CONTINUOUS PREFERENCE RELATIONS WHICH ARE OBSERVABLE IN MARKETS

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I. *Introduction*

Economic theories have been largely built on the basis of rational behaviors by rational economic agents. In the consumption sector of an economy one thus analyzes behaviors of satisfaction maximizing consumers whose preference patterns exhibit a consistency in the sense that preference relations are transitive or acyclic. A conspicuous deviation from this approach to economic theory has appeared in the past decade in the works of Sonnenschein [12], Mas-Colell [6], Gale-Mas-Colell [3], Shafer [10], and Shafer-Sonnenschein [11] in general equilibrium analysis of market economies. In all of these works the assumptions of transitivity and acyclicity of preference relations were dropped. And it was shown that the nonemptiness of demand sets of individuals and the existence of a competitive equilibrium of an economy can be proven as long as the convexity assumptions on continuous preference relations and on consumption sets are retained. (Shafer-Sonnenschein [11] does not require preference relations to be convex.)

In view of the results obtained in a continuum of agents' models that not only the assumption of convexity of preference relations (Aumann [1]) but also that of consumption sets (Mas-Colell [7], and Yamazaki [13, 14]) is not required in establishing existence of a competitive equilibrium if the transitivity or the acyclicity of preference relations is assumed, a natural question suggests itself: Can we altogether relax the convexity requirements and the consistency of preference relations?

Whereas it is interesting to find meaningful sufficient conditions which guarantee the nonemptiness of individual demand sets (for positive market prices) without requiring convexity, transitivity, or acyclicity, we note the fact that a characterization of continuous preference relations which give rise to nonempty demand sets for any positive price vector is "almost trivially" available (see Mukherji [8] and our Lemma 1 below). On the other hand, as noted in [13], so long as demand sets of individual consumers are nonempty on compact budget sets, the consistency requirement for preference relations is not essential in the existence problem of large economies. Thus the largest class of continuous preference relations of any interest to us consists of those which give rise to nonempty demand sets on compact budget sets. In fact, from a Samuelsonian point of view, unless preferences are "visible" or "observable" in markets through the individual demands that they generate, they lie beyond the scope of analysis of market economies.

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The above considerations lead us to the investigation of mathematical properties possessed by the class $\mathscr{P}$ of all continuous preference relations which are observable in markets. In order to show that it is not entirely pointless to study about $\mathscr{P}$, we shall present in the Appendix an example of a continuous preference relation which is observable in markets but which does not satisfy any of the known sufficient conditions for existence of maximal elements in budget sets with respect to the preference relation. The present paper contains two remarks on the mathematical properties of $\mathscr{P}$.

A topology on $\mathscr{P}$ will be introduced in the manner of Hildenbrand [5] and Grodal [4]. Let $\mathscr{R}$ denote the set of all closed subsets of $R^l \times R^l$. To induce a topology on the set of all continuous preference relations which are irreflexive and transitive, Grodal [4] used an injection from the set into $\mathscr{R}$. In the present paper I use an injection from $\mathscr{P}$ into $\mathscr{X} \times \mathscr{R}$ where $\mathscr{X}$ is the set of all consumption sets. If one restricts $\mathscr{P}$ to the set of irreflexive preference relations, then the topology introduced here coincides with the one in [4] and [5]. In this topology $\mathscr{P}$ becomes a compact metrizable space (the first remark). The second remark is concerned with the density of continuous preference relations which induce locally upper hemicontinuous demand correspondences. When consumption sets are not necessarily convex, one cannot expect to have upper hemicontinuous individual demand correspondences in general. The best one can hope for is that a preference relation induces a locally upper hemicontinuous demand correspondence. We shall show that, by restrcicting the set $\mathscr{P}$ somewhat, the set of preference relations giving rise to locally upper hemicontinuous demand correspondences is dense at any given price vector p and any wealth level w.

II. *Notation, Definitions and the Statement of Results*

R^l denotes the l-dimensional Euclidean space. If x belongs to R^l, then $x=(x^1, \dots, x^l)$. R^l represents the *commodity space* and an element x of R^l a *commodity bundle*. For any x and y in R^l we take $x>y$ to mean $x^j>y^j$ for all j, and $x\geq y$ to mean $x^j\geq y^j$ for all j. If Q is a subset of R^l and x is an element of R^l, $x\leq Q$ means $x\leq y$ for every y in Q. The inner product $x^1y^1+\dots+x^ly^l$ of two members x, y of R^l is denoted $x\cdot y$. The symbol \ will be used for set-theoretic subtraction.

A *consumption set* X is a nonempty subset of the commodity space which is closed and bounded from below. Given a vector b in R^l and a compact set K of R^l, we denote by $\mathscr{X}$ the set of all consumption sets X which have the properties: $b\leq X$ and $X\cap K\neq\phi$. b and K are fixed once for all in this paper. A commodity need not be infinitely divisible; however, this does not mean that the commodity space should be further restricted. In fact the divisibility and/or indivisibility of commodities will be expressed through the shapes of consumption sets.

We define a *preference relation* as a pair $(X, \succ)$ where X is a consumption set in $\mathscr{X}$ and $\succ$ is a subset of $X\times X$. Instead of $(x, y)\in\succ$ we shall use the notation $x\succ y$. Thus $x\not\succ y$ means $(x, y)\notin\succ$. A preference relation $(X, \succ)$ is said to be *continuous* if $\succ$ is open relative to $X\times X$. The set of all continuous preference relations are denoted by $\mathscr{P}^\circ$.

$R^l{}_{++}$ denotes the set of all vectors p with $p^j>0$ for all j. Given $(p, w)\in R^l{}_{++}\times R$, we define the *budget set* for X

$$B(p, w; X)=\{x\in X \mid p\cdot x\leq w\},$$

and the *demand set* for $(X, \succ)$

$D(p, w; X, \succ) = \{x \in B(p, w; X) \mid z \not\succ x \text{ for every } z \text{ in } B(p, w; X)\}$.

We denote by $\mathscr{P}$ the set of all continuous preference relations which are observable in markets, i.e.,

$\mathscr{P} = \{(X, \succ) \in \mathscr{P}^\circ \mid \text{for any } (p, w) \text{ in } R^l_{++} \times R,\ D(p, w; X, \succ) \neq \phi$
whenever $B(p, w; X) \neq \phi\}$.

Given a vector p of R^l_{++} and a nonempty finite subset Q of R^l define

$\alpha(p, Q) = \max\ \{p \cdot x \mid x \in Q\}$.

We shall introduce a property of preference relations which we shall refer to as the property (FD):

(FD) Given a nonempty finite subset Q of X and a vector p in R^l_{++}, there exists an $x^* \in B(p, \alpha(p, Q); X)$ such that $z \not\succ x^*$ for every z in Q.

Let

$\mathscr{P}_{FD} = \{(X, \succ) \in \mathscr{P}^\circ \mid (X, \succ) \text{ satisfies } (FD)\}$.

One can get a characterization of the set $\mathscr{P}$ very easily (cf. Mukherji [8]). We shall state this as a lemma.

LEMMA 1: $\mathscr{P} = \mathscr{P}_{FD}$.

REMARK: Let $\mathscr{P}_t$ be the set of all continuous, irreflexive and transitive preference relations. It is well known that $\mathscr{P}_t \subset \mathscr{P}$. (See Schmeidler [9, Lemma 2].) A preference relation $(X, \succ)$ is said to be acyclic if, for any finite subset $\{x_1, \ldots, x_k\}$ of X, $[x_1 \succ x_2, \ldots,$ and $x_{k-1} \succ x_k]$ implies $x_k \not\succ x_1$. Let $\mathscr{P}_{ac}$ denote the set of all continuous and acyclic preference relations. Then, $\mathscr{P}_{ac} \subset \mathscr{P}$. (See Bergstrom [2]; note also that the proof given in [9, Lemma 2] for a continuous, irreflexive and transitive relation can be applied to the case of a continuous and acyclic preference relation without any modifications.) Let $\mathscr{P}_{co}$ denote the set of all continuous and convex preference relations. It is also known that $\mathscr{P}_{co} \subset \mathscr{P}$. (See Sonnenschein [12], Mas-Colell [6], and Shafer [10].) In general all of these inclusion relations are strict. In fact we shall exhibit in the Appendix an example of a continuous preference relation which is not acyclic, nor transitive, nor convex, but which is observable in markets.

Given a preference relation $(X, \succ)$ define

$F_\succ = \{(x, y) \in X \times X \mid x \not\succ y\}$.

If $(X, \succ)$ is a member of $\mathscr{P}^\circ$, $F_\succ$ is a closed subset of $R^l \times R^l$. Let $\mathscr{R}$ denote the space of all closed subsets of $R^l \times R^l$ endowed with the topology of closed convergence. In this topology a sequence (F_n) in $\mathscr{R}$ converges to F, denoted $\mathrm{Lim}\ F_n = F$, if and only if Lim Sup $F_n = F =$ Lim Inf F_n, where Lim Sup $F_n = \{x \in R^l \mid$ for any neighborhood U of x and for any n, there exists $m \geq n$ such that $U \cap F_m \neq \phi\}$, and Lim Inf $F_n = \{x \in R^l \mid$ for any neighborhood U of x, there exists n such that for all $m \geq n$ one has $U \cap F_m \neq \phi\}$. In this topology $\mathscr{R}$ becomes a compact metrizable space (see Hildenbrand [5, Theorem 2, p. 19]). The set $\mathscr{X}$ is also endowed with the topology of closed convergence. Again it is a compact metrizable space (Hildenbrand [5, p. 86 and p. 97]). Let us define a natural map ι of $\mathscr{P}$ into $\mathscr{X} \times \mathscr{R}$, where the latter is endowed with the product topology, by $(X, \succ) \mapsto (X, F_\succ)$. Then, it is immediate that the map $\iota: \mathscr{P} \to \mathscr{X} \times \mathscr{R}$ is an injection. We thus endow the set $\mathscr{P}$ with the topology induced by the injection ι. If a preference relation $(X, \succ)$ is irreflexive, i.e., $x \not\succ x$ for every x in X, the set $\{x \in R^l \mid (x, x) \in F_\succ\}$ determines the consumption set X.

Hence, the topology used by Hildenbrand [5] and Grodal [4] coincides with ours on the set of all continuous and irreflexive preference relations.

Our first remark is the following:

PROPOSITION 1: *$\mathscr{P}$ is a compact, metrizable space.*

We now turn to the problem of denseness of continuous preference relations which induce locally upper hemicontinuous demand correspondences: More precisely, given $(p, w)\in R^l_{++}\times R$, define

$$\mathscr{P}(p, w)=\{(X, \succ)\in\mathscr{P} \mid B(p, w; X)\neq\phi\},$$
$$\mathscr{P}_{uhc}(p, w)=\{(X, \succ)\in\mathscr{P}(p, w) \mid D(p, w; X, \succ) \text{ is upper hemicontinuous at } (p, w)\};$$

then the problem is whether $\mathscr{P}_{uhc}(p, w)$ is dense in $\mathscr{P}(p, w)$. Unfortunately there is one catch in this problem which is related to the question of extendability of a preference relation $(X, \succ)$ in $\mathscr{P}$ to $(X', \succ')$ in $\mathscr{P}$ for any given $X'\in\mathscr{X}$ with $X\subset X'$. Whereas it is an interesting question to be asked, it is beyond the scope of our present note. Thus we restrict the set $\mathscr{P}$ to $\mathscr{P}^*$:

$$\mathscr{P}^*=\{(X, \succ)\in\mathscr{P} \mid \text{there exists a preference relation } (X^*, \succ^*) \text{ in } \mathscr{P} \text{ such that } (X, \succ) \text{ is a restriction of } (X^*, \succ^*) \text{ to } X, \text{ where } X^*=\{x\in R^l \mid b\leq x\}\}.$$

The sets $\mathscr{P}^*(p, w)$ and $\mathscr{P}_{uhc}^*(p, w)$ are defined to be restrictions of $\mathscr{P}(p, w)$ and $\mathscr{P}_{uhc}(p, w)$ to $\mathscr{P}^*$. Then, our second remark is given by the following:

PROPOSITION 2: *$\mathscr{P}^*_{uhc}(p, w)$ is dense in $\mathscr{P}^*(p, w)$.*

III. *Proofs*

Proof of Lemma 1:

To show $\mathscr{P}\subset\mathscr{P}_{FD}$. Let $(X, \succ)\in\mathscr{P}$. Let Q be a nonempty finite subset of X. $B(p, \alpha(p, Q); X)\neq\phi$ implies that there exists an x^* in $B(p, \alpha(p, Q); X)$ such that $z\not\succ x^*$ for every z in $B(p, \alpha(p, Q); X)$. But by the definition of $\alpha(p, Q)$, Q is contained in $B(p, \alpha(p, Q); X)$. Thus $(X, \succ)\in\mathscr{P}_{FD}$.

To show $\mathscr{P}_{FD}\subset\mathscr{P}$. Let $(X, \succ)\in\mathscr{P}_{FD}$. Let $(p, w)\in R^l_{++}\times R$ be such that $B(p, w; X)\neq\phi$. Suppose $D(p, w; X, \succ)=\phi$. Then, if we define $R(x)=\{z\in B(p, w; X) \mid x\succ z\}$, $R(x)$ are open relative to $B(p, w; X)$ by continuity of $\succ$ and $\{R(x)\}_{x\in B(p, w; X)}$ forms an open cover of $B(p, w; X)$. The compactness of $B(p, w; X)$ implies existence of a nonempty finite subset Q of $B(p, w; X)$ such that $\{R(x)\}_{x\in Q}$ is a subcover. By (*FD*), there exists an x^* in $B(p, \alpha(p, Q); X)\subset B(p, w; X)$ such that $x\not\succ x^*$ for every x in Q. It follows that $x^*\notin R(x)$ for x in Q. This is contrary to the fact that $\{R(x)\}_{x\in Q}$ is a subcover. This proves that $(X, \succ)\in\mathscr{P}$. ■

Proof of Proposition 1:

Since $\mathscr{X}\times\mathscr{R}$ is compact, it is enough to show that $\iota(\mathscr{P})$ is a closed set. Let $(X_n, \succ_n)_{n=1,2,\ldots}$ be a sequence in $\mathscr{P}$, $\iota((X_n, \succ_n))=(X_n, F_n)$, and $\text{Lim}\,(X_n, F_n)=(X, F)$. Define $\succ=\{(x, y)\in X\times X \mid (x, y)\notin F\}$. We must show that $(X, \succ)\in\mathscr{P}$ and $\iota((X, \succ))=(X, F)$. This latter is true because $F_{\succ}=F$ by definition of $\succ$. (X, F) belongs to $\mathscr{X}\times\mathscr{R}$ as it is compact. To show that $(X, \succ)$ belongs to $\mathscr{P}$ it is enough to prove, by Lemma 1, that

$(X, \succ)$ satisfies (FD). Let $Q=\{z_1, \ldots, z_k\}$ be a nonempty finite subset of X. We must show that, given a vector p in R^l_{++}, there exists an x^* in $B(p, \alpha(p, Q); X)$ such that $z \not\succ x^*$ for every z in Q. Since $Q \subset X = \text{Lim } X_n$, there exist sequences $(z_{in})_{n=1,2,\ldots}$, $i=1, \ldots, k$, such that $z_{in} \to z_i$ and $z_{in} \in X_n$ for every i and for n large enough. Set $Q_n=\{z_{1n}, \ldots, z_{kn}\}$ and $w_n=\alpha(p, Q_n)$. $(X_n, \succ_n) \in \mathscr{P}$ implies that there exists an x_n in $B(p, w_n; X_n)$ such that $z_{in} \not\succ_n x_n$ for every i and for n large enough. Fix $\varepsilon > 0$ arbitrarily, and set $w=\alpha(p, Q)$. As $z_{in} \to z_i$ for every i, there exists a positive integer N such that $n \geq N$ implies $z_{in} \in B(p, w+\varepsilon) = \{x \in R^l \mid p \cdot x \leq w+\varepsilon\}$ for every i and n. Thus for $n \geq N$, x_n belongs to $B(p, w+\varepsilon)$. By compactness of the set $B(p, w+\varepsilon)$, there is a convergent subsequence of the sequence (x_n). Without loss of generality one can assume that the sequence (x_n) itself converges to an x^* in $B(p, w+\varepsilon)$. Since $x_n \in X_n$ residually, $x^* \in X$. The fact that $z_{in} \to z_i$ for every i implies that $w_n \to w$. Thus, from $p \cdot x_n \leq w_n$ and $x_n \to x^*$ it follows that $p \cdot x^* \leq w$. Hence, $x^* \in B(p, w; X)$.

We now show that $z_i \not\succ x^*$ for every z_i in Q. If not, for some i $z_i \succ x^*$; by continuity of $\succ$, we would have $z_{in} \succ x_n$ for n sufficiently large, a contradiction. Therefore, $(X, \succ)$ satisfies the property (FD).

Q.E.D.

Proof of Proposition 2:

Given a price vector $p \neq 0$ in R^l, denote the hyperplane to which p is normal by $H(p, w)$, $w \in R$, i.e.,

$$H(p, w)=\{x \in R^l \mid p \cdot x = w\}.$$

The open sphere and the open half sphere centered at x with radius δ, denoted $S(x, \delta)$ and $HS_p(x, \delta)$ respectively, are defined by

$$S(x, \delta)=\{z \in R^l \mid \| x-z \| < \delta\}$$

$$HS_p(x, \delta)=\{z \in S(x, \delta) \mid p \cdot z < p \cdot x\},$$

where $\|\cdot\|$ denotes the Euclidean norm in R^l. Let X be in $\mathscr{X}$. A consumption vector x in X is said to have local cheaper points in X if for every $\delta > 0$ $HS_p(x, \delta) \cap X \neq \phi$. Let $C_p(X)$ be the set of all consumption vectors in X that do not have local cheaper points in X, i.e.,

$$C_p(X)=\{x \in X \mid HS_p(x, \delta) \cap X = \phi \text{ for some } \delta > 0\}.$$

Also, define

$$C_{p,w}(X)=C_p(X) \cap H(p, w).$$

Given a hyperplane $H(p, w)$, hyperplanes $H(p, w+\varepsilon\| p \|)$ and $H(p, w-\varepsilon\|p\|)$, denoted for short $H(p, w \pm \varepsilon \| p \|)$, are exactly ε away from $H(p, w)$, i.e.,

$$\inf \{\| x-y \| \mid x \in H(p, w) \text{ and } y \in H(p, w \pm \varepsilon \| p \|)\} = \varepsilon.$$

So, let us denote $w \pm \varepsilon \| p \|$ by $w(\pm \varepsilon)$ for a given $\varepsilon > 0$. We shall need the following result:

LEMMA 2: If $C_{p,w(\pm\varepsilon)}(X)=\phi$, then $C_{p,w}(X_\varepsilon)=\phi$ where $X_\varepsilon=\{z \in R^l \mid \| z-x \| \leq \varepsilon \text{ for some } x \text{ in } X\}$.

Proof: Assume $C_{p,w(\pm\varepsilon)}(X)=\phi$ and $x \in H(p, w) \cap X_\varepsilon$. We want to show that for any $\delta > 0$ we have $HS_p(x, \delta) \cap X_\varepsilon \neq \phi$. Since $x \in X_\varepsilon$, there exists an $x^* \in X$ such that $\| x-x^* \| \leq \varepsilon$. We consider three cases.

Case I. There exists an $x^* \in X$ such that $\| x-x^* \| < \varepsilon$.

Put $\delta^* = \varepsilon - \| x-x^* \| > 0$. Then, we have $S(x, \delta^*) \subset X_\varepsilon$ because if $z \in S(x, \delta^*)$ then $\| x-z \| < \delta^*$ and hence

$$\| z-x^* \| \leq \| z-x \| + \| x-x^* \| < \delta^* + \| x-x^* \| = \varepsilon;$$

thus $z \in X_\varepsilon$. But $S(x, \delta^*) \subset X_\varepsilon$ implies that $HS_p(x, \delta) \cap X_\varepsilon \neq \phi$ for any $\delta > 0$.

Case II. For every $x' \in X$, $\| x-x' \| \geq \varepsilon$ but there exists an $x^* \in X$ such that $\| x-x^* \| = \varepsilon$ and $\inf \{\| x^*-z \| \mid z \in H(p, w)\} < \varepsilon$.

In this case there exists a z in $H(p, w)$ such that $\| x^*-z \| < \varepsilon$. Set $z(t) = tz + (1-t)x$; then $z(t) \in X_\varepsilon$ for $0 < t < 1$. Indeed,

$$\| x^*-z(t) \| = \| (1-t)(x^*-x) + t(x^*-z) \| \leq (1-t) \| x^*-x \| + t \| x^*-z \| < \varepsilon.$$

Note also that $z(t) \in H(p, w)$ as $x, z \in H(p, w)$. Since every $z(t)$, for $0 < t < 1$, is an interior point of X_ε, there is a $\lambda(t)$ for every t such that $0 < \lambda(t) < 1$ and $\gamma z(t) \in X_\varepsilon$ for every γ such that $\lambda(t) \leq \gamma \leq 1$. Put $\delta(n) = \max \{\lambda(1/n), 1-1/n\}$ and $y(n) = z(1/n)$, i.e., $y(n) = (1/n)z + (1-1/n)x$, for each $n = 1, 2, \ldots$. Then define a sequence (y_n) by $y_n = \delta(n) y(n)$. We then have $y_n \to x$, $y_n \in X_\varepsilon$ and $p \cdot y_n < w$; this last inequality is true because $y(n) \in H(p, w)$ implies that $p \cdot y_n = \delta(n)(p \cdot y(n)) < w$. Thus for any given $\delta > 0$ we have $y_n \in HS_p(x, \delta)$ for n large enough and hence $HS_p(x, \delta) \cap X_\varepsilon \neq \phi$.

Case III. This case covers the remaining possibility: For every x' in X $\|x-x'\| \geq \varepsilon$ and $\inf \{\|x'-z\| \mid z \in H(p, w)\} \geq \varepsilon$, but there exists an $x^* \in X$ such that $\|x-x^*\| = \varepsilon$ and $\inf \{\|x^*-z\| \mid z \in H(p, w)\} = \varepsilon$.

In this case x^* lies on the hyperplane $H(p, w(\varepsilon))$ or $H(p, w(-\varepsilon))$. Thus, by the hypothesis, there exists a sequence (x^*_n) such that $x^*_n \in X$ and $p \cdot x^*_n < p \cdot x^*$ for each n, and $x^*_n \to x^*$. Let $pr: R^l \to H(p, w)$ be the perpendicular projection of R^l onto $H(p, w)$, and put $pr(x^*_n) = x_n$ so that $\|x^*_n - x_n\| = \inf \{\|x^*_n - z\| \mid z \in H(p, w)\}$. Apply the arguments in Case II to x_n to obtain $y_n \in X_\varepsilon$ such that $p \cdot y_n < w$ and $\|x_n - y_n\| < 1/n$ for each n. Since the projection pr is continuous, the facts that $x^*_n \to x^*$ and $pr(x^*) = x$ imply that $x_n \to x$. Thus, for any given $\delta > 0$, $y_n \in HS_p(x, \delta)$ for sufficiently large n. ■

We now proceed to prove Proposition 2. Let $(X, \succ)$ be in $\mathscr{P}^*(p, w) \backslash \mathscr{P}^*_{uhc}(p, w)$, and (ε_n) a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. For each n define $X_n = \{x \in R^l \mid \|x-y\| \leq \varepsilon_n$ for some y in $X\}$. Since $(X, \succ)$ is in $\mathscr{P}^*(p, w)$, there is $(X^*, \succ^*)$ in $\mathscr{P}$ such that $X^* = \{x \in R^l \mid b \leq x\}$, and $\succ$ is the restriction of $\succ^*$ to X, i.e., $F_\succ = F_{\succ^*} \cap X \times X$. So define for each n

$$F_n = F_{\succ^*} \cap X_n \times X_n,$$
$$\succ_n = \{(x, y) \in X_n \times X_n \mid (x, y) \notin F_n\}.$$

Then, $(X_n, \succ_n) \in \mathscr{P}^*$. Since $X_1 \supset \ldots \supset X_n \supset X_{n+1} \supset \ldots \supset X$, the fact that $(X, \succ) \in \mathscr{P}^*(p, w)$ implies that $(X_n, \succ_n) \in \mathscr{P}^*(p, w)$ for every n. It can also be verified that Lim $X_n = X$ and Lim $F_n = F_\succ$. We thus obtained a sequence $(X_n, \succ_n)$ in $\mathscr{P}^*(p, w)$ converging to $(X, \succ)$. It remains to show that $(X_n, \succ_n)$ can be chosen to be in $\mathscr{P}^*_{uhc}(p, w)$.

Define

$$J_p(X) = \{w' \in R \mid H(p, w') \cap C_p(X) \neq \phi\}.$$

Then by Lemma in [13], $J_p(X)$ is a countable set. So enumerate the elements of $J_p(X)$ as $w_1, w_2, \ldots, w_k, \ldots$. We can choose the above ε_n so that $w(\pm\varepsilon_n) \neq w_k$ for every k and n. It then follows from Lemma 2 that $w \notin J_p(X_n)$ for every n. Thus, by the standard arguments one can show that $D(p, w; X_n, \succ_n)$ is upper hemicontinuous at (p, w), that is, $(X_n, \succ_n) \in \mathscr{P}^*_{uhc}(p, w)$ for each n.

Q.E.D.

IV. *A Final Remark*

In view of the proof of Proposition 2 what is essentially true is that at any given pair (p, w) of a price vector $p \in R^l_{++}$ and a wealth level $w \in R$, the set of consumption sets X such that w is not in the "critical set of X," i.e., $w \notin J_p(X)$, is dense. It indicates a possibility of considering a "dispersed consumption sets distribution" in overcoming the nonconvexity of consumption sets in the equilibrium existence problem as oposed to considering the dispersion of endowments or wealth (cf. Yamazaki [13, 14]). This final remark may also be of some interest to the "smoothing by aggregation" problem literature where the dispersion of preference relations is a basic force responsible for the "smoothing" effects.

Appendix

In this appendix we describe an example announced in Section 2. Let $X = R_+$. We shall define a continuous relation $\succ$ on X by way of a continuous function v: $R_+ \times R_+ \to R$. Define $v(x, y) = (y-1)(x-1)(x-5)$ for each (x, y) in $R_+ \times R_+$. Define $\succ_v$ on R_+ by

$$\succ_v = \{(x, y) \in R_+ \times R_+ \mid v(x, y) > 0\}.$$

By continuity of the function v, $\succ_v$ is open in $R_+ \times R_+$. We shall show first that the preference relation $(R_+, \succ_v)$ is observable in markets, i.e., the demand sets on nonempty compact budget sets are nonempty. Given $(p, w) \in R_{++} \times R_+$, we shall show that $D(p, w; R_+, \succ_v)$ is nonempty. Define $g(y) = \max \{v(x, y) | x \in B(p, w; R_+)\}$. Note that if $y \in B(p, w; R_+)$ and $g(y) \leq 0$, then $y \in D(p, w; R_+, \succ_v)$.

If $0 \leq w/p < 1$, then $g(y) = v(w/p, y)$ for any y satisfying $0 \leq y \leq w/p$. Thus, we have $g(y) < 0$ and hence $y \in D(p, w; R_+, \succ_v)$. If $1 \leq w/p < 3$, then we have $g(y) = v(0, y)$ for $y \geq 1$ and $g(y) = v(w/p, y)$ for $y < 1$. In particular, $g(1) = v(0, 1) = 0$ so that $p \cdot 1 \leq w$ implies that $1 \in D(p, w; R_+, \succ_v)$. If $3 \leq w/p < 6$, then we have $g(y) = v(0, y)$ for $y \geq 1$ and $g(y) = v(3, y)$ for $y < 1$. Again $1 \in D(p, w; R_+, \succ_v)$. If $w/p \geq 6$, then we have $g(y) = v(w/p, y)$ for $y \geq 1$ and $g(y) = v(3, y)$ for $y < 1$. But $g(1) = v(w/p, 1) = $ p and $p \cdot 1 < w$ imply that $1 \in D(p, w; R_+, \succ_v)$. Thus we have shown that for any $(p, w) \in R_{++} \times R_+$, $D(p, w; R_+, \succ_v) \neq \phi$.

We now show that (i) $(R_+, \succ_v)$ is not convex, (ii) $(R_+, \succ_v)$ is not transitive, and (iii) $(R_+, \succ_v)$ is not acyclic:

(i) $(R_+, \succ_v)$ is not convex.

$$\{x \in R_+ \mid x \succ_v 2\} = [0, 1) \cup (5, \infty),$$

which is not a convex set.

(ii) $(R_+, \succ_v)$ is not transitive.

Let $x = 3$, $y = 1/2$ and $z = 2$; then $v(x, y) = v(3, 1/2) = 2 > 0$, $v(y, z) = v(1/2, 2) = 9/4 > 0$, and $v(x, z) = v(3, 2) = -4 < 0$. Thus, $x \succ_v y$ and $y \succ_v z$ and $x \not\succ_v z$.

(iii) $(R_+, \succ_v)$ is not acyclic.

Let $x = 6$, $y = 8$ and $z = 7$; then, $v(x, y) = v(6, 8) = 35 > 0$, $v(y, z) = v(8, 7) = 126 > 0$, and $v(z, x) = v(7, 6) = 60 > 0$. Thus $x \succ_v y$, $y \succ_v z$, and $z \succ_v x$.

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