NORMAL APPROXIMATION THEOREM ON THE SIZE DISTRIBUTION OF "BLOCKING" COALITIONS†

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I. *Introduction*

Consider a Pareto optimal allocation in a finite exchange economy, which is not a Walrasian allocation. It cannot be improved upon, by definition of Pareto optimality, by the coalition of all agents in the economy. However, the well-known limit theorem on the core of an economy (in its strong form) (see Edgeworth [6], Debreu and Scarf [4], and Hildenbrand [11]) essentially says that the allocation is not in the core of the economy and hence it is improved upon by some coalition of agents, if the number of agents in the economy is large enough. We are thus motivated to study the size distribution of those coalitions that improve upon the Pareto optimal allocation. (From now on we frequently refer to such coalitions as "blocking" coalitions.) Of course, we realize that the concept of "to improve upon" is not a descriptive one. (See Hildenbrand [11] on this point.) Therefore, no empirical work is conceivable on the size distribution of blocking coalitions. We set out in this paper a purely theoretical investigation of this distribution.

The paper is concerned with the asymptotic size distribution of coalitions that improve upon a Pareto optimal allocation in a finite exchange economy. We shall show, imprecisely speaking, that the size distribution of blocking coalitions, rescaled so that the mean size is zero and one unit is the standard deviation of the size distribution of coalitions in the economy, converges (pointwise) to the normal (0,1) distribution as the number of agents in the economy increases indefinitely. To give a precise statement of this result, we now turn to formal description of the model. We are indebted to previous work done by Mas-Colell [13]. The tools and methods introduced by him will be extensively exploited in this paper.

II. *Formal Model, Pareto Prices, and "Blocking" Coalitions*

The *consumption set* is $P=\{x=(x^1,\ldots,x^l)\in R^l \mid x^h>0,\ h=1,\ldots,l\}$. We shall restrict ourselves to a class of *smooth, strictly convex preferences* on P (see Debreu [2]). This means that preferences are representable by a twice continuously differentiable (C^2) utility function u on P such that (i) $D_h u(x)>0$, $h=1,\ldots,l$, for all $x\in P$, (ii) $\{y\in P \mid u(y)\geqq u(x)\}$ is strictly convex, and (iii) the closure of every indifference set is contained in P. $\mathscr{P}$ denotes the set of all

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preference relations satisfying these conditions. Given $\succsim \in \mathscr{P}$, define a continuously differentiable (C^1) function $g_{\succsim}: P \to S$, where $S=\{p\in P|\ \|p\|=1\}$, by $g_{\succsim}(x)=(1/\|Du(x)\|)Du(x)$ where u is a C^2 utility function for $\succsim$ satisfying (i)–(iii). The topology on $\mathscr{P}$ is the one induced by the topology of uniform convergence on compact sets of the $g_{\succsim}$ functions and their first partial derivatives. Then $\mathscr{P}$ becomes a separable metric space. (See H. Dierker [5].) The space of consumption characteristics is $\mathscr{P}\times P$ with the product topology.

An *economy* is a mapping $\mathscr{E}$ of a finite set A of economic agents into the space $\mathscr{P}\times P$ of agents' characteristics. If we consider just one economy, we shall write $(\succsim_a, e(a))$ for $\mathscr{E}(a)$. An *allocation* for the economy $\mathscr{E}$ is a mapping f of A into P. An allocation f for $\mathscr{E}$ is said to be *attainable* if $\sum_{a\in A}f(a)=\sum_{a\in A}e(a)$. Any nonempty subset C of A is called a *coalition*. An allocation f for $\mathscr{E}$ is said to be *improved upon* (or "blocked") by coalition C if there exists an allocation g for $\mathscr{E}$ such that $g(a)\succ_a f(a)$ for every $a\in C$ and $\sum_{a\in C}g(a)\leqq\sum_{a\in C}e(a)$. A coalition which can improve upon a given allocation f is sometimes referred to as a *blocking coalition*. An attainable allocation f is *Pareto optimal* if it is not improved upon by A. The set of all attainable allocations for the economy $\mathscr{E}$ that no coalition can improve upon is called the *core* of the economy.

Let $f:A\to P$ be a Pareto optimal allocation for $\mathscr{E}$. Since every agent of the economy $\mathscr{E}$ has smooth preference, one can associate with f the uniquely determined price vector $p(f)\in S$, called the *Pareto price vector*, having the property that $p(f)\cdot x\geqq p(f)\cdot f(a)$ whenever $x\succsim_a f(a)$, or simply $p(f)=g_{\succsim}(f(a))$, for every $a\in A$. Since preferences are strictly monotone by (i), the allocation f is *Walrasian* if and only if $p(f)\cdot f(a)=p(f)\cdot e(a)$ for every $a\in A$. Thus individual budget deviations $p(f)\cdot(f(a)-e(a))$ can be used to form an index that measures how far the allocation f is from being Walrasian. Several indices may be considered for this purpose corresponding to different norms in the space of finite sequences. Here, let us take two indices.

$$\sigma_\infty(f)=\max_{a\in A}|p(f)\cdot f(a)-p(f)\cdot e(a)|,\text{ and}$$

$$\sigma(f)=\sqrt{\frac{\sum_{a\in A}(p(f)\cdot f(a)-p(f)\cdot e(a))^2}{\#A}}$$

We shall use the index given by $\sigma(f)$ in the main theorem. Of course, $\sigma_\infty(f)$ is the stronger measure of the gap of the allocation f from being Walrasian (that is, $\sigma_\infty(f)\geqq\sigma(f)$ and the convergence of σ_∞ to zero means uniform convergence to zero of individual budget deviation of every agent). It is an open question as to whether $\sigma(f)$ can be replaced by $\sigma_\infty(f)$ or not in the main theorem. (Compare this with Mas-Colell [13, Remark 5].) $(\sigma(f))^2$ may be viewed as the variance of the income distribution induced by the allocation f.

Now, let $(\mathscr{E}_n)$ be a sequence of economies with characteristics in a compact subset Q of $\mathscr{P}\times P$ and let $\#A_n\to\infty$. (A_n is the domain of $\mathscr{E}_n$.) Consider in every economy $\mathscr{E}_n$ a Pareto optimal allocation f_n. Let p_n be the Pareto price vector associated with f_n, i.e., $p_n=p(f_n)$. Assume that the sequence (f_n) is bounded (uniformly in n) and f_n's are bounded away from being Walrasian, say $\sigma(f_n)\geqq\delta>0$ for all n. Then consider the following result of Grodal [9] that generalizes a result of Debreu [3]:

PROPOSITION (Grodal): *Given a compact set Q of $\mathscr{P}\times P$, there exists a constant q such that for every economy $\mathscr{E}: A\to Q$ and every allocation f in the core of the economy $\mathscr{E}$ we have:*

$$\sigma_\infty(f) \leqq \frac{q}{\#A} .$$

Since $\sigma(f_n) \geqq \delta$ implies $\sigma_\infty(f_n) \geqq \delta$, it follows from this proposition that there is n_0 such that for $n \geqq n_0$ f_n does not belong to the core of the economy $\mathscr{E}_n$. Hence for $n \geqq n_0$ there are coalitions of the economy $\mathscr{E}_n$ that improve upon f_n.

In this context several questions concerning the number of blocking coalitions naturally suggest themselves. In fact Mas-Colell asked in his stimulating paper [13]: "How many blocking coalitions are there?" It follows from his result that the fraction of those coalitions that improve upon the allocation f_n tends to 1/2 with $n \to \infty$. This number 1/2 is obviously an upper bound because for a Pareto optimal allocation for $\mathscr{E}_n$ either a coalition C or its complement $A_n \backslash C$ cannot improve upon the allocation. A similar question was asked by the paper of Greenberg, Weber, and Yamazaki [8]. They asked: "How many blocking coalitions are there when their relative size is restricted between $0 \leqq a < b \leqq 1$? Their results imply that the fraction of those coalitions with relative size between $0 \leqq a < b \leqq 1$ that improve upon the allocation f_n also tends to 1/2 with $n \to \infty$. We ask in this paper a yet another question in the same context: "What is the asymptotic size distribution of coalitions that improve upon the Pareto optimal allocation f_n?"

The question is distinctively different from those posed earlier. The way it is answered in this paper, however, is related to the earlier ones. This point will become clear in the next section.

III. *Size Distribution of Blocking Coalitions*

Given a finite set C, $\#C$ denotes its cardinality. When C is a coalition, $\#C$ is called the size of the coalition.

Let an economy $\mathscr{E}: A \to \mathscr{P} \times P$ and a Pareto optimal allocation $f: A \to P$ for $\mathscr{E}$ be given. For a nonnegative integer $m \leqq n$ where $n = \#A$, define

$$C_m(A) = \{C \subset A \mid \#C = m\},$$

$$C_m{}^B(\mathscr{E}, f) = \{C \in C_m(A) \mid C \text{ improves upon } f\}.$$

$C_m(A)$ is the set of coalitions having the size m, and $C_m{}^B(\mathscr{E}, f)$ represents the set of coalitions of size m, that can improve upon f. When the allocation f is bounded away from being Walrasian in the sense that $\sigma(f)$ is bounded away from zero, it makes sense to define the *size distribution of blocking coalitions* for $\mathscr{E}$ and f. It is defined by

$$F(\gamma : \mathscr{E}, f) = \frac{\sum_{0 \leqq m \leqq \gamma} \#C_m{}^B(\mathscr{E}, f)}{\sum_{0 \leqq m \leqq n} \#C_m{}^B(\mathscr{E}, f)} .$$

for $\gamma \in [0, n]$. $F(\gamma : \mathscr{E}, f)$ is the fraction of those blocking coalitions whose size is less than or equal to γ. It will be convenient for our purposes to rescale the size distribution of blocking coalitions. By definition, *the size distribution function of blocking coalitions* for $\mathscr{E}$ and f is

$$F^*(\beta : \mathscr{E}, f) = \frac{\sum_{0 \leqq m \leqq \beta \frac{\sqrt{n}}{2} + \frac{n}{2}} \#C_m{}^B(\mathscr{E}, f)}{\sum_{0 \leqq m \leqq n} \#C_m{}^B(\mathscr{E}, f)}$$

for $\beta \in (-\infty, \infty)$. Thus, in the rescaled size distribution, coalitions of (rescaled) size less than or equal to β represent coalitions of (actual) size less than or equal to $\frac{\beta\sqrt{n}}{2}+\frac{n}{2}$. In other words, the rescaling is done so that zero is the mean size $\frac{n}{2}$ and one unit is $\frac{\sqrt{n}}{2}$ which is the standard deviation of the size distribution of coalitions.

Let us now state the main result of the paper.

THEOREM: *Let Q and X respectively denote a compact subset of the consumption characteristics $\mathscr{P} \times P$ and the consumption set P. Let real numbers $\varepsilon > 0$ and $\beta \in (-\infty, \infty)$ be given. Then, there is a positive constant M such that for any economy $\mathscr{E}: A \to Q$ and Pareto optimal allocation $f: A \to X$, we have:*

$$\text{if } \sigma(f) > M/\sqrt{n}\text{, then } |F^*(\beta: \mathscr{E}, f) - \Phi(\beta)| < \varepsilon$$

where $\Phi(\cdot)$ is the normal $(0,1)$ distribution.

In the context of the sequences $(\mathscr{E}_n)$ and (f_n) of economies and Pareto optimal allocations in the previous section, the theorem implies that the size distribution function of blocking coalitions $F^*(\beta: \mathscr{E}_n, f_n)$ converges to $\Phi(\beta)$ pointwise with $n \to \infty$ where $\Phi(\cdot)$ is the normal (0,1) distribution.

Let us explain ideas of our proof. The main idea is best understood by decomposing the fraction $F(\gamma: \mathscr{E}_n, f_n)$ into three parts:

$$F(\gamma: \mathscr{E}_n, f_n) = \frac{\sum_{0 \leqq m \leqq \gamma} \# C_m{}^B(\mathscr{E}_n, f_n)}{\sum_{0 \leqq m \leqq \gamma} \# C_m(A_n)} \cdot \frac{\sum_{0 \leqq m \leqq \gamma} \# C_m(A_n)}{\sum_{0 \leqq m \leqq n} \# C_m(A_n)} \cdot \frac{\sum_{0 \leqq m \leqq n} \# C_m(A_n)}{\sum_{0 \leqq m \leqq n} \# C_m{}^B(\mathscr{E}_n, f_n)},$$

that is, the fraction of those blocking coalitions which have size less than or equal to γ is the product of $F_{1,n}(\gamma) \equiv$ (the fraction of those coalitions with size less than or equal to γ that improve upon f_n), $F_{2,n}(\gamma) \equiv$ (the fraction of those coalitions whose size is less than or equal to γ), and $F_{3,n}(\gamma) \equiv$ (the inverse of the fraction of those coalitions which improve upon f_n). The term $F_{3,n}(\gamma)$ does not depend upon γ and its inverse was studied by Mas-Colell [13]. We thus know $F_{3,n}(\gamma) \to 2$ with $n \to \infty$. The second term $F_{2,n}(\gamma)$ has the binomial distribution with probabilities of "success" and "failure" both equal to $\frac{1}{2}$. Thus, as is well known, $F_{2,n}(\gamma)$ can be approximated by a normal distribution for large enough n. The first term $F_{1,n}(\gamma)$ is related to the fractions studied by [8] but is different. Actually it is not possible to apply Mas-Colell's methods to study $F_{1,n}(\gamma)$ because for any fixed γ, γ/n becomes negligible for large enough n. The consideration of the first and the second terms leads us to the particular rescaling adopted in this paper. Thus instead of decomposing $F(\gamma: \mathscr{E}_n, f_n)$, decompose $F^*(\beta: \mathscr{E}_n, f_n)$ and denote the terms corresponding to $F_{i,n}(\gamma)$ by $F^*_{i,n}(\gamma)$, $i = 1,2,3$. Then, one can show that $F^*_{1,n}(\beta) \to \frac{1}{2}$ with $n \to \infty$ (Proposition 1). On the other hand, we already know that $F^*_{3,n}(\beta) \to 2$ with $n \to \infty$. Hence, for large enough n, the first and the third term tend to cancel each other out. We are approximately left with the term $F^*_{2,n}(\beta)$, which is a rescaled binomial distribution, and we know that $F^*_{2,n}(\beta) \to \Phi(\beta)$ with $n \to \infty$. This completes a heuristic proof.

If we had adopted a rescaling so that one unit is n instead of $\frac{\sqrt{n}}{2}$, then the first term would have coincided with the fraction studied in [8]. However, their rescaling (in other words, to look at *relative size*) is not appropriate for our purposes. Moreover, it is easier to study the term $F^*_{1,n}(\beta)$ than the corresponding fraction in terms of relative sizes. (This is due to the fact that $\frac{\beta\sqrt{n}}{2}+\frac{n}{2}$ becomes very close to $\frac{n}{2}$ as $n\to\infty$. Cf. Mas-Colell [13] and Yamazaki [14].)

We now present a formal proof. The proof uses the following two results.

PROPOSITION 1: *Let $0<\eta<1$ be given. Under the conditions of the THEOREM, there is a constant H such that if $\sigma(f)>H/\sqrt{n}$, then one has:*

$$(1)\quad \gamma(\beta:\mathscr{E},f)\equiv\frac{\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m{}^B(\mathscr{E},f)}{\frac{1}{2}\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m(A)}>\eta;$$

$$(2)\quad \gamma^c(\beta:\mathscr{E},f)\equiv\frac{\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#\{C_m(A)\backslash C_m{}^B(\mathscr{E},f)\}}{\frac{1}{2}\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m(A)}>\eta,$$

where $n=\#A$.

PROPOSITION 2: (Mas-Colell [13, Theorem, p. 208]): *Let $0<\eta<1$ be given. Under the conditions of the THEOREM, there is a constant L such that if $\sigma(f)>L/\sqrt{n}$, then one has:*

$$1\geqq\tau(\mathscr{E},f)\equiv\frac{\sum_{0\leqq m\leqq n}\#C_m{}^B(\mathscr{E},f)}{\frac{1}{2}\sum_{0\leqq m\leqq n}\#C_m(A)}>\eta$$

where $n=\#A$.

Proof of the THEOREM

Let $\mathscr{E}:A\to Q$ be an economy with $\#A=n$ and $f:A\to X$ a Pareto optimal allocation for $\mathscr{E}$. Let $\varepsilon>0$ and β be given. By definition of $F^*(\beta:\mathscr{E},f)$ we can write $F^*(\beta:\mathscr{E},f)=T_1T_2T_3{}^{-1}$ where

$$T_1=\frac{\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m{}^B(\mathscr{E},f)}{\frac{1}{2}\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m(A)}$$

$$T_2=\frac{\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\#C_m(A)}{\sum_{0\leqq m\leqq n}\#C_m(A)},\ \text{and}$$

$$T_3=\frac{\sum_{0\leqq m\leqq n}\#C_m{}^B(\mathscr{E},f)}{\frac{1}{2}\sum_{0\leqq m\leqq n}\#C_m(A)}.$$

Let $0<\delta<1$ be such that $\delta<\frac{\varepsilon}{3+\varepsilon}$. Put $\eta=1-\delta$. By Propositions 1 and 2 there are constants H and L such that $\sigma(f)>H/\sqrt{n}$ (resp. $>L/\sqrt{n}$) entails $\gamma(\beta:\mathscr{E},f)$, $\gamma^c(\beta:\mathscr{E},f)>\eta$ (resp. $\tau(\mathscr{E},f)>\eta$). Put $M'=\max\{H,L\}$. Then $\sigma(f)>M'/\sqrt{n}$ implies $|T_i-1|<\delta$ for $i=1,3$. On the other hand, by a well-known De Moivre-Laplace Limit Theorem (see, for example, Feller [7, p. 186 and p. 244]) there exists a positive number N such that for all $n\geqq N$ one has $|T_2-\Phi(\beta)|<\delta$. Thus, there is a constant $M>M'$ such that $\sigma(f)>M/\sqrt{n}$ entails $|T_i-1|<\delta$ for $i=1,3$ and $|T_2-\Phi(\beta)|<\delta$. Hence we obtain

$$
\begin{aligned}
|T_1T_2T_3^{-1}-\Phi(\beta)| &= \left|\frac{T_2}{T_3}(T_1-1)+T_2\left(\frac{1}{T_3}-1\right)+(T_2-\Phi(\beta))\right| \\
&\leqq \left|\frac{T_1-1}{T_3}\right|+\left|\frac{1}{T_3}-1\right|+|T_2-\Phi(\beta)| \\
&\leqq \frac{|T_1-1|}{|T_3|}+\frac{|T_3-1|}{|T_3|}+|T_2-\Phi(\beta)| \\
&< \frac{\delta}{1-\delta}+\frac{\delta}{1-\delta}+\frac{\varepsilon}{3}<\varepsilon\,.
\end{aligned}
$$

This completes the proof of the theorem.

Q.E.D.

Proof of Proposition 1

[Step I] The proof given here is a variant of the one in Mas-Colell [13].

Define constants α, G, r, and K from Lemma A.1–A.3 in the Appendix as in Mas-Colell [13, p. 212]. Choose ξ so that $\frac{\eta}{1+\eta}+\frac{1}{2}<\xi<1$. Let $\bar{t}>0$ be such that $1-\Phi(\bar{t})=\frac{1+\xi}{4}$.

Let n_0 be such that for $n\geqq n_0$ one has $\frac{\beta\sqrt{n}}{2}+\frac{n}{2}>\frac{n}{4}+1$. Set $b_n(\beta)=\sum_{0\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}\binom{n}{m}$, and $b_n=\sum_{0\leqq m\leqq\frac{n}{4}}\binom{n}{m}$. It follows from the properties of binomial coefficients that $\frac{b_n}{b_n(\beta)}\to 0$ with $n\to\infty$. Hence there exists a positive integer $n_1\geqq n_0$ such that one has $\frac{b_n}{b_n(\beta)}<\frac{1-\eta}{2}$ for all $n\geqq n_1$.

Now, choose a positive number J so that it satisfies the following two inequalities:

$$\frac{2K\bar{t}}{\alpha J^2}\leqq\frac{1-\xi}{4} \quad\text{and}\quad Ge^{-(r\alpha^2J^2/4)}\leqq\frac{1-\xi}{2}.$$

We finally set $H_0=\alpha J^2/\bar{t}$. In this step we shall show that $\sigma(f)>H_0/\sqrt{n}$ entails $\gamma(\beta:\mathscr{C},f)>\eta$ for $n\geqq n_1$. Put $\sigma^2=(\sigma(f))^2$ and $\sigma_m^2=m(1-\frac{m}{n})\sigma^2$. With $p=p(f)$ and $y_a=e(a)-f(a)$, define the following families of coalitions:

$$\mathscr{C}_m^1=\{C\in C_m(A)|\ \|\textstyle\sum_{a\in C}y_a\|\leqq\frac{\alpha Jm}{\sqrt{n}}\},$$

$$\mathscr{C}_m^2=\{C\in C_m(A)|p\cdot(\frac{\sum_{a\in C}y_a}{\|\sum_{a\in C}y_a\|})\geqq\frac{J}{\sqrt{n}}\},$$

$$\mathscr{C}_m^3=\{C\in C_m(A)|\textstyle\sum_{a\in C}p\cdot y_a\geqq\sigma_m\bar{t}\},$$

$$\mathscr{C}_m^4=\{C\in C_m(A)|p\cdot(\frac{\sum_{a\in C}y_a}{\|\sum_{a\in C}y_a\|})\leqq-\frac{J}{\sqrt{n}}\},$$

$$\mathscr{C}_m^5=\{C\in C_m(A)|\textstyle\sum_{a\in C}p\cdot y_a\leqq-\sigma_m\bar{t}\}.$$

By Lemma A.1 in the Appendix a coalition C of size m or its complement $A\backslash C$ (of size $n-m$) improves upon f depending upon whether C belongs to $\mathscr{C}_m^1\cap\mathscr{C}_m^2$ or $\mathscr{C}_m^1\cap\mathscr{C}_m^4$. (For this argument see [13, p. 213].) Thus, if we put

$$\theta=\begin{cases}\dfrac{\#\{\bigcup_{\frac{n}{4}\leqq m\leqq\frac{\beta\sqrt{n}}{2}+\frac{n}{2}}(\mathscr{C}_m^1\cap\mathscr{C}_m^2)\}}{\frac{b_n(\beta)}{2}}, & \text{if } \beta\leqq 0;\\[2ex] \dfrac{\#\{\bigcup_{\frac{n}{4}\leqq m\leqq\frac{n}{2}}(\mathscr{C}_m^1\cap\mathscr{C}_m^2)\}+\#\{\bigcup_{-\frac{\beta\sqrt{n}}{2}+\frac{n}{2}\leqq m<\frac{n}{2}}(\mathscr{C}_m^1\cap\mathscr{C}_m^4)\}}{\frac{b_n(\beta)}{2}}, & \text{if } \beta>0;\end{cases}$$

then $\gamma(\beta:\mathscr{E},f)\geqq\theta$. Put $\Pi_m{}^j=\#\mathscr{C}_m{}^j/\binom{n}{m}$ for $j=1,\ldots,5$. From Lemma A.2 in the Appendix one obtains (as in [13, p. 214]) $\Pi_m{}^1\geqq\frac{1+\xi}{2}$ for any integer m such that $\frac{n}{4}\leqq m\leqq\frac{n}{2}$.

On the other hand, we have $\sigma_m\bar{t}\geqq\frac{H_0m\bar{t}}{n}$ for $m\leqq\frac{n}{2}$. Thus, it follows from the definition of constants that $\mathscr{C}_m^1\cap\mathscr{C}_m^3\subset\mathscr{C}_m^2$ and $\mathscr{C}_m^1\cap\mathscr{C}_m^5\subset\mathscr{C}_m^4$. These imply that $\Pi_m{}^2\geqq\Pi_m{}^1+\Pi_m{}^3-1$ and $\Pi_m{}^4\geqq\Pi_m{}^1+\Pi_m{}^5-1$. Let $d_a=p\cdot y_a$ in Lemma A.3 in the Appendix. We then obtain for each integer m such that $\frac{n}{4}\leqq m\leqq\frac{n}{2}$

$$\Pi_m{}^3\geqq-\frac{K}{\sigma\sqrt{m}}+1-\Phi(\bar{t})\geqq-\frac{K\sqrt{n}}{H_0\sqrt{m}}+\frac{1+\xi}{4}\geqq-\frac{2K}{H_0}+\frac{1+\xi}{4}$$
$$\geqq-\frac{2K\bar{t}}{\alpha J^2}+\frac{1+\xi}{4}\geqq\frac{\xi}{2};$$

similarly, one obtains

$$\Pi_m{}^5\geqq\frac{\xi}{2}.$$

Hence it follows

$$\Pi_m{}^2\geqq\frac{2\xi-1}{2}\text{ and }\Pi_m{}^4\geqq\frac{2\xi-1}{2}.$$

Therefore, $\Pi_m{}^j$ for $j=1,2,4$ are no less than $\frac{2\xi-1}{2}$ for each integer m such that $\frac{n}{4}\leqq m\leqq\frac{n}{2}$. It follows that

$$\theta\geqq(2\xi-1)\left(1-\frac{b_n}{b_n(\beta)}\right)>\frac{(2\xi-1)(1+\eta)}{2}>\eta\text{ for }n\geqq n_1.$$

[Step II] If $C\in C_m{}^B(\mathscr{E},f)$, then there is $g:C\to P$ such that $g(a)\succ_a f(a)$ for all $a\in A$ and $\sum_{a\in C}g(a)\leqq\sum_{a\in C}e(a)$. The former implies that $p\cdot g(a)>p\cdot f(a)$ (recall that $p=p(f)$) for all $a\in C$. It follows that $\sum_{a\in C}p\cdot f(a)<\sum_{a\in C}p\cdot g(a)\leqq\sum_{a\in C}p\cdot e(a)$. Thus one obtains $\sum_{a\in C}p\cdot y_a>0$ (recall $y_a=e(a)-f(a)$). It means that if $m=\#C$, then $\sum_{a\in C}p\cdot y_a<0$ implies $C\notin C_m{}^B(\mathscr{E},f)$ and $\sum_{a\in C}p\cdot y_a>0$ implies $A\backslash C\notin C_{n-m}{}^B(\mathscr{E},f)$. With this in mind, define

$$\mathscr{C}_m{}^6=\{C\in C_m(A)\,|\,\textstyle\sum_{a\in C}p\cdot y_a<0\},$$
$$\mathscr{C}_m{}^7=\{C\in C_m(A)\,|\,\textstyle\sum_{a\in C}p\cdot y_a>0\},$$

$$\Pi_m{}^6 = \# \mathscr{C}_m{}^6 / \binom{n}{m},\ \Pi_m{}^7 = \# \mathscr{C}_m{}^7 / \binom{n}{m}.$$

If we put

$$\theta^c = \begin{cases} \dfrac{\#\{\bigcup_{\frac{n}{4} \leqq m \leqq \frac{\beta\sqrt{n}}{2} + \frac{n}{2}} \mathscr{C}_m{}^6\}}{\frac{b_n(\beta)}{2}}, & \text{if } \beta \leqq 0; \\[2ex] \dfrac{\#\{\bigcup_{\frac{n}{4} \leqq m \leqq \frac{n}{2}} \mathscr{C}_m{}^6\} + \#\{\bigcup_{\frac{-\beta\sqrt{n}}{2} + \frac{n}{2} \leqq m < \frac{n}{2}} \mathscr{C}_m{}^7\}}{\frac{b_n(\beta)}{2}}, & \text{if } \beta > 0; \end{cases}$$

then $\gamma^c(\beta: \mathscr{E}, f) \geqq \theta^c$. Let $K, \xi, \sigma, \sigma_m, n_1$ be defined as in Step I. Put $H_1 = \dfrac{2K}{1-\xi}$.

Suppose $\sigma(f) > \dfrac{H_1}{\sqrt{n}}$ and $n \geqq n_1$; we shall show that this implies $\gamma^c(\beta: \mathscr{E}, f) > \eta$. By setting $t=0$ and $d_a = p \cdot y_a$ in Lemma A.3 one obtains

$$\Pi_m{}^6 \geqq \tfrac{1}{2} - \frac{K}{\sqrt{m}\,\sigma} \geqq \frac{2\xi - 1}{2},$$

and similarly,

$$\Pi_m{}^7 \geqq \frac{2\xi - 1}{2},$$

for each integer m with $\dfrac{n}{4} \leqq m \leqq \dfrac{n}{2}$. Hence it follows that

$$\theta^c \geqq (2\xi - 1)\left(1 - \frac{b_n}{b_n(\beta)}\right)$$
$$\geqq (2\xi - 1)\left(\frac{1+\eta}{2}\right) > \eta \text{ for } n \geqq n_1.$$

This finishes the argument of Step II.

Since $X \subset P$ is compact, there exists a real number s such that $x \in X$ implies $\|x\| \leqq s$. Let $H = \max\{H_0, H_1, 2s\sqrt{n_1}\}$. Then, for this H, the arguments of Step I and II show that the assertions (1) and (2) hold true.

Q.E.D.

APPENDIX

We collect here some of the known results that were used in the proofs. We simply give references to their proofs.

LEMMA A.1: *Let $Q \subset \mathscr{P} \times P$ and $X \subset P$ be compact sets. There is $\alpha > 0$ such that for all $(\succsim, \omega) \in Q$, $x \in X$, and $y \in R^l$, if $g_{\succsim}(x) \cdot y \neq 0$, then $x + \alpha(g_{\succsim}(x) \cdot y / \|y\|)(1/\|y\|) y \succ x$.*

See Mas-Colell [13, p. 211].

LEMMA A.2: *Let $T \subset R^l$ be nonempty and compact. There are constants $G>0$ and $r>0$ such that if $x_a \in T$, $a \in A$, $\#A=n$, and $\sum_{a \in A} x_a = 0$, then for every $\varepsilon>0$ and $m \leqq n$,*

$$\#\{C \subset A \mid \#C=m,\ \|\sum_{a \in C} x_a\| \geqq \varepsilon m\} \leqq \binom{n}{m} G e^{-r\varepsilon^2 m}.$$

See Hoeffding [12, Th.1, Section 6].

LEMMA A.3: *Let $T \subset R$ be nonempty and compact. Then there is a constant $K>0$ such that if $d_a \in T$, $a \in A$, $\#A=n$, $\sum_{a \in A} d_a = 0$, and $m \leqq n/2$, then, letting $\sigma^2=(1/n)\sum_{a \in A} d_a^2, \sigma_m^2 = m(1\text{-}m/n)\sigma^2$, $F_m(t)=(1/\binom{n}{m})\#\{C \subset A \mid \#C=m,\ \sum_{a \in C} d_a < \sigma_m t\}$, and $\Phi(t)$ be the normal (0,1) distribution, we have $\sup_{t \in R} |F_m(t)-\Phi(t)| \leqq K\sqrt{m}\,\sigma$.*

See Bikelis [1].

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