F} \in \mathscr{S}$ is called *a production economy with competitive firm structures* or simply a (*competitive*) *production economy* and is denoted by $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F} \in \mathscr{S}})$. Since the mean production set correspondence of the economy exhibits superadditivity, the way in which total profits are divided in equilibrium becomes part of the notion of equilibrium itself. We adopt the notion of equilibrium proposed independently by Boehm (1972), Oddou (1972), and Sondermann (1974) for our model of production with competitive firm structures.

The competitive equilibrium of a production economy with competitive firm structures is given by a price vector p^*, a consumption plan f^*, a competitive firm structure $\mathscr{F}^*$, a production plan y^* and a profit distribution π^*, which satisfy certain conditions.

(E.1) $f^*(a) \in \phi(a, p^*, p^* \cdot e(a) + \pi^*(a))$, ν-a.e. in A.

(E.2) 1) $p^* \cdot y^*(F) = \sup p^* \cdot Y_{\mathscr{F}^*}(F)$, $\tau_{\mathscr{F}^*}$-a.e. in $\mathscr{F}^*$.

2) $\int_{\mathscr{F}^*} p^* \cdot y^*(F) = \max p^* \cdot Y(A)$.

(E.3) 1) $\int_C \pi^* = \int_{\mathfrak{S}} p^* \cdot y^* d\tau_{\mathscr{F}^*}$ for $(C, \mathfrak{S}) \in (\mathscr{A}_{\mathscr{F}^*}, \sum_{\mathscr{F}^*})$ such that $\mathfrak{S} = Pr_{\mathscr{F}^*}(C)$.

2) $\int_C \pi^* \geq \nu_{p^*}(C)$ for every $C \in \mathscr{A}$.

(E.4) 1) $\int_A f^* \leq \int_{\mathscr{F}^*} y^* + \int_A e$;

2) $p^* \cdot \int_A f^* = p^* \cdot \int_{\mathscr{F}^*} y^* + p^* \cdot \int_A e$.

The conditions (E.1), (E.2), and (E.4) are familiar. The condition (E.3) 1) requires that the aggregate profits distributed over a coalition C of primary economic agents, measured in units per capita of all the primary agents, should equal to the sum of profits, measured in the same way, attained by the firms formed by the coalition C at the equilibrium. The condition (E.3) 2) requires that there should be no incentive for any coalition of primary economic agents to alter the firm structure of the economy realized at the equilibrium.

Given a production economy with competitive firm structures $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$, a *competitive equilibrium* of $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$ is a quintuple $(p^*, f^*, \mathscr{F}^*, y^*, \pi^*)\in(R^l_+\setminus\{0\}, \mathscr{L}_X, \mathscr{S}, L_1(\mathscr{F}^*, \sum_{\mathscr{F}^*}, \tau_{\mathscr{F}^*}; R^l), L_1(A, \mathscr{A}, \nu))$ which satisfies conditions (E.1)–(E.4). f^* is called a *competitive allocation.*

For technical reasons we introduce the notion of a *competitive quasi-equilibrium*, which is defined as above, but (E.1) replaced by

(E.1′) $f^*(a)\in\psi(a, p^*, p^*\cdot e(a)+\pi^*(a))$, ν-a.e. in A.

In this case f^* is called a *quasi-competitive allocation.*

For the relations between these two concepts of equilibrium one can refer to Debreu (1962) or Hildenbrand (1970).

Let $(p^*, f^*, \mathscr{F}^*, y^*, \pi^*)$ be a competitive equilibrium of a production economy with competitive firm structures $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$. If we set $\bar{y}^*=\int_{\mathscr{F}^*}y^*d\tau_{\mathscr{F}^*}$, then we can easily verify that $(f^*, \bar{y}^*, \pi^*)$ satisfies the conditions of an equilibrium in Sondermann's production economy (1974) where his superadditive production set correspondence $Y: \mathscr{A}\to R^l$ is interpreted as the mean production set correspondence of the economy.

We now state our main result:

Theorem 5.1: Let $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$ be a production economy with competitive firm structures satisfying (X.1)–(X.3), (Y.1)–(Y.7),

(Z.1) $\boldsymbol{ac}\,Y(A)\cap R^l_+=\{0\}$.

(Z.2) $\boldsymbol{ac}\,Y(A)\cap-\boldsymbol{ac}\,Y(A)=\{0\}$, *and*

(Z.3) $\{\boldsymbol{ac}\,Y(A)+e(a)\}\cap X(a)\neq\phi$, *$\nu$-a.e. in A.*

Then, there exists a competitive quasi-equilibrium of $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$.

IV. *Core and Competitive Allocations of a Production Economy with Competitive Firm Structures*

1. *Core Allocations*

An allocation $f: A\to R^l$ is called *attainable with respect to a firm structure $\mathscr{F}\in\mathscr{S}$*, if $\int_A f\in\int_A e+\int_{\mathscr{F}}Y_{\mathscr{F}}$. Let $(\mathscr{E}, (Y_{\mathscr{T}})_{\mathscr{T}\in\mathscr{S}})$ be a competitive production economy. An allocation f is said to be *attainable for the economy*, or simply *attainable*, if there exists a firm structure $\mathscr{F}\in\mathscr{S}$ with respect to which f is attainable: in other words, f is attainable if

$$\int_A f\in\int_A e+Y(A).$$

An allocation f is *dominated via coalition* C of primary economic agents, if there exists another allocation g such that

(C.1) $g(a) \succ_a f(a)$, ν-a.e. in C;

(C.2) $\nu(C)>0$ and $\int_C g \in \int e + Y(C)$.

The set of all attainable allocations for the production economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$ that are not dominated via any coalition of primary economic agents is called the *core* of the economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$.

It is easy to show the following:

Proposition 1.1: *Every competitive allocation belongs to the core of* $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$.

Proof: Let $(p, f, \mathscr{F}, y, \pi)\in(R^l_+\setminus\{0\}, \mathscr{L}_X, \mathscr{S}, L_1(\mathscr{F}, \sum_{\mathscr{F}}, \tau_{\mathscr{F}}; R^l), L_1(A, \mathscr{A}, \nu))$ be a competitive equilibrium of $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$. According to (E.4) 1) of the definition of a competitive equilibrium, f is attainable. Suppose that f is dominated via coalition C with $\nu(C)>0$. This means that there exist an allocation $g\in\mathscr{L}_X$, a firm structure $\mathscr{F}'\in\mathscr{S}_C$ and a production assignment $y\in\mathscr{L}_{Y_{\mathscr{F}'}}$, such that

(1) $g(a) \succ_a f(a)$, ν-a.e. in C;

(2) $\int_C g = \int_C e + \bar{z}$ where $\bar{z} = \int_{Pr_{\mathscr{F}'}(C)} y d\tau_{\mathscr{F}'}$.

From (1) and (E.1) it follows that, ν-a.e. in C,

(3) $p\cdot e(a) + \pi(a) < p\cdot g(a)$,

which implies

(4) $\int_C p\cdot e + \int_C \pi < \int_C p\cdot g$.

Since $p\in R^l_+$ and $z\in Y(C)$, (2) implies

(5) $p\cdot\int_C g - p\cdot\int_C e = p\cdot z \leq \sup p\cdot Y(C)$.

(4) and (5) imply $\int_C \pi < \sup p\cdot Y(C)$, a contradiction to (E.3) 2).

Q.E.D.

Unfortunately we cannot hope to establish the converse of this proposition. Indeed V. Boehm (1973) has given an example of production economy with an atomless measure space of economic agents and a superadditive mean production set correspondence, in which an allocation in the core need not be a competitive allocation. Since the mean production set correspondence generated in a production economy with competitive firm structures is in fact superadditive, as shown in the proposition III.3.3, Boehm's example does apply to economies considered here.

In a production model, in which a microeconomic description of the production sector is not given, it is not possible to probe into economic factors working as an obstacle to obtaining the equivalence between the core and the set of competitive allocations. Nor is it possible to see, even on an intuitive basis, to what extent a core allocation "can be" a competitive allocation. Although we cannot give a full answer to these questions, we shall turn our attention to the problem for a possible answer. For this purpose we shall introduce the concept of production economy with a fixed firm structure.

2. *Production Economy With a Fixed Firm Structure*

Given a competitive production economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$, we define a *competitive production economy with a fixed firm structure* $\mathscr{F}\in\mathscr{S}$ by $(\mathscr{E}_{\mathscr{F}}, Y_{\mathscr{F}})$ where $\mathscr{E}_{\mathscr{F}}$ is the map from $(A, \mathscr{A}_{\mathscr{F}}, \nu_{\mathscr{F}})$ into $\mathscr{P}\times R^l$ defined by $\mathscr{E}_{\mathscr{F}}(a)=\mathscr{E}(a)$ for every $a\in A$. A production economy $(\mathscr{E}_{\mathscr{F}}, Y_{\mathscr{F}})$ may be interpreted as an economy in which the only possible firm structure is $\mathscr{F}$. Thus, feasible coalitions of primary economic agents are restricted to those in $\mathscr{A}_{\mathscr{F}}$ as the result of formation of "syndicates" among primary economic agents for the purpose of production.

The mean production set correspondence of such an economy is given by the mean production set correspondence of the economy with a firm structure $\mathscr{F}$, i.e., by $Y_{\mathscr{F}}: \mathscr{A}_{\mathscr{F}}\to R^l$, which is already defined in the section III. 3. As shown there $Y_{\mathscr{F}}$ is countably additive and $\nu_{\mathscr{F}}$-continuous. Therefore, as in the production economy of W. Hildenbrand (1970), the individual profit distribution can be defined unambiguously. Indeed, we define for every coalition of primary economic agents C in $\mathscr{A}_{\mathscr{F}}$ $\mu_{\mathscr{F}}(C, p)=\sup p\cdot Y_{\mathscr{F}}(C)$ for each price vector p. It is easily checked that $\mu_{\mathscr{F}}(\cdot, p)$ is countably additive on $\mathscr{A}_{\mathscr{F}}$. Further, $\mu_{\mathscr{F}}(\cdot, p)$ is absolutely continuous with respect to the measure $\nu_{\mathscr{F}}$. Consequently, by the Radon-Nikodym Theorem, there exists an $\mathscr{A}_{\mathscr{F}}$-measurable function $\pi_{\mathscr{F}}(\cdot, p)$ of A into R such that for every coalition $C\in\mathscr{A}_{\mathscr{F}}$ we have $\mu_{\mathscr{F}}(C, p)=\int_C\pi_{\mathscr{F}}(\cdot, p)$. The function $\pi_{\mathscr{F}}(\cdot, p)$ determines an individual profit distribution for the economy $(\mathscr{E}_{\mathscr{F}}, Y_{\mathscr{F}})$ uniquely up to $\nu_{\mathscr{F}}$-equivalence.

Given a firm structure $\mathscr{F}^*\in\mathscr{S}$, a *competitive equilibrium of* $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$ is a triplet $(p^*, f^*, y^*)\in(R^l_+\backslash\{0\}, \mathscr{L}_X, \mathscr{L}_{Y_{\mathscr{F}^*}})$ satisfying the conditions (E.1), (E.2) 1) and 2), (E. 3) 1), and (E.4) 1) and 2), where $\pi^*(a)$ is replaced by $\pi_{\mathscr{F}^*}(a, p^*)$ as defined above, $Y(A)$ by $Y_{\mathscr{F}^*}(A)$, and $(A, \mathscr{A}, \nu)$ by $(A, \mathscr{A}_{\mathscr{F}^*}, \nu_{\mathscr{F}^*})$. A *competitive quasi-equilibrium of* $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$ is defined as above with the exception of (E.1) which is replaced by (E.1′). An allocation f: $A\to R^l$ is said to be *attainable for* $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$ if it is attainable with respect to the firm structure $\mathscr{F}^*$. The definition of the *core allocations of* $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$ is the same as that of $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$ except that we replace $Y(C)$ in (C.2) by $Y_{\mathscr{F}^*}(C)$ and that coalitions are restricted to those belonging to $\mathscr{A}_{\mathscr{F}^*}$, that is, the measure space $(A, \mathscr{A}, \nu)$ is replaced by $(A, \mathscr{A}_{\mathscr{F}^*}, \nu_{\mathscr{F}^*})$.

The proof of the proposition 1.1 with some minor changes establishes the following:

Proposition 2.1: *Let $\mathscr{F}^*\in\mathscr{S}$. Every competitive allocation of $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$ belongs to the core of $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$.*

Moreover, we can establish the following result without difficulty.

Theorem 2.2: *Let $\mathscr{F}^*\in\mathscr{S}$, and assume that the measure space $(A, \mathscr{A}_{\mathscr{F}^*}, \nu_{\mathscr{F}^*})$ is atomless. Then, every allocation in the core of $(\mathscr{E}_{\mathscr{F}^*}. Y_{\mathscr{F}^*})$ is a competitive quasi-equilibrium of $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$.*

This result follows from a basic result of W. Hildenbrand (1974, Theorem 1, pp.

216–9) if we replace Y by $Y_{\mathscr{F}^*}$ and $(A, \mathscr{A}, \nu)$ by $(A, \mathscr{A}_{\mathscr{F}^*}, \nu_{\mathscr{F}^*})$. On the other hand, it extends his result in the following sense. Let us interpret the production set correspondence $Y: \mathscr{A} \to R^l$ in the Hildenbrand model of a production economy as the mean production set correspondence. Since it is assumed to be countably additive, convex-valued, and ν-continuous, it follows from Hildenbrand (1974, Theorem 8, p. 77) that there is a correspondence γ of A into R^l such that $\int_C \gamma \subset Y(C)$, and $cl\left(\int_C \gamma\right) = cl(Y(C))$ for every $C \in \mathscr{A}$. The correspondence $\gamma: A \to R^l$ can be regarded as the production set assignment to each primary economic agent as a production unit. Thus the Hildenbrand model of a production economy may be considered as a competitive production economy with the fixed firm structure $\mathscr{F} = \{\{a\} \mid a \in A\}$. In this sense the Hildenbrand's production economy is a Hicksian economy with a continuum of agents. The theorem above gives a more general statement than does Theorem 1 of Hildenbrand (1974, p. 216). It says that whenever the firm structure of an economy is fixed, the set of core allocations is contained in the set of quasi-competetitive allocations. In an economy where a competitive quasi-equilibrium is actually a competitive equilibrium, the set of core allocations exactly equals that of competitive allocations so far as the firm structure of the economy is fixed.

The existence of a competitive quasi-equilibrium of $(\mathscr{E}_{\mathscr{F}}, Y_{\mathscr{F}})$ for $\mathscr{F} \in \mathscr{S}$ can be proved without difficulty using Sondermann (1974, Lemma 8.1, pp. 286–7) under the assumptions (X.1)–(X.3), (Y.1), (Y.2),

(Y.5′) $Y_{\mathscr{F}}(A)$ is closed,

and (Z.1)–(Z.3) in which $Y(A)$ is replaced by $Y_{\mathscr{F}}(A)$.

3. *A Remark on the Core Allocations*

From now on we restrict ourselves to the set of allocations of an economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F} \in \mathscr{S}})$ that are $\mathscr{A}_{\mathscr{F}}$-measurable for each $\mathscr{F} \in \mathscr{S}$. Given an allocation $f: A \to R^l$ in the core of $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F} \in \mathscr{S}})$, there is a firm structure $\mathscr{F}^* \in \mathscr{S}$ with respect to which f is attainable, i.e., $\int_A f \in \int_A e + \int_{\mathscr{F}^*} Y_{\mathscr{F}^*}$. It follows that the allocation f also belongs to the set of core allocations of the production economy $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$. According to the theorem 2.2, f is a quasi-competitive allocation of $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$. Therefore, there exist a price vector p and a production assignment $y: \mathscr{F}^* \to R^l$ such that (p, y, f) is a competitive quasi-equilibrium of $(\mathscr{E}_{\mathscr{F}^*}, Y_{\mathscr{F}^*})$. Nevertheless, under the market price vector p, the economy may be able to do better with a different firm structure, that is, it is possible that the total profits $p \cdot \int_{\mathscr{F}^*} y$ under the firm structure $\mathscr{F}^*$ is less than the maximum possible, i.e., $p \cdot \int_{\mathscr{F}^*} y < \max p \cdot Y(A)$. If we do have the equality $p \cdot \int_{\mathscr{F}^*} y = \max p \cdot Y(A)$, however, it can be easily checked that p, f, $\mathscr{F}^*$, y, and $\pi \in \mathscr{C}(p)$ give rise to a competitive quasi-equilibrium of the economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F} \in \mathscr{S}})$.

More specifically, let $\mathscr{S}_f$ be the set of competitive firm structures with respect to which an allocation f is attainable. Let $W_f[\mathscr{F}]$ be the set of ordered pairs $(p, y) \in R^l_+ \setminus \{0\} + \mathscr{L}_{Y_{\mathscr{F}}}$ such that (p, y, f) is a competitive quasi-equilibrium of $(\mathscr{E}_{\mathscr{F}}, Y_{\mathscr{F}})$. Then, given an allocation f in the core of $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$, f is a quasi-competitive allocation of $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$ if there exist a firm structure $\mathscr{F}^*\in\mathscr{S}_f$ and a pair $(p^*, y^*)\in W_f[\mathscr{F}^*]$ such that $p^*\cdot\int_{\mathscr{F}^*} y=\max p^*\cdot Y(A)$.

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APPENDIX

1. *Proof of Proposition 2.1*

(1) $\mathfrak{S}\in\sum_{\mathscr{F}_1}$ implies that $Pr_{\mathscr{F}_1}{}^{-1}(\mathfrak{S})=Pr_{\mathscr{F}_2}{}^{-1}(\mathfrak{S})\in\mathscr{A}$. It follows that $\mathfrak{S}\in\sum_{\mathscr{F}_2}$.

(2) Let $\mathscr{F}_{A\backslash C_2}$ and $\mathscr{F}_{C_2\backslash C_1}$ be partitions of $A\backslash C_2$ and $C_2\backslash C_1$, consisting of elements of $\mathscr{N}$, respectively. Set $\mathfrak{S}_1=Pr_{\mathscr{F}_1}(C_1)(=\{F\in\mathscr{F}_1 \mid C_1\cap F\neq\phi\})$. Note that as $\mathscr{F}_1\in\mathscr{S}_{C_1}$ we have $C_1=Pr_{\mathscr{F}_1}{}^{-1}(\mathfrak{S}_1)$. Now define a firm structure $\mathscr{F}_2$ by $\mathscr{F}_2=\mathfrak{S}_1\cup\mathscr{F}_{A\backslash C_2}\cup\mathscr{F}_{C_2\backslash C_1}$. Then, $\mathscr{F}_2\in\mathscr{S}$. By (1) we have $\mathfrak{S}_1\in\sum_{\mathscr{F}_2}$. Since $Pr_{\mathscr{F}_2}{}^{-1}(\mathscr{F}_{C_2\backslash C_1})=C_2\backslash C_1\in\mathscr{A}$, one has $\mathfrak{S}_2=\mathfrak{S}_1\cup\mathscr{F}_{C_2\backslash C_1}\in\sum_{\mathscr{F}_2}$ and $Pr_{\mathscr{F}_2}{}^{-1}(\mathfrak{S}_2)=C_2$. Therefore, $\mathscr{F}_2\in\mathscr{S}_{C_2}$. Since $\mathfrak{S}_1$ belongs to $\sum_{\mathscr{F}_2}$, it follows that one has $\mathscr{F}_2\in\mathscr{S}_{C_1}$ and $\mathscr{F}_2\in[\mathscr{F}_1]_{C_1}$.

(3) Let $\mathscr{F}_{A\backslash C_1\cup C_2}$ be a partition of $A\backslash(C_1\cup C_2)$ consisting of elements of $\mathscr{N}$. Define $\mathscr{F}=\mathfrak{S}_1\cup\mathfrak{S}_2\cup\mathscr{F}_{A\backslash C_1\cup C_2}$. Then $\mathscr{F}\in\mathscr{S}$. By definition $\mathscr{F}$ belongs to $\mathscr{S}_{C_1\cup C_2}$. Since one has $\mathfrak{S}_1$, $\mathfrak{S}_2\subset\mathscr{F}$, it follows from (1) $\mathfrak{S}_1$, $\mathfrak{S}_2\in\sum_{\mathscr{F}}$. Hence, we also have $\mathfrak{S}_1\cup\mathfrak{S}_2\in\sum_{\mathscr{F}}$.

Q.E.D.

2. *Proof of Lemma 3.1*

Let $\bar{y}$ be in $\int_{\mathfrak{S}} Y_{\mathscr{F}_1}d\tau_{\mathscr{F}_1}$; then there is y_1 in $\mathscr{L}_{Y_{\mathscr{F}_1}}$ such that $\bar{y}=\int_{\mathfrak{S}} y_1 d\tau_{\mathscr{F}_1}$. Define $y_2:\mathscr{F}_2\to R^l$ by $y_2(F)=y_1(F)$ for F in $\mathfrak{S}$ and $y_2(F)=0$ otherwise. Let B be a Borel set in R^l. By measurability of y_1, $\mathfrak{S}\cap y_1{}^{-1}(\mathrm{B})$ belongs to $\sum_{\mathscr{F}_1}$. Since one has, by definition of y_2, $\mathfrak{S}\cap y_1{}^{-1}(\mathrm{B})=\mathfrak{S}\cap y_2{}^{-1}(\mathrm{B})\subset\mathscr{F}_2$, (1) of the proposition 2.1 implies that $\mathfrak{S}\cap y_2{}^{-1}(\mathrm{B})$ belongs to $\sum_{\mathscr{F}_2}$. Then from the definition of y_2 it follows that $y_2{}^{-1}(\mathrm{B})$ belongs to $\sum_{\mathscr{F}_2}$, showing the measureability of y_2. Also by the definition one has (i) $y_2(F)\in Y_{\mathscr{F}_2}(F)$ for $\tau_{\mathscr{F}_2}$-a.e. in $\mathscr{F}_2$ and (ii) $\int_{\mathfrak{S}} y_1 d\tau_{\mathscr{F}_1}=\int_{\mathfrak{S}} y_2 d\tau_{\mathscr{F}_2}$. It follows that y_2 is in $\mathscr{L}_{Y_{\mathscr{F}_2}}$ and one has $\bar{y}=\int_{\mathfrak{S}} y_2 d\tau_{\mathscr{F}_2}$. Therefore $\bar{y}$ belongs to $\int_{\mathfrak{S}} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$. Since we can interchange the role of indices 1 and 2, we have completed the proof.

Q.E.D.

3. *Proof of Lemma 3.2*

(1) Since one has $0\in Y_{\mathscr{F}}(F)$ for every $F\in\mathscr{F}$ and $\mathscr{F}\in\mathscr{S}_C$, it follows $0\in\int_{\mathfrak{S}} Y_{\mathscr{F}} d\tau_{\mathscr{F}}$ where $\mathfrak{S}=Pr_{\mathscr{F}}(C)$. Therefore one has $0\in Y(C)$. If $\nu(C)=0$, then $\tau_{\mathscr{F}}(\mathfrak{S})=0$. It follows $\int_{\mathfrak{S}} y d\tau_{\mathscr{F}}=0$ for every $y\in\mathscr{L}_{Y_{\mathscr{F}}}$. Hence one obtains $Y(C)=\{0\}$ if $\nu(C)=0$.

(2) Given $\mathscr{F}_1\in\mathscr{S}_{C_1}$, by (2) of the proposition 2.1, there exists $\mathscr{F}_2\in\mathscr{S}_{C_2}$ such that $\mathscr{F}_2\in[\mathscr{F}_1]_{C_1}$. This together with the lemma 3.1 implies that $\int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1}=\int_{\mathfrak{S}_1} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$, where $\mathfrak{S}_1\in\sum_{\mathscr{F}_1},\sum_{\mathscr{F}_2}$, and $\cup\mathfrak{S}_1=C_1$. Let $\mathfrak{S}_2=Pr_{\mathscr{F}_2}(C_2\backslash C_1)$. Since one has $0\in\int_{\mathfrak{S}_2} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$, it follows $\int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1}\subset\int_{\mathfrak{S}_1\cup\mathfrak{S}_2} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$. Therefore, one obtains $Y(C_1)\subset Y(C_2)$.

Q.E.D.

4. *Proof of Proposition 3.3*

Let C_1 and C_2 be any disjoint sets in $\mathscr{A}$. Let $\bar{y}_1$ belongs to $Y(C_1)$ and $\bar{y}_2$ to $Y(C_2)$. We need to show that $\bar{y}_1+\bar{y}_2$ is in $Y(C_1\cup C_2)$. By definitions of $Y(C_1)$ and $Y(C_2)$ one has $\bar{y}_1\in\int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1}$, and $\bar{y}_2\in\int_{\mathfrak{S}_2} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$, for some $\mathscr{F}_1\in\mathscr{S}_{C_1}$, $\mathscr{F}_2\in\mathscr{S}_{C_2}$, where $\mathfrak{S}_1\in\sum_{\mathscr{F}_1}$, $\mathfrak{S}_2\in\sum_{\mathscr{F}_2}$, $\cup\mathfrak{S}_1=C_1$, and $\cup\mathfrak{S}_2=C_2$. By (3) of the proposition 2.1 there exists $\mathscr{F}\in\mathscr{S}_{C_1\cup C_2}$ such that $\mathfrak{S}_1,\mathfrak{S}_2\in\sum_{\mathscr{F}}$. Then it follows from the lemma 3.1 that $\int_{\mathfrak{S}_1} Y_{\mathscr{F}} d\tau_{\mathscr{F}}=\int_{\mathfrak{S}_1} Y_{\mathscr{F}_1} d\tau_{\mathscr{F}_1}$ and $\int_{\mathfrak{S}_2} Y_{\mathscr{F}} d\tau_{\mathscr{F}}=\int_{\mathfrak{S}_2} Y_{\mathscr{F}_2} d\tau_{\mathscr{F}_2}$. Therefore, we have $\bar{y}_1+\bar{y}_2\in\int_{\mathfrak{S}_1\cup\mathfrak{S}_2} Y_{\mathscr{F}} d\tau_{\mathscr{F}}\subset Y(C_1\cup C_2)$.

Q.E.D.

5. *Proof of Lemma 3.4*

(1) Let C be a set in $\mathscr{A}$ with $\nu(C)=1$. Let $\bar{y}\in Y(A)$; then, one has $\bar{y}=\int_{\mathscr{F}} y$ for some $\mathscr{F}\in\mathscr{S}$ and $y\in\mathscr{L}_{Y_{\mathscr{F}}}$. By (Y.3) there exist a firm structure $\mathscr{F}'\in\mathscr{S}_C$ and $y'\in\mathscr{L}_{Y_{\mathscr{F}'}}$ such that one has $\bar{y}=\int_{\mathscr{F}'} y'$. Since $\mathscr{F}'$ is in $\mathscr{S}_C$, there exists a set $\mathfrak{S}\in\sum_{\mathscr{F}'}$ such that one has $C=Pr_{\mathscr{F}'}^{-1}(\mathfrak{S})$. It follows from $\tau_{\mathscr{F}'}(\mathfrak{S})=\nu\circ Pr_{\mathscr{F}'}^{-1}(\mathfrak{S})=\nu(C)=1$ that $\int_{\mathscr{F}'} y' d\tau_{\mathscr{F}}=\int_{\mathfrak{S}} y' d\tau_{\mathscr{F}'}$. Therefore, one obtains that $\bar{y}\in\int_{\mathfrak{S}} Y_{\mathscr{F}'} d\tau_{\mathscr{F}'}\subset Y(C)$. On the other hand, by (2) of the lemma 3.2 we have $Y(C)\subset Y(A)$. It follows that $Y(C)=Y(A)$ if $\nu(C)=1$.

(2) Let $(C_n)_n$ be an increasing sequence in $\mathscr{A}$ such that $\cup_n C_n=A$. Let $\mathscr{F}$ be a firm structure in $\mathscr{S}$. By (Y.4) there exist a firm structure $\mathscr{F}'$ in $\mathscr{S}$, a subsequence $(C_k)_k$ of the sequence $(C_n)_n$ with $\cup_k C_k=A$, and an associated sequence $(A_k)_k$ in $\mathscr{A}_{\mathscr{F}'}$ such that one has $\int_{\mathscr{F}} Y_{\mathscr{F}}=\int_{\mathscr{F}'} Y_{\mathscr{F}'}$, $A_k\subset C_k$, and $\nu(C_k\backslash A_k)=0$ for each k. This implies $\nu(\cup_k A_k)=$

$\nu(A)$. It follows from (Y. 3) that one has $\int_{\cup_k \mathfrak{S}_k} Y_{\mathscr{T}'} = \int_{\mathscr{T}'} Y_{\mathscr{T}'}$ where $Pr_{\mathscr{T}'}^{-1}(\mathfrak{S}_k) = A_k$ for each k, as one has $\tau_{\mathscr{T}'}(\cup_k \mathfrak{S}_k) = \nu(\cup_k A_k) = 1$. Hence, $Y_{\mathscr{T}'}(A_k) = \int_{\mathfrak{S}_k} Y_{\mathscr{T}'} \to Y_{\mathscr{T}'}(\cup_k A_k) = \int_{\cup_k \mathfrak{S}_k} Y_{\mathscr{T}'} = Y_{\mathscr{T}'}(A)$ and $Y_{\mathscr{T}'}(A_k) \subset Y(A_k) \subset Y(C_k)$ imply that $Y_{\mathscr{T}'}(A) = Y_{\mathscr{T}}(A) \subset \lim_n \sup Y(C_n)$. Since this is true for each $\mathscr{T}$ in $\mathscr{S}$, one obtains $Y(A) = \bigcup_{\mathscr{T} \in \mathscr{S}} Y_{\mathscr{T}}(A) \subset \lim_n \sup Y(C_n)$.

On the other hand by (Y.5) and (2) of the lemma 3.2 one has $\lim_n \sup Y(C_n) \subset Y(A)$.

Q.E.D.

6. *Proof of Lemma 4.1*

(1) (Y.7) implies the balancedness of v_p. Hence, $\mathscr{C}(p)$ is nonempty [see Kannai (1969) or Schmeidler (1967)].

(2) Let μ be in $\mathscr{C}(p)$ and $(C_n)_n$ be an increasing sequence in $\mathscr{A}$ with $\cup_n C_n = A$. Then by definition of $\mathscr{C}(p)$, one has $v_p(C_n) \leq \mu(C_n) \leq \mu(A) = v_p(A)$ for every $n = 1, 2, \ldots$. Now (2) of the lemma 3.4 implies that $v_p(C_n)$ tends to $v_p(A)$. Thus, $\mu(C_n)$ converges to $\mu(A)$. Therefore, μ is countably additive.

(3) Let C be a set in $\mathscr{A}$ with $\nu(C) = 0$. (1) of the lemma 3.4 implies that $v_p(A \backslash C) = v_p(A)$. Hence, one has $0 \leq \mu(C) = \mu(A) - \mu(A \backslash C) \leq v_p(A) - v_p(A \backslash C) = 0$; that is, μ is absolutely continuous with respect to the measure ν.

(4) Let μ be an element of $\mathscr{C}$(p). Suppose there were $C \in \mathscr{A}_{\mathscr{T}^*}$ with $\mathfrak{S} \in \sum_{\mathscr{T}^*}$ and $\cup \mathfrak{S} = C$ such that $\mu(C) > \int_{\mathfrak{S}} p \cdot y d\tau_{\mathscr{T}^*} \geq 0$. By definition of $\mathscr{C}$(p), we have $\mu(C) \geq v_p(C) = \sup p \cdot Y(C) = \int_{\mathfrak{S}} p \cdot y d\tau_{\mathscr{T}^*}$, and $\mu(A) = \max p \cdot Y(A) = \int_{\mathscr{T}^*} p \cdot y$. Therefore, we would have $\int_{\mathscr{T}^*} p \cdot y = \mu(A) = \mu(A \backslash C) + \mu(C) > \int_{\mathscr{T}^* \backslash \mathfrak{S}} p \cdot y d\tau_{\mathscr{T}^*} + \int_{\mathfrak{S}} p \cdot y d\tau_{\mathscr{T}^*} = \int_{\mathscr{T}^*} p \cdot y$, a contradiction.

Q.E.D.

7. *Proof of the Theorem*

The proof is a simple and straightforward application of the existence proofs employed by D. Sondermann (1974), and T. Ichiishi (1977).

Lemma 7.1: *Let* $K \subset P^+$ be a compact set in R^l. Then, the set

$$Y_K = \{\bar{y} \in Y(A) \mid p \cdot \bar{y} \geq 0 \text{ for some } p \in K\}$$

is compact in R^l.

Proof: See Sondermann (1974, Lemma 6.3 (1), p. 278).

Corollary 7.2: *The correspondence* $p \mapsto S(p)$ *is convex- and compact-valued and upper semi-continuous (u.s.c.) on every compact subset* K *of* P^+.

Proof: By the lemma 7.1 S(p) is contained in the compact set Y_K. Therefore, the corollary follows from Debreu (1959, 3.5 (3)).

Q.E.D.

Lemma 7.3: *Let* $K \subset P^+$ *be compact. Then, there exists* $M > 0$ *such that, for every* C *in*

$\mathscr{A}$ *and for any p, q in K, one has*

$$|\nu_p(C)-\nu_q(C)|\leq||p-q||M.$$

Proof: See Sondermann (1974, Lemma 6.3(2), pp. 278–279).

Proof of Theorem 5.1

1. Let ∂ be an arbitrary positive number and K an arbitrary compact subset of P^+. Set $W(p)=co \bigcup_{\substack{q\in K\\ ||p-q||<\partial}} \mathscr{C}(q)$, where co H denotes the convex hull of H, and define a correspondence $C_{\partial,K}: K\to L_1(A, \mathscr{A}, \nu)$ by $C_{\partial,K}(p)=cl\,W(p)$. Then, $C_{\partial,K}$ is strongly lower semi-continuous (l.s.c.) on K, convex-valued, and strongly closed-valued. Hence, by Michael (1956), there exists a strongly continuous selection, $\pi_{\partial,K}: K\to L_1(A, \mathscr{A}, \nu)$ with $\pi_{\partial,K}(p)\in C_{\partial,K}(p)$ for every p in K.

Now, for every p in K and every $\pi_{\partial,K}(p)$ in $C_{\partial,K}(p)$, we have

$$\left|\int_A \pi_{\partial,K}(p)(a)\nu(da)-\int_A \pi_p(a)\nu(da)\right|$$
$$\leq\left|\int_A \pi_{\partial,K}(p)-\sum_i\alpha_i\int_A\pi_{q_i}\right|+\left|\sum_i\alpha_i\int_A\pi_{q_i}-\int_A\pi_p\right|,$$

where $\sum_i\alpha_i\pi_{q_i}\in W(p)$ with $\alpha_i\geq 0$ for each i and $\sum_i\alpha_i=1$ approximates $\pi_{\partial,K}(p)$ and $\pi_p\in\mathscr{C}(p)$. Indeed, since $\pi_{\partial,K}(p)$ belongs to the closure of $W(p)$ in the strong topology, we can pick an element π in $W(p)$ which is arbitrarily close to $\pi_{\partial,K}(p)$ in the strong topology; then, by definition of $W(p)$, π can be written as $\sum_i\alpha_i\pi_{q_i}$ with π_{q_i} in $\mathscr{C}(q_i)$, q_i in K, and $||p-q_i||<\partial$ for each i. Hence, the first term in the right hand-side of the above inequality can be made arbitrarily small. On the other hand, by the lemma 7.3, there exists a positive number M_K such that $\int_A(\pi_p-\pi_q)\leq||p-q||M_K$ for any p, q in K. Then

$$\left|\sum_i\alpha_i\int_A\pi_{q_i}-\int_A\pi_p\right|\leq\sum_i\alpha_i\left|\int_A(\pi_{q_i}-\pi_p)\right|$$
$$\leq\sum_i\alpha_i||q_i-p||M_K<\partial M_K.$$

Therefore, we have $\left|\int_A\pi_{\partial,K}(p)(a)\nu(da)-\int_A\pi_p(a)\nu(da)\right|<\partial M_K.$

2. Let $(P^k)_{k\geq 1}$ be an increasing sequence of nonempty, convex, compact subsets of the interior of P such that $\cup_k P^k=\text{int}\,P$. By the lemma 7.1 the set Y_{P^k} is compact for every k. Hence, by the lemma 7.3, for each k there exists a positive number M_k such that, for any p, q in P^k, one has $|\nu_p(C)-\nu_q(C)|\leq||p-q||M_k$ for every C in $\mathscr{A}$. Now, take a decreasing sequence $(\partial_k)_k$ of positive numbers so that $\partial_k\to 0$ and $\partial_k M_k<1/k$ for each k. Set $C^k=C_{\partial_k,P^k}$, and $\pi_p{}^k=\pi_{\partial_k,P^k}(p)$. Then by the previous step one obtains

$$\left|\int_A\pi_p{}^k-\int_A\pi_p\right|<1/k \quad \text{for every } p \text{ in } P^k.$$

For each k, let U^k be the least closed convex cone, with vertex 0, containing P^k, and let T^k denote its dual cone, i.e., $T^k=(U^k)^\circ$. Define a sequence of correspondences F^k from P^k into T^k by

$$F^k(p)=G^k(p)-S(p)-\int_A e-(1/k, \ldots, 1/k)$$

where

$$G^k(p)=\int_A \psi(a, p, p\cdot e(a)+\pi_p{}^k(a))\nu(da).$$

It is easy to check that F^k is u.s.c. on P^k (using the corollary 7.2), nonempty-valued, convex- and compact-valued, and that for every $\bar{z}$ in $F^k(p)$, $p\cdot \bar{z}\leq 0$. Therefore, for every k, there exists p_k in P^k, by Debreu (1956), such that $T^k\cap F^k(p_k)\neq\phi$. This means that, for each k, there exist a price vector p_k in P^k, a consumption plan f^k in $\mathscr{L}_X$, and a total production $\bar{y}^k$ in $S(p_k)$ such that

$$f^k(a)\in\psi(a, p_k, p_k\cdot e(a)+\pi_{p_k}{}^k(a)),\ \nu\text{-a.e. in } A,$$

$$\bar{z}^k=\int_A f^k-\bar{y}^k-\int_A e-(1/k, \ldots, 1/k)\ \in\ T^k.$$

Put $\bar{x}^k=\int_A f^k$. By the same argument as in Sondermann (1974, p. 283) we can show that $(\bar{x}^k)_k$ and $(\bar{y}^k)_k$ are bounded. Therefore, we can extract convergent subsequences, still denoted by the index k, such that

$$\lim_k\int_A f^k=b,\ \lim_k \bar{y}^k=\bar{y},\ \lim_k p_k=p^*.$$

By Debreu (1962, Lemma 3), $\bar{y}$ is in $S(p^*)$ and hence p^* is in P_Y.

3. Put $K=\{p_1, p_2, \ldots\}\cup\{p^*\}$. K is compact in P^+. Hence by the lemma 7.3 there exists a positive number M such that for any p_k in K

$$|\, \nu_{p^*}(C)-\nu_{p_k}(C)\,|\leq||\,p^*-p_k\,||\,M\quad\text{for every } C \text{ in } \mathscr{A}.$$

Let $(C_n)_n$ be an arbitrarily chosen decreasing sequence in $\mathscr{A}$ such that $\cap_n C_n=\phi$. Fix $\varepsilon>0$ arbitrarily. Since one has $\lim_k p_k=p^*$, there is an integer k_1 such that $||p^*-p_k||<\varepsilon/M$ for $k\geq k_1$. Then for any C in $\mathscr{A}$ one has $|\,\nu_{p^*}(C)-\nu_{p_k}(C)\,|<\varepsilon$ for $k\geq k_1$. Let k_2 be such that $1/k<\varepsilon$ for $k\geq k_2$. Put $k_0=\max(k_1, k_2)$. Then, for $k\geq k_0$, $\nu_{p^*}(A\backslash C_n)-2\varepsilon<\nu_{p_k}(A\backslash C_n)-\varepsilon\leq\int_{A\backslash C_n}\pi_{p_k}-\varepsilon<\int_{A\backslash C_n}\pi_{p_k}{}^k$. Therefore, one obtains for all k sufficiently large $\int_{C_n}\pi_{p_k}{}^k=\int_A\pi_{p_k}{}^k-\int_{A\backslash C_n}\pi_{p_k}{}^k<\int_A\pi_{p_k}-\nu_{p^*}(A\backslash C_n)+3\varepsilon<\nu_{p^*}(A)-\nu_{p^*}(A\backslash C_n)+2\varepsilon$. Since $\nu_{p^*}(A\backslash C_n)$ converges to $\nu_{p^*}(A)$, by (2) of the lemma 3.4, $\int_{C_n}\pi_{p_k}{}^k$ converges to 0 uniformly in k. As $\{\pi_{p_k}{}^k\}$ is bounded, it follows from Dunford and Schwartz (1958, Theorem IV. 8.9, p. 292) that $\{\pi_{p_k}{}^k\}$ is weakly sequentially compact.

4. Therefore, one can extract convergent subsequences, again denoted by the indices k, such that $\lim_k p_k=p^*$, $\lim_k\int_A f^k=b$, $\lim_k \bar{y}^k=\bar{y}\in S(p^*)$, and $\lim_k \pi_{p_k}{}^k=\pi_{p^*}$ (weakly). Define the profit distribution $\pi^*: A\to R_+$ by $\pi^*(a)=\pi_{p^*}(a)$ if $p^*\cdot\bar{y}>0$, and $=0$ if $p^*\cdot\bar{y}=0$. If p^* is in P_Y^+, then $\left|\int_A\pi^*-\nu_{p^*}(A)\right|\leq\left|\int_A\pi_{p^*}-\int_A\pi_{p_k}{}^k\right|+\left|\int_A\pi_{p_k}{}^k-\int_A\pi_{p_k}\right|+\left|\int_A\pi_{p_k}-\nu_{p^*}(A)\right|$. Since the right hand-side of this inequality can be made arbitrarily small by

taking sufficiently large k, we obtain $\int_A \pi^* = v_{p^*}(A)$. One also has $\int_C \pi^* + 3\varepsilon > \int_C \pi_{p_k}{}^k + 2\varepsilon > \int_C \pi_{p_k} + \varepsilon \geq v_{p_k}(C) + \varepsilon \geq v_{p^*}(C)$ for k sufficiently large. Therefore, π^* belongs to $\mathscr{C}(p^*)$ for p^* in P_Y^+.

If p^* is in $P_Y \backslash P_Y^+$, then $\pi^*(a)=0$ for every a in A, and $\int_A \left| \pi^* - \pi_{p_k}{}^k \right| = \int_A \pi_{p_k}{}^k < \int_A \pi_{p_k} + 1/k$ for any π_{p_k} in $\mathscr{C}(p_k)$, where the last term is equal to $p_k \cdot \bar{y}^k + 1/k$ and converges to $p^* \cdot \bar{y} = 0$. Consequently, π^* belongs to $\mathscr{C}(p^*)$ and $\pi_{p_k}{}^k$ converges to π^* strongly if p^* is in $P_Y \backslash P_Y^+$.

Thus, in both cases, $\pi_{p_k}{}^k$ converges weakly to π^*, and π^* belongs to $\mathscr{C}(p^*)$.

5. We do not know, in general, whether the sequences $(f^k)_k$ and $(\pi_{p_k}{}^k)_k$ are convergent in the strong topology; however, we know that both sequences are bounded below by a ν-integrable function and we have shown that the sequences of their integrals are convergent: $\lim_k \int_A f^k = b$, $\lim_k \int_A \pi_{p_k}{}^k = \int_A \pi^* = p^* \cdot \bar{y}$. We now apply Fatou's lemma in l-dimension [see Hildenbrand (1974, Lemma 3, p. 69)] to the sequence $(f^k, \pi_{p_k}{}^k)_k$. Consequently there exist a function f^* of A into R^l and a function π of A into R_+ such that

(1) $\int_A f^* \leq b$ and $\int_A \pi \leq p^* \cdot \bar{y}$;

(2) ν-a.e. in A, $(f^*(a), \pi(a))$ is a cluster point in R^{l+1} of the sequence $(f^k(a), \pi_{p_k}{}^k(a))_k$.

We shall show that

(3) ν-a.e. in A, $\pi_{p_k}{}^k(a)$ converges to $\pi^*(a)$.

Put $\lambda_k(C) = \int_C \pi_{p_k}{}^k$. By the previous step, $\pi_{p_k}{}^k$ converges to π^* weakly. Hence, by Dunford and Schwartz (1958, IV. 13.25, p. 342) the limit $\lim_k \lambda_k(C)$ exists for each C in $\mathscr{A}$. Then by the Vitali-Hahn-Saks theorem [see Dunford and Schwartz (1958, Theorem III. 7.2. and Corollary III. 7.3, pp. 158–159)] the set function $\lambda(C) = \lim_k \lambda_k(C)$ is countably additive on $\mathscr{A}$. Since $\{\pi_{p_k}{}^k\}_k$ is bounded, by the way Fatou's lemma is proved in Hildenbrand (1974, pp. 69–73), we can take π in (1) to satisfy the equality $\int_A \pi = p^* \cdot \bar{y}$. Therefore, passing to a subsequence of $(f^k, \pi_{p_k}{}^k)_k$, we may assume that π is also a weak limit of $(\pi_{p_k}{}^k)_k$. Then, by the uniqueness of weak limit [Dunford and Schwartz (1958, Lemma II. 3. 26, p. 68)], π^* differs from π in a ν-null set. Hence (2) implies (3).

We shall now show that the function f^* determined in (1) has the proprty: ν-a.e. in A, $f^*(a)$ belongs to $\phi(a, p^*, p^* \cdot e(a) + \pi^*(a))$. Since ν-a.e. in A, $f^k(a)$ is in $X(a)$, which by assumption is a closed set, (2) implies

(4) ν-a.e. in A, $f^*(a)$ belongs to $X(a)$.

For every $k=1, 2, \ldots$, we have ν-a.e. in A, $p_k \cdot f^k(a) \leq p_k \cdot e(a) + \pi_{p_k}{}^k(a)$. Thus (2) and (3) imply $p^* \cdot f^*(a) \leq p^* \cdot e(a) + \pi^*(a)$, ν-a.e. in A, which together with (4) proves

(5) ν-a.e. in A, $f^*(a)$ belongs to $B(a, p^*, p^* \cdot e(a) + \pi^*(a))$.

Consider the case $\inf p^* \cdot X(a) < p^* \cdot e(a) + \pi^*(a)$, i.e., there is a feasible consumption x in $X(a)$

such that $p^*\cdot x<p^*\cdot e(a)+\pi^*(a)$. Thus, according to (3) we have for k large enough $p_k\cdot x<p_k\cdot e(a)+\pi_{p_k}{}^k(a)$. Consequently, $f^k(a)\in\psi(a, p_k, p_k\cdot e(a)+\pi_{p_k}{}^k(a))$, ν-a.e. in A, implies $x\precsim_a f^k(a)$. The set $\{z\in X(a)\mid x\precsim_a z\}$ is by assumption closed. Hence, (2) implies $x\precsim_a f^*(a)$. Thus we proved

(6) ν-a.e. in A, $x\in X(a)$ and $p^*\cdot x<p^*\cdot e(a)+\pi^*(a)$ imply $x\precsim_a f^*(a)$.

But in the case $\inf p^*\cdot X(a)<p^*\cdot e(a)+\pi^*(a)$ every vector $x\in X(a)$ with $p^*\cdot x=p^*\cdot e(a)+\pi^*(a)$ is a limit of vectors x_n with $p^*\cdot x_n<p^*\cdot e(a)+\pi^*(a)$, since $X(a)$ is by assumption convex. Thus we proved, ν-a.e. in A, $\inf p^*\cdot X(a)<p^*\cdot e(a)+\pi^*(a)$ implies that $f^*(a)$ is a maximal element in $B(a, p^*, p^*\cdot e(a)+\pi^*(a))$. This together with (5) proves that

(7) ν-a.e. in A, $f^*(a)$ belongs to $\psi(a, p^*, p^*\cdot e(a)+\pi^*(a))$.

Now, let $\mathscr{F}^*\in\mathscr{S}$ and $y^*\in\mathscr{L}_{Y_{\mathscr{F}^*}}$ be such that $\bar{y}=\int_{\mathscr{F}^*}y^*$. Then exactly as in Sondermann (1974, p. 285) we can show that

$$\int_A f^*\leq\int_{\mathscr{F}^*}y^*+\int_A e,\ \text{and}\ \ p^*\cdot\int_A f^*=p^*\cdot\int_{\mathscr{F}^*}y^*+p^*\cdot\int_A e.$$

This completes the proof that $(p^*, f^*, \mathscr{F}^*, y^*, \pi^*)$ is a competitive quasi-equilibrium for the production economy $(\mathscr{E}, (Y_{\mathscr{F}})_{\mathscr{F}\in\mathscr{S}})$.

Q.E.D.

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