

PRICE FORMATION AND COOPERATIVE
BEHAVIOR OF FIRMS:
A LIMIT THEOREM ON COMPETITION AMONG FIRMS†

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I. *Introduction*

Since A. Cournot presented a model of duopoly, a non-cooperative situation has been considered as an important case of competition among firms. Cournot considered a simple case of two firms and presented a concept of stable equilibrium, which is known as the Cournot Equilibrium, where each firm cannot make its profit greater by changing its amount of product as long as the other firm does not change its amount of product.

In our opinion, however, there are two points to be completed in his analysis. The first is that cooperative behaviors of firms should be included. In fact, it is well known that the Cournot Equilibrium is not efficient. Namely, the profits of both firms can be increased if they collude. In this sense, the Cournot Equilibrium is not stable.

The second is that the theory must be developed in the frame-work of general equilibrium theory. It is quite important to show how firms can get the demand for their products in the product-market and how they can get the supply of factors for production in the factor-market. Also, the distribution of profits from firms to share holders must be described.

Unlike Cournot's analysis, in this paper we consider cooperative behaviors of firms. Our economic model is a so-called Arrow-Debreu model in [1] and [3]. In our economy, consumers are price-taking agents, and they maximize their utility within their budgets. Consumers are assumed not to take any cooperative behavior. On the other hand, firms are price-making agents, and they make a coalition to maximize their profits. Therefore, only firms are active agents in our economy. This supposition is quite plausible from a realistic point of view. In fact, collusive behavior of firms can be more often observed in the real world rather than collusive behavior of consumers. The purpose of this paper is to investigate what is a "stable" agreement among all the firms in the economy where cooperative behaviors of firms exist, and to verify what allocation is realized in the economy under such a "stable" agreement.

When it comes to a cooperative agreement among firms, a difficult problem is how profits are divided among firms. In fact, in a small economy, we cannot find a decisive way of profit division among firms, and as a result we cannot distinctly know what allocation is realized in the economy. However, if the economy is large, the allocation which is realized under a "stable" agreement on profit division among all the firms can be shown to be a Walrasian allocation. This argument exactly corresponds to the limit theorem on the

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equivalence between Edgeworth allocation and Walrasian equilibrium, which has been shown, such as, by H. Scarf [9], G. Debreu and H. Scarf [4], and R.J. Aumann [2].

To sum up, our conclusion is as follows: As long as consumers behave as price-taking agents, and if the economy is large, a “stable” agreement among all the firms realizes a competitive allocation in the sense of Walras in the economy. We should note that our limit theorem is essentially different from that of J.J. Gabszewicz and J-P. Vial [5], where a non-cooperative case is studied.

II. Model and Notation

We shall use the following notation: R^k denotes a k -dimensional Euclidean space, where k is a positive integer. R_+^k is the non-negative orthant of R^k , and R_{++}^k is the interior of R_+^k . Also, $R_-^k = -R_+^k$ and $R_{--}^k = -R_{++}^k$. In particular, when $k=1$, we write R , R_+ , R_{++} , R_- , and R_{--} instead of R^1 , R_+^1 , R_{++}^1 , R_-^1 , and R_{--}^1 . Subscripts attached to vectors will be used exclusively to denote coordinates. Following standard practice, for x and y in R^k we take $x > y$ to mean $x_i > y_i$ for all i ; $x \geq y$ to mean $x_i \geq y_i$ for all i ; and $x \gg y$ to mean $x \geq y$ but not $x = y$. The integral of a vector function is to be taken as the vector of integrals of the components. The scalar product $\sum_{i=1}^k x_i y_i$ of two members x and y of R^k is denoted by $x \cdot y$. The symbol O^k denotes the origin of R^k . The symbol \sim will be used for set-theoretic subtraction, whereas the symbol $-$ will be reserved for ordinary algebraic subtraction.

We shall consider a private ownership economy with a fixed list of firms and consumers. Let (A, \mathcal{A}, μ) be a finite positive measure space of economic agents, i.e., the elements of the set A are interpreted as economic agents, the class \mathcal{A} as a collection of sets of economic agents, and the number $\mu(C)$ for each $C \in \mathcal{A}$ as the size of set C . The mathematical structure of (A, \mathcal{A}, μ) is as follows: A is an arbitrary set, \mathcal{A} is a σ -algebra of subsets of A (i.e., is a class of subsets of A containing ϕ and A , and is closed under the operations of complementation, countable union, and countable intersection), and μ is a countably additive set function of \mathcal{A} into R_+ such that $\mu(A) < \infty$. In the economy there are two kinds of economic agents, that is, consumers and firms. The sets of all the consumers and all the firms are denoted by S and T respectively. It is assumed that

$$S \cup T = A, S \cap T = \phi, \text{ and } S, T \in \mathcal{A}.$$

In what follows, an element of the consumer set S is always denoted by s , and an element of the firm set T is always denoted by t . Let us define two classes as follows:

$$\mathcal{S} = \{C \in \mathcal{A} \mid C \subset S\} \text{ and } \mathcal{T} = \{C \in \mathcal{A} \mid C \subset T\}.$$

Define two measures $\mu_1(U) = \mu(U)$ for $U \in \mathcal{S}$ and $\mu_2(V) = \mu(V)$ for $V \in \mathcal{T}$. Then we can easily verify that the classes \mathcal{S} and \mathcal{T} are σ -algebras. Therefore, (S, \mathcal{S}, μ_1) and (T, \mathcal{T}, μ_2) are proved to be finite positive measure spaces.

In our economic model there are finite kinds of different commodities. The number of commodities is denoted by a positive integer l . Define a mapping $X: S \rightarrow 2^{R^l}$. The set $X(s)$ is interpreted as the commodity consumption set of each consumer $s \in S$. And for each consumer $s \in S$ we define an irreflexive binary relation \succ_s on $X(s)$, which represents each consumer's preferences of commodity consumption.

Let f° denote an integrable function of S into R^l whose image $f^\circ(s)$ is interpreted as the

amounts of commodities initially held by consumer $s \in S$.

In the case of firms, a mapping $Y: T \rightarrow 2^R$ is defined. The set $Y(t)$ is interpreted as the production possibility set of firm $t \in T$.

Furthermore, we define a product measure space $(S \times T, \mathcal{S} \times \mathcal{T}, \mu_1 \times \mu_2)$, which is the Cartesian product of the measure spaces (S, \mathcal{S}, μ_1) and (T, \mathcal{T}, μ_2) . The profits of firms in this economy are all distributed to consumers in a historically determined way. That is, there is a measurable function $\theta: S \times T \rightarrow R_+$ such that

$$\int_S \theta(s, t) ds = 1 \text{ for every } t \in T,$$

with $\theta(s, t)$ standing for the relative share of consumer $s \in S$ in the profit of firm $t \in T$. Namely, if firm t earns profit $\pi(t)$, then $\theta(s, t) \cdot \pi(t)$ must be paid by firm t to consumer s as a profit dividend.

Throughout this paper, we shall use the following notations to denote integrals: For example, if h, h' , and h'' are integrable functions on the measure spaces (A, \mathcal{A}, μ) , (S, \mathcal{S}, μ_1) , and (T, \mathcal{T}, μ_2) respectively, then their integrals are always denoted by $\int h d\mu$, $\int h' ds$, and $\int h'' dt$ respectively.

III. Market Structure and Cooperative Behavior of Firms

We shall consider an economy in which consumers are price-taking agents and firms are price-making agents. Therefore, it is only firms that actively behave in the economy. Our problem is what allocation will be realized in such an economy.

In order to describe an allocation in the economy, we use a pair (f, g) of integrable functions such that $f: S \rightarrow R^l$ and $g: T \rightarrow R^l$. The image $f(s)$ denotes the amount of commodities allotted to consumer $s \in S$, and the image $g(t)$ denotes the production activity by firm $t \in T$. Of course, we have only to take into account allocations which are technologically feasible.

Definition 1.

An allocation (f, g) is *feasible between* $U \in \mathcal{S}$ and $V \in \mathcal{T}$ if the following conditions are satisfied:

- (i) $f(s) \in X(s)$ *a.e.* (almost everywhere) in U .
- (ii) $g(t) \in Y(t)$ *a.e.* in V .
- (iii) $\int_U (f - f^0) ds \leq \int_V g dt$.

In particular, When $U = S$ and $V = T$, we call it a *feasible allocation for the entire economy*.

In the above definition, condition (i) says that allotment $f(s)$ of commodities is acceptable for each consumer $s \in U$. Condition (ii) means that production activity $g(t)$ is possible for each firm $t \in V$. Condition (iii) guarantees that coalition V of firms can sustain the allotment $f(s)$ of commodities to every consumer $s \in U$.

Definition 1 simply says that allocation (f, g) can be technologically realized between consumers in U and firms in V . But it needs to be realizable in the market as well as in the technological sense. We assume that all the consumers buy and sell commodities in the market so as to maximize their utility within their budgets. Therefore, each consumer's

behavior in the market is described by his demand function.

Each consumer's budget depends on both profit dividends from firms and a price system. We use an integrable function $\pi : T \rightarrow R_+$ to denote profits of firms. The image $\pi(t)$ denotes the amount of profit which firm $t \in T$ promises to pay. Each consumer's demand for commodities is defined as follows: For a price vector $p \in R_+^I$ and for a promise of profit payments by firms $\pi : T \rightarrow R_+$, the demand of consumer $s \in S$ is

$$D(p, \pi, s) = \{x \in B(p, \pi, s) \mid \text{not } y \succ_s x \text{ for any } y \in B(p, \pi, s)\},$$

where

$$B(p, \pi, s) = \left\{ x \in X(s) \mid p \cdot x \leq p \cdot f^o(s) + \int_T \theta(s, t) \cdot \pi(t) dt \right\}.$$

To sum up, each consumer $s \in S$ behaves in the market in the following way: When each firm $t \in T$ promises to pay the amount $\pi(t)$ of profit, and when a price system $p \in R_+^I$ is announced to him, he selects a point $x \in D(p, \pi, s)$, and buy or sell the amount $(x - f^o(s))$ of commodities in the market.

On the other hand, firms are price-making agents who try to earn profits as much as possible. In general, we should consider the cooperative behavior of firms. Some firms may form a coalition to earn profits. Let V be such a coalition of firms. If any amount of profit is earned by coalition V , then it will be divided in some way among the firms in the coalition. Since firms do not have any resources, or any commodities, they need to buy from consumers some resources as inputs for production. Also, after producing some goods, they have to sell them in order to earn profits. In short, they must get both supply of factors and demand for products in the market. To do so, coalition V needs to make contracts with some consumers. Let U be a group of such consumers that coalition V wants to transact with. Since consumers are price-taking agents, coalition V has only to present prices of commodities to each consumer in order to make a contract. In that case, coalition V does not necessarily have to announce same prices to all the consumers. Announced prices can be different from consumer to consumer, because consumers are assumed not to take any collusive behavior. In this sense, there may exist a kind of price discrimination among consumers. Therefore, different consumers may transact with coalition V at different prices.

In this way, coalition V separately makes a contract with each consumer in U . To describe these contracts, we use a measurable function $p : S \rightarrow R_+^I$, whose image $p(s)$ denotes the prices of commodities under which consumer $s \in U$ transacts with coalition V . As was pointed out earlier, consumers' decisions depend on a promise of profit payment $\pi : T \rightarrow R_+$ which firms make. Therefore, coalition V can also influence consumers by changing the promise of profit payment $\pi(t)$ of firm $t \in V$. So, we refer to a pair (p, π) of functions such that $p : S \rightarrow R_+^I$ and $\pi : T \rightarrow R_+$ as a *marketing strategy*.

Definition 2.

A feasible allocation (f, g) between $U \in \mathcal{S}$ and $V \in \mathcal{T}$ is *realizable in the market* if there exists a marketing strategy (p, π) such that

(i) $f(s) \in D(p(s), \pi, s)$ a.e. in U

and

(ii) $\int_V \pi dt \leq \int_U p(s) \cdot (f(s) - f^o(s)) ds.$

We call such a pair (f, g) a *realizable allocation for V inducing U with a marketing strategy (p, π)* . Also such a quadruplet $(f, g; p, \pi)$ is called a *realizable contract configuration between U and V* .

In particular, when $U=S$ and $V=T$, we call such a pair (f, g) a *realizable allocation for the entire economy with a marketing strategy (p, π)* . And, such a quadruplet $(f, g; p, \pi)$ is called a *realizable contract configuration of the entire economy*.

This definition shows conditions which guarantee that a feasible allocation (f, g) between coalitions U and V can be realized in the market by coalition V of firms inducing a group U of consumers with a marketing strategy (p, π) . Condition (i) simply means that all the consumers in U are taking prices as given. Namely, each consumer $s \in U$ is maximizing his utility within his budget when the prices $p(s)$ of commodities are announced to him by coalition V . At that time, the net amount $(f(s) - f^0(s))$ of commodities are traded between coalition V and consumer $s \in U$. As a result of the trades with consumers in U , by virtue of condition (iii) in Definition 1, coalition V can get enough supply of factors for production and can produce enough amount of goods to satisfy the demand of consumers in U .

On the other hand, coalition V can earn the amount $p(s) \cdot (f(s) - f^0(s))$ of profit in the trade with consumer $s \in U$, because that trade is done under the price system $p(s)$. Therefore, the total amount of profit earned by coalition V is $\int_U p(s) \cdot (f(s) - f^0(s)) ds$. Hence, condition (ii) implies that the total amount $\int_V \pi dt$ of profit which the firms in coalition V promise to pay is not greater than the total amount of profit actually earned by coalition V in the trades with consumers in U .

Under the conditions of Definition 2, coalition V can induce a group U of consumers to participate in the coalition with a marketing strategy (p, π) . Namely, the consumers and firms in V and U form an autarchic subeconomy. But, we have to note the following: The consumers in U are not perfectly independent of other agents not in $U \cup V$, because condition (i) depends on the promise of profit payment which the firms in $T \sim V$ make. However, a justification for the independency of group $U \cup V$ can be made by the following lemma.

Lemma 3.1

Let $(f, g; p, \pi)$ be a realizable contract configuration between $U \in \mathcal{S}$ and $V \in \mathcal{T}$. Then the following holds.

$$\int_{s \in U} \left[\int_V \theta \cdot \pi dt \right] ds \leq \int_U \left[\int_{T \sim V} \theta \cdot \pi dt \right] ds.$$

Proof: By virtue of the definition of demand function, the condition (i) in Definition 2 implies that

$$(3.1) \quad p(s) \cdot f(s) \leq p(s) \cdot f^0(s) + \int_T \theta(s, t) \cdot \pi(t) dt \quad a.e. \text{ in } U.$$

Integrating this inequality over U , we have

$$(3.2) \quad \int_U p(s) \cdot (f(s) - f^0(s)) ds \leq \int_U \left[\int_T \theta \cdot \pi dt \right] ds.$$

By virtue of condition (ii) in Definition 2, this inequality implies that

$$\begin{aligned} \int_V \pi dt &\leq \int_U \left[\int_T \theta \cdot \pi dt \right] ds \\ &= \int_U \left[\int_{T \sim V} \theta \cdot \pi dt \right] ds + \int_U \left[\int_V \theta \cdot \pi dt \right] ds \\ &= \int_U \left[\int_{T \sim V} \theta \cdot \pi dt \right] ds - \int_{S \sim U} \left[\int_V \theta \cdot \pi dt \right] ds + \int_S \left[\int_V \theta \cdot \pi dt \right] ds. \end{aligned}$$

Since $\int_S \left[\int_V \theta \cdot \pi dt \right] ds = \int_V \left[\int_S \theta \cdot \pi ds \right] dt = \int_V \pi dt$ by Fubini's theorem, we have the conclusion of this lemma.

Q.E.D.

In the inequality of this lemma, the left-hand side is the dividend of profit which must be paid to the consumers in $S \sim U$, who have shares in the profits of firms in V . And the right-hand side is the dividend of profit to be paid to the consumers in U , who have shares in the profits of the firms in $T \sim V$. Therefore, this lemma implies that group $U \cup V$ does not have any debt to the complementary group $A \sim (U \cup V)$ in the sense of social accounting.

Moreover, the following properties concerning a realizable contract configuration of the entire economy should be noted.

Lemma 3.2

If $(f, g; p, \pi)$ is a realizable contract configuration of the entire economy, then

$$(i) \quad p(s) \cdot f(s) = p(s) \cdot f^\circ(s) + \int_T \theta(s, t) \cdot \pi(t) dt \quad a.e. \text{ in } S,$$

and

$$(ii) \quad \int_T \pi dt = \int_S p(s) \cdot (f(s) - f^\circ(s)) ds.$$

Proof: Since $(f, g; p, \pi)$ is a realizable contract configuration of the entire economy, we can put $U=S$ and $V=T$ in the inequalities (3.1) and (3.2) in the proof of Lemma 3.1 and in condition (ii) of Definition 2. Namely, we have the following:

$$(3.3) \quad p(s) \cdot f(s) \leq p(s) \cdot f^\circ(s) + \int_T \theta(s, t) \cdot \pi(t) dt \quad a.e. \text{ in } S.$$

$$(3.4) \quad \int_S p(s) \cdot (f(s) - f^\circ(s)) ds \leq \int_S \left[\int_T \theta \cdot \pi dt \right] ds.$$

$$(3.5) \quad \int_T \pi dt \leq \int_S p(s) \cdot (f(s) - f^\circ(s)) ds.$$

Since $\int_S \left[\int_T \theta \cdot \pi dt \right] ds = \int_T \left[\int_S \theta \cdot \pi ds \right] dt = \int_T \pi dt$ by Fubini's theorem, (3.4) and (3.5) imply that

$$\int_T \pi dt = \int_S p(s) \cdot (f(s) - f^\circ(s)) ds = \int_S \left[\int_T \theta \cdot \pi dt \right] ds.$$

The first equality of this implies property (ii) of this lemma. Also, the second equality implies property (i) of this lemma together with (3.3).

Q.E.D.

In this lemma property (i) says that all the consumers are exhausting their budgets. And, property (ii) means that profits are completely distributed among firms.

We now confine our attention to realizable contract configurations of the entire economy. A contract configuration of the entire economy can be regarded as an agreement about profit

division among all the firms in the economy. Namely, coalition T of all the firms announces prices $p(s)$ to each consumer $s \in S$ and transact with him, and all the profits which arise in the trades with all the consumers in the economy are divided among all the firms in the coalition. However, even if a contract configuration of the entire economy is realizable in the market, it may be unsatisfactory for some firms, and they may try to improve it upon. In fact, if it is unsatisfactory, they will and can form a coalition, make a new contract configuration, and better their situations in the following way.

Definition 3.

A realizable contract configuration $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ of the entire economy can be improved upon by a coalition V of firms inducing a group U of consumers if there exists a realizable contract configuration $(f, g; p, \pi)$ between U and V such that

- (i) $f(s) \succ_s \hat{f}(s)$ a.e. in U ,
 - (ii) $\pi(t) > \hat{\pi}(t)$ a.e. in V ,
- and such that
- (iii) $\pi(t) = \hat{\pi}(t)$ a.e. in $T \sim V$.

In this definition, condition (i) means that every consumer in U can achieve a better situation by making a new contract with coalition V . In other words, coalition V can induce every consumer in U to participate in the coalition by proposing a better contract to him with a new marketing strategy (p, π) . Condition (ii) means that every firm can actually get greater profit as the result of the new trades with the consumers in U . Condition (iii) says that coalition V is supposing that the firms outside the coalition, or in $T \sim V$, do not change their promises of profit payment. Accordingly, this definition says that, as long as each firm $t \in T \sim V$ promises to pay the amount $\hat{\pi}(t)$ of profit, coalition V can form an autarchic sub-economy by inducing the consumers in U to participate in the coalition in order that the profits of the firms in coalition V may become larger.

Definition 3 suggests that firms are active agents and consumers are passive agents in this economy. In fact, consumers consider announced prices of commodities as given in the market. In this sense, consumers are price-taking agents in this economy. But, we are assuming that consumers are always ready to make contracts with coalitions of firms who announce more favorable prices of commodities to them. So, firms cannot announce arbitrary prices of commodities to consumers, because if they announce unfavorable prices to consumers, any contract with consumers will not be realized. If a coalition of firms fails to induce any consumers to participate in the coalition, no profit will arise in the coalition. Therefore, condition (i) is a fatal one that must be satisfied when firms try to form a new coalition.

There may be some realizable contract configurations of the entire economy that cannot be destroyed by any coalition of firms.

Definition 4.

A realizable contract configuration of the entire economy is called an *equilibrium of the economy* if it cannot be improved upon by any coalition of firms with positive measure.

We have to note that this definition simply says that a realizable contract configuration

cannot be improved upon by *any coalition of firms* inducing some consumers if it is an equilibrium of the economy. Therefore, it may happen that an allocation of a contract configuration, which is an equilibrium of the economy in the sense of the above definition, can be improved upon by *a group only of consumers* if consumers are no longer price-taking agents. In this sense, we are considering *a firm-leading economy* and asking what allocations are "stable" in such an economy.

Our purpose is to find what allocation is realized under a "stable" agreement among all the firms. As we shall show later, only Walrasian allocation can be realized if the economy is large.

IV. Competitive Contract Configuration

A so-called Walrasian equilibrium is a special form of realizable contract configurations of the entire economy. We can define a Walrasian equilibrium in the economy by using the definition of contract configuration.

Definition 5.

A realizable contract configuration $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ of the entire economy is called to be *competitive* if there exists a price vector $\bar{p} \in R_+^l$ such that

$$(i) \quad \hat{p}(s) = \bar{p} \quad \text{a.e. in } S,$$

and such that

$$(ii) \quad \hat{\pi}(t) = \bar{p} \cdot \hat{g}(t) = \text{Sup} \{ \bar{p} \cdot y \mid y \in Y(t) \} \quad \text{a.e. in } T.$$

The allocation (\hat{f}, \hat{g}) of such a contract configuration is called *a competitive allocation*. Also, we call such a price vector \bar{p} *a competitive price vector*.

In this definition, condition (i) means that every consumer transacts under the same price system \bar{p} , that is, every consumer is maximizing his utility at the prices \bar{p} of commodities. Condition (ii) says that profits are divided among firms corresponding to their production activities, that is, the amount $\hat{\pi}(t)$ of profit distributed to firm $t \in T$ is exactly equal to the amount $\bar{p} \cdot \hat{g}(t)$ which is the maximum value that the firm can attain by production activities under the price system \bar{p} . Of course, in this case, we may consider every firm as a price-taker who is maximizing his profit under the price system \bar{p} .

We now show that the competitive contract configuration is quite "stable" in the following sense.

Theorem 1.

Any competitive contract configuration cannot be improved upon by any coalition of firms that has positive measure. Namely, every competitive contract configuration is an equilibrium of the economy in the sense of Definition 4.

Proof: Let $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ be a competitive contract configuration, and $\bar{p} \in R_+^l$ be a competitive price vector associated with it. Suppose the contract configuration can be improved upon by a coalition of firms. Then, according to Definition 3, there exists a realizable

contract configuration $(f, g; p, \pi)$ between a coalition $V \in \mathcal{S} : \mu(V) > 0$ and a group $U \in \mathcal{S}$ such that

$$(4.1) \quad f(s) \succ_s \hat{f}(s) \quad a.e. \text{ in } U,$$

$$(4.2) \quad \pi(t) > \hat{\pi}(t) \quad a.e. \text{ in } V,$$

and such that

$$(4.3) \quad \pi(t) = \hat{\pi}(t) \quad a.e. \text{ in } T \sim V.$$

First, we consider the case that $\mu(U) = 0$. Definition 1(iii) implies that

$$(4.4) \quad \int_V g dt \geq 0.$$

Also, Definition 2(ii) implies that

$$(4.5) \quad \int_V \pi dt \leq 0.$$

On the other hand, Definition 5(ii) implies that

$$\hat{\pi}(t) = \bar{p} \cdot \hat{g}(t) \geq \bar{p} \cdot g(t) \quad a.e. \text{ in } V.$$

Integrating this over V , we obtain

$$(4.6) \quad \int_V \hat{\pi} dt \geq \bar{p} \cdot \int_V g dt.$$

Inequalities (4.4), (4.5), and (4.6) imply that

$$\int_V \pi dt \leq 0 \leq \int_V \hat{\pi} dt.$$

This, however, contradicts inequality (4.2), because $\mu(V) > 0$.

Next, we shall consider the case that $\mu(U) > 0$. By virtue of Definition 5(i), expression (4.1) implies that

$$\bar{p} \cdot f(s) > \bar{p} \cdot f^\circ(s) + \int_T \theta(s, t) \cdot \hat{\pi}(t) dt \quad a.e. \text{ in } S.$$

Since $\mu(U) > 0$, integrating this inequality over U , we have

$$(4.7) \quad \bar{p} \cdot \int_U f ds > \bar{p} \cdot \int_U f^\circ ds + \int_U \left[\int_T \theta \cdot \hat{\pi} dt \right] ds.$$

On the other hand, the feasibility of allocation (f, g) implies, by Definition 1(iii), that

$$(4.8) \quad \bar{p} \cdot \int_O (f - f^\circ) ds \leq \bar{p} \cdot \int_V g dt$$

Also, since $\pi(t) = \hat{\pi}(t)$ a.e. in $T \sim V$, we have

$$(4.9) \quad \int_U \left[\int_{T \sim V} \theta \cdot \pi dt \right] ds = \int_U \left[\int_{T \sim V} \theta \cdot \hat{\pi} dt \right] ds \\ = \int_U \left[\int_T \theta \cdot \hat{\pi} dt \right] ds - \int_U \left[\int_V \theta \cdot \hat{\pi} dt \right] ds.$$

Moreover, since $\pi(t) > \hat{\pi}(t) \geq \bar{p} \cdot g(t)$ a.e. in V , we have

$$(4.10) \quad \int_{S \sim U} \left[\int_V \theta \cdot \pi dt \right] ds \geq \int_{S \sim U} \left[\int_V \theta \cdot \hat{\pi} dt \right] ds = \int_S \left[\int_V \theta \cdot \hat{\pi} dt \right] ds - \int_U \left[\int_V \theta \cdot \hat{\pi} dt \right] ds \\ = \int_V \hat{\pi} dt - \int_U \left[\int_V \theta \cdot \hat{\pi} dt \right] ds \\ \geq \bar{p} \cdot \int_V g dt - \int_U \left[\int_V \theta \cdot \hat{\pi} dt \right] ds,$$

because $\int_S \left[\int_V \theta \cdot \hat{\pi} dt \right] ds = \int_V \left[\int_S \theta \cdot \hat{\pi} ds \right] dt = \int_V \hat{\pi} dt$ by Fubini's theorem. Therefore, inequalities (4.9) and (4.10) imply, by virtue of Lemma 3.1,

$$(4.11) \quad \bar{p} \cdot \int_V g ds \leq \int_U \left[\int_T \theta \cdot \hat{\pi} dt \right] ds.$$

Hence, it follows from (4.8) and (4.11) that

$$\bar{p} \cdot \int_U f ds \leq \bar{p} \cdot \int_U f^o ds + \int_U \left[\int_T \theta \cdot \hat{\pi} dt \right] ds,$$

which contradicts inequality (4.7).

In any case, we can derive a contradiction. Therefore, the contract configuration $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ cannot be improved upon by any coalition of firms that has positive measure.

Q.E.D.

V. Assumptions and a Limit Theorem

In order to make the converse of Theorem 1 hold, we need some assumptions on preference relations and demand functions.

First we assume the following on preference relations.

Assumption 5.1

For almost every consumer $s \in S$, the following hold:

- (i) The consumption set $X(s)$ is closed, convex, and $f^o(s) \in \text{Int}X(s)$.
- (ii) The preference relation \succ_s is transitive, *i.e.*,
 $x \succ_s y$ and $y \succ_s z$ imply $x \succ_s z$.
- (iii) The preference relation \succ_s is continuous, *i.e.*,
the set $\{x \in X(s) \mid \text{not } x \succ_s y\}$ is closed for all $y \in X(s)$.
- (iv) The preference relation \succ_s is not locally satiated, *i.e.*,
if $x \in B(p, \pi, s)$ and $p \cdot x < p \cdot f^o(s) + \int_T \theta \cdot \pi dt$, then we have $y \succ_s x$ for all $y \in D(p, \pi, s)$.

Also, we make the following assumption on demand functions, which is rather stringent, but familiar. Beforehand, let us define a function space.

$$L_1^+ = \{ \pi \mid \pi \text{ is an integrable function of } T \text{ into } R_+ \},$$

which is endowed with a topology by a norm $\| \pi \| = \int_T | \pi(t) | dt$.

Assumption 5.2

For almost every consumer $s \in S$, the following hold:

- (i) For given $\pi \in L_1^+$, $D(p, \pi, s) \neq \emptyset$ if and only if $p \in R_{++}^l$
- (ii) The demand set $D(p, \pi, s)$ is a singleton for each $p \in R_{++}^l$ and $\pi \in L_1^+$.
- (iii) The correspondence $D(p, \pi, s)$ of $R_{++}^l \times L_1^+$ into R^l is upper-hemicontinuous, that is, together with condition (ii) in this assumption, $D(p, \pi, s)$ is continuous with respect to $p \in R_{++}^l$ and $\pi \in L_1^+$.

Of course, we know that this assumption can be derived if we make some assumptions in addition to Assumption 5.1. However, such an argument is very tedious, and we should

assume the result rather than derive it.

Next we shall assume the measurability on preference relations and demand correspondences. In advance, let us define a mapping $Q:S \rightarrow 2^{R^2}$ by

$$Q(s) = \{(x, x') \in X(s) \times X(s) \mid x \succ_s x'\} \quad \text{for each } s \in S.$$

Assumption 5.3¹

(i) The mapping $Q:S \rightarrow 2^{R^2}$ is \mathcal{S} -measurable, i.e.,

$$G_Q = \{(s, z) \in S \times R^2 \mid z \in Q(s)\} \in \mathcal{S} \times \mathcal{B}(R^2).$$

(ii) The mapping $Y:T \rightarrow 2^{R^1}$ is \mathcal{T} -measurable, i.e.,

$$G_Y = \{(t, y) \in T \times R^1 \mid y \in Y(t)\} \in \mathcal{T} \times \mathcal{B}(R^1).$$

(iii) For a fixed $\pi \in L_1^+$, the mapping $D(., \pi, .):R_+^1 \times S \rightarrow 2^{R^1}$ is $\mathcal{B}(R^1) \times \mathcal{S}$ -measurable, i.e.,

$$G_D(\pi) = \{(p, s, x) \in R_+^1 \times S \times R^1 \mid x \in D(p, \pi, s)\} \in \mathcal{B}(R^1) \times \mathcal{S} \times \mathcal{B}(R^1).$$

This assumption is purely technical and it has no economic meanings. Detailed arguments on this sort of measurability have been done by W. Hildenbrand [8, Chp. 1, especially see Thm. 1 on p. 96 and Thm. 2 on p. 102].

Under these assumptions, we have the following theorem.

Theorem 2. (a limit theorem)

Under Assumptions 5.1, 5.2, and 5.3, if the measure space (A, \mathcal{A}, μ) is non-atomic, then for any realizable contract configuration $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ that cannot be improved upon by any coalition of firms with positive measure, there exists a marketing strategy $(\tilde{p}, \tilde{\pi})$ such that the contract configuration $(\hat{f}, \hat{g}; \tilde{p}, \tilde{\pi})$ is competitive. Namely, under any equilibrium of the economy in the sense of Definition 4, a competitive allocation is realized.

This theorem says that it is a competitive allocation that is realized in a large economy by a stable agreement among all the firms. In other words, if the economy is large, the perfect competition among firms realizes a Walrasian allocation in the economy as long as consumers are price-taking agents. So, this theorem is a kind of limit theorem.

However, in the above theorem, it does not generally hold that $(\hat{p}, \hat{\pi}) = (\tilde{p}, \tilde{\pi})$. Namely even if a contract configuration is an equilibrium of the economy, a unique price system does not always hold in the economy under the contract configuration. This is partly because the demand correspondence of each consumer is not invertible.

The proof of Theorem 2 will be given in the next section.

VI. Proof of the Limit Theorem

The method of proof used here is essentially one that has been developed by W. Hildenbrand [6], [7], and K. Vind [10].

Let $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ be a realizable contract configuration of the entire economy which cannot be improved upon by any coalition of firms that has positive measure. Define a mapping

¹ $\mathcal{B}(R^k)$ denotes the family of Borel sets in R^k for each positive integer k . $\mathcal{S} \times \mathcal{B}(R^2)$, $\mathcal{T} \times \mathcal{B}(R^1)$, and $\mathcal{B}(R^1) \times \mathcal{S} \times \mathcal{B}(R^1)$ are the product σ -algebras generated by \mathcal{S} , \mathcal{T} , and $\mathcal{B}(R^k)$.

$F: S \rightarrow 2^{R^l}$ as

$$F(s) = \{x \in X(s) \mid x \succ_s \hat{f}(s)\} \cap [\cup_{p \in R^l_+} D(p, \hat{\pi}, s)] \text{ for each } s \in S.$$

Define a mapping $E: S \rightarrow 2^{R^l}$ as

$$E(s) = \{x - f^0(s) \mid x \in F(s)\} \text{ for each } s \in S.$$

Define a mapping $H: A \rightarrow 2^{R^{l+1}}$ as

$$H(a) = \begin{cases} \{(x, \alpha) \mid x \in E(a), \alpha = -\int_T \theta(a, t) \cdot \hat{\pi}(t) dt\} & \text{when } a \in S, \\ \{(y, \beta) \mid -y \in Y(a), \beta = \hat{\pi}(a)\} & \text{when } a \in T. \end{cases}$$

Moreover, define a mapping $\hat{H}: A \rightarrow 2^{R^{l+1}}$ as

$$\hat{H}(a) = H(a) \cup \{0^{l+1}\} \text{ for each } a \in A.$$

Define a set $L(\hat{H})$ of integrable functions as

$$L(\hat{H}) = \{h \mid h \text{ is an integrable function of } A \text{ into } R^{l+1} \text{ such that } h(a) \in \hat{H}(a) \text{ a.e. in } A\}.$$

Obviously, $L(\hat{H}) \neq \phi$ since $0^{l+1} \in \hat{H}(a)$ a.e. in A .

Define a subset Z of R^{l+1} as

$$Z = \left\{ \int_C h d\mu \mid h \in L(\hat{H}), C \in \mathcal{A}, \mu(C) > 0 \right\}$$

Then, we have the following lemma on the set Z .

Lemma 6.1

The set Z is a convex subset of R^{l+1} with $Z \cap R^{l+1}_- = \phi$.

Proof: The convexity of the set Z is immediately derived from Lemma *A* of K. Vind [10], since the measure space (A, \mathcal{A}, μ) is non-atomic.

Suppose there exists a point $z \in Z$ such that $z < 0^{l+1}$, that is, there exists a function $h \in L(\hat{H})$ and there exists a set $C \in \mathcal{A}$ with $\mu(C) > 0$ such that

$$(6.1) \quad \int_C h d\mu < 0^{l+1}.$$

Of course, we can assume that $h(a) \in H(a)$ a.e. in C . Let $U = S \cap C$ and $V = T \cap C$. Also, we can assume that $\mu(V) > 0$. In fact, if $\mu(V) = 0$, define a mapping $h': A \rightarrow R^{l+1}$ as

$$h'(a) = \begin{cases} h(a) & \text{when } a \in S, \\ (-\hat{g}(a), \hat{\pi}(a)) & \text{when } a \in T. \end{cases}$$

Then, $h' \in L(\hat{H})$. Also, we can choose a set $V' \in \mathcal{F}$ with $\mu(V') > 0$ such that

$$\int_{(C \cup V')} h' d\mu < 0^{l+1},$$

because the measure space (A, \mathcal{A}, μ) is non-atomic. Therefore, in inequality (6.1), the function h and the set C can be replaced by the function h' and the set $C \cup V'$. Hence we can assume that $\mu(V) > 0$.

Next, define integrable functions $f: S \rightarrow R^l$ and $g: T \rightarrow R^l$ as

$$f(s) = (h_1(s), \dots, h_l(s)) + f^\circ(s) \quad \text{for each } s \in S$$

and

$$g(t) = -(h_1(t), \dots, h_l(t)) \quad \text{for each } t \in T$$

Also, the following hold:

$$-\int_T \theta(s, t) \cdot \hat{\pi}(t) dt = h_{l+1}(s) \quad \text{for almost every } s \in U,$$

and

$$\hat{\pi}(t) = h_{l+1}(t) \quad \text{for almost every } t \in V.$$

In the above, h_i denotes the i -th coordinate of the function h . Then, by (6.1) we obtain

$$(6.2) \quad \int_U f ds - \int_U f^\circ ds < \int_V g dt$$

and

$$(6.3) \quad -\int_U \left[\int_T \theta \cdot \hat{\pi} t dt \right] ds < -\int_V \hat{\pi} dt.$$

We shall show that there exists a measurable function $p: S \rightarrow R_+^l$ such that $f(s) \in D(p(s), \hat{\pi}, s)$ a.e. in U . By Assumption 5.3 (iii), the following set,

$$(R_+^l \times [(S \sim U) \times R^l \cup \{(s, x) \in U \times R^l \mid x = f(s)\}]) \cap G_D(\hat{\pi}).$$

belongs to $\mathcal{B}(R^l) \times \mathcal{S} \times \mathcal{B}(R^l)$. Therefore, by the Measurable Choice Theorem (See, for example, [8, p. 54]), there exist measurable functions $f': S \rightarrow R^l$ and $p: S \rightarrow R_+^l$ such that $f'(s) \in D(p(s), \hat{\pi}, s)$ a.e. in U . Since $f'(s) = f(s)$ a.e. in U , we have that $f(s) \in D(p(s), \hat{\pi}, s)$ a.e. in U .

Define an integrable function $\pi^k: T \rightarrow R_+$ for each $k = 1, 2, \dots$ by

$$\pi^k(t) = \begin{cases} \hat{\pi}(t) + 1/k & \text{when } t \in V \\ \hat{\pi}(t) & \text{when } t \in T \sim V. \end{cases}$$

By Assumption 5.2 (i), it follows that $D(p(s), \pi^k, s) \neq \emptyset$ a.e. in U for all $k = 1, 2, \dots$, because $D(p(s), \hat{\pi}, s) \neq \emptyset$, that is, $p(s) > 0^l$ a.e. in U . By virtue of Assumption 5.2 (ii), we can define a measurable function $f^k: S \rightarrow R^l$ for each $k = 1, 2, \dots$ by

$$f^k(s) = \begin{cases} D(p(s), \pi^k, s) & \text{when } s \in U \\ f(s) & \text{when } s \in S \sim U. \end{cases}$$

Then, by Assumption 5.1 (iv), we have that for all $k = 1, 2, \dots$ $f^k(s) = f(s)$ or $f^k(s) \succ_s f(s)$ a.e. in U . In any case, by Assumption 5.1 (ii), $f^k(s) \succ_s \hat{f}(s)$ a.e. in U , because $f(s) \succ_s \hat{f}(s)$ a.e. in U . Moreover, we have to note that $\{f^k\}_{k=1}^\infty$ converges to f a.e. in U because of Assumption 5.2 (iii). Therefore, by Egoroff's theorem, for arbitrary positive number ϵ there exists a measurable set $W_\epsilon \subset U$ with $\mu(W_\epsilon) < \epsilon$ such that $\{f^k\}_{k=1}^\infty$ converges to f uniformly on $U \sim W_\epsilon$. Hence, for such a number ϵ , we can assume that the integral of f^k over $U \sim W_\epsilon$ exists for each k , and that $\int_{(U \sim W_\epsilon)} f^k ds$ converges to $\int_{(U \sim W_\epsilon)} f ds$.

On the other hand, the following holds:

$$(6.4) \quad \left\| \int_U (f - f^\circ) ds - \int_{(U \sim W_\epsilon)} (f^k - f^\circ) ds \right\|$$

$$\begin{aligned} &= \left\| \int_{W_\epsilon} (f - f^\circ) ds + \int_{(U \sim W_\epsilon)} (f - f^k) ds \right\| \\ &\leq \left\| \int_{W_\epsilon} (f - f^\circ) ds \right\| + \left\| \int_{(U \sim W_\epsilon)} (f - f^k) ds \right\|, \end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm in R^l . Since the functions f and f° are integrable, $\left\| \int_{W_\epsilon} (f - f^\circ) ds \right\|$ goes to zero as ϵ goes to zero. Also, as was shown, for a fixed arbitrary number ϵ , $\left\| \int_{(U \sim W_\epsilon)} (f - f^k) ds \right\|$ goes to zero as k goes to infinity. Therefore, (6.2) and (6.4) imply that there exist sufficiently small ϵ and sufficiently large k such that

$$(6.5) \quad \int_{(U \sim W_\epsilon)} (f^k - f^\circ) ds < \int_V g dt.$$

Furthermore, the following hold:

$$\begin{aligned} &\left| \int_U \left[\int_T \theta \cdot \hat{\pi} dt \right] ds - \int_{(U \sim W_\epsilon)} \left[\int_T \theta \cdot \pi^k dt \right] ds \right| \\ &= \left| \int_{W_\epsilon} \left[\int_T \theta \cdot \hat{\pi} dt \right] ds - \int_U \left[\int_V (1/k) dt \right] ds + \int_{W_\epsilon} \left[\int_V (1/k) dt \right] ds \right| \\ &\leq \left| \int_{W_\epsilon} \left[\int_T \theta \cdot \hat{\pi} dt \right] ds \right| + \mu(U) \cdot \mu(V)/k + \mu(W_\epsilon) \cdot (V)/k \end{aligned}$$

and

$$\begin{aligned} \left| \int_V \hat{\pi} dt - \int_V \pi^k dt \right| &= \int_V (1/k) dt \\ &= \mu(V)/k. \end{aligned}$$

These inequalities imply together with (6.3) that there exist sufficiently small ϵ and sufficiently large k such that

$$(6.6) \quad \int_{(U \sim W_\epsilon)} \left[\int_T \theta \cdot \pi^k dt \right] ds > \int_V \pi^k dt.$$

For fixed ϵ and k satisfying (6.5) and (6.6), re-define the function f^k by

$$f^k(s) = \begin{cases} D(p(s), \pi^k, s) & \text{when } s \in U \sim W_\epsilon \\ f(s) & \text{when } s \in S \sim (U \sim W_\epsilon). \end{cases}$$

Then, the function f^k is integrable and, by Assumption 5.1 (iv),

$$p(s) \cdot f^k(s) = p(s) \cdot f^\circ(s) + \int_T \theta \cdot \pi^k dt \quad \text{a.e. in } U \sim W_\epsilon.$$

Integrating this over $U \sim W_\epsilon$, we have

$$\int_{(U \sim W_\epsilon)} p(s) \cdot (f^k(s) - f^\circ(s)) ds = \int_{(U \sim W_\epsilon)} \left[\int_T \theta \cdot \pi^k dt \right] ds$$

Therefore, by (6.6), we obtain

$$(6.7) \quad \int_{(U \sim W_\epsilon)} p(s) \cdot (f^k(s) - f^\circ(s)) ds > \int_V \pi^k dt.$$

Hence, (6.5) and (6.7) imply that coalition V of firms can improve upon the contract configuration $(\hat{f}; \hat{g}; \hat{p}, \hat{\pi})$ the of entire economy, that is, can construct another contract configuration $(f^k, g; p, \pi^k)$ with group $U \sim W_\epsilon$ of consumers. However, this is a contra-

diction to the premise that $(\hat{f}, \hat{g}; \hat{p}, \hat{\pi})$ cannot be improved upon by any coalition of firms with positive measure.

Q.E.D.

Lemma 6.2

There exists a vector $q = (\bar{p}, \gamma) \in R_+^l \times R_+$ with $q \neq 0^{l+1}$ such that

$$(i) \quad \bar{p} \cdot \hat{f}(s) = \bar{p} \cdot f^o(s) + \gamma \cdot \int_T \theta \cdot \hat{\pi} dt \quad a.e. \text{ in } S$$

and that

$$(ii) \quad \bar{p} \cdot \hat{g}(t) = \gamma \cdot \hat{\pi}(t) \quad a.e. \text{ in } T.$$

Proof: Since Lemma 6.1 holds, it follows from a well-known separation theorem that there exists a vector $q = (\bar{p}, \gamma) \in R_+^l \times R_+$ with $q \neq 0^{l+1}$ such that $q \cdot z \geq 0$ for all $z \in Z$. Namely, by definition of set Z , $q \cdot \int_C h d\mu \geq 0$ for all $C \in \mathcal{A}$ with $\mu(C) > 0$ and for all $h \in L(\hat{H})$.

In particular, when $C = A$, we have $q \cdot \int_A h d\mu \geq 0$ for all $h \in L(\hat{H})$.

Since $0^{l+1} \in \hat{H}(a)$ a.e. in A , we have

$$(6.8) \quad \text{Inf} \left\{ q \cdot \int_A h d\mu \mid h \in L(\hat{H}) \right\} = 0.$$

On the other hand, let us define the following mappings; a mapping $F' : S \rightarrow 2^{R^{2l}}$, where for each $s \in S$

$$F'(s) = [R_+^l \times \{x \in X(s) \mid x \succ_s \hat{f}(s)\}] \cap \{(p, x) \in R_+^l \times R^l \mid x \in D(p, \hat{\pi}, s)\},$$

a mapping $E' : S \rightarrow 2^{R^{2l}}$ where for each $s \in S$,

$$E'(s) = \{(p, x - f^o(s)) \mid (p, x) \in F'(s)\},$$

a mapping $H' : A \rightarrow 2^{R^{2l+1}}$, where

$$H'(a) = \begin{cases} \{(p, x, \alpha) \mid (p, x) \in E'(a), \alpha = - \int_T \theta(a, t) \cdot \hat{\pi}(t) dt\} & \text{when } a \in S, \\ \{(p, y, \beta) \mid (p, -y) \in R_+^l \times Y(a), \beta = \hat{\pi}(a)\} & \text{when } a \in T, \end{cases}$$

and a mapping $\hat{H}' : A \rightarrow 2^{R^{2l+1}}$, where

$$\hat{H}'(a) = H'(a) \cup \{0^{2l+1}\} \quad \text{for each } a \in A.$$

Then, Assumption 5.3 implies that the graph of F' is measurable, and that the graph of \hat{H}' is also measurable. Therefore, we have (See [8, p. 63, Prop. 6])

$$(6.9) \quad \text{Inf} \left\{ (0^l, q) \cdot \int_A h' d\mu \mid h' \in L(\hat{H}') \right\} = \int_A \text{Inf} \left\{ (0^l, q) \cdot z' \mid z' \in \hat{H}'(a) \right\} d\mu,$$

where $L(\hat{H}') = \{h' \mid h' \text{ is an integrable function of } A \text{ into } R^{2l+1} \text{ such that } h'(a) \in \hat{H}'(a) \text{ a.e. in } A\}$. Since, by definitions of \hat{H} and \hat{H}' ,

$$\text{Inf} \left\{ (0^l, q) \cdot \int_A h' d\mu \mid h' \in L(\hat{H}') \right\} \geq \text{Inf} \left\{ q \cdot \int_A h d\mu \mid h \in L(\hat{H}) \right\}$$

and

$$\text{Inf } \{(0', q) \cdot z' \mid z' \in \hat{H}'(a)\} = \text{Inf } \{q \cdot z \mid z \in \hat{H}(a)\} \quad a.e. \text{ in } A,$$

(6.8) and (6.9) imply that $\int_A \text{Inf } \{q \cdot z \mid z \in \hat{H}(a)\} d\mu = 0$. That is, $\text{Inf } \{q \cdot z \mid z \in \hat{H}(a)\} = 0$ a.e. in A , since $0^{l+1} \in \hat{H}(a)$ a.e. in A . Hence,

$$(6.10) \quad q \cdot z \geq 0 \text{ for all } z \in \hat{H}(a) \text{ a.e. in } A.$$

For firms in T , (6.10) implies, since $\hat{g}(t) \in Y(t)$ a.e. in T , that

$$(6.11) \quad -\bar{p} \cdot \hat{g}(t) + \gamma \cdot \hat{\pi}(t) \geq 0 \quad a.e. \text{ in } T.$$

For consumers in S , if $\hat{f}(s) = f^\circ(s)$, then $\int_T \theta(s, t) \cdot \hat{\pi}(t) dt = 0$ for such $s \in S$ because $\hat{f}(s) \in D(\hat{p}(s), \hat{\pi}, s)$ and because of Assumption 5.1 (iv). Therefore we have $\bar{p} \cdot \hat{f}(s) = \bar{p} \cdot f^\circ(s) + \gamma \cdot \int_T \theta(s, t) \cdot \hat{\pi}(t) dt$ for such $s \in S$. Moreover, if $\hat{f}(s) \neq f^\circ(s)$, we can choose $\hat{p}^* \in R_{++}^l$ arbitrarily close to $\hat{p}(s)$ such that $\hat{p}^* \cdot \hat{f}(s) < \hat{p}^* \cdot f^\circ(s) + \int_T \theta(s, t) \cdot \hat{\pi}(t) dt$, since $\hat{p}(s) \in R_{++}^l$ because of Assumption 5.2 (i). This implies, by Assumption 5.1 (iv), that there exists $\hat{x}^* \in D(\hat{p}^*, \hat{\pi}, s)$ such that $\hat{x}^* \succ_s \hat{f}(s)$. Therefore, $\hat{x}^* \in F(s)$ for such $s \in S$, that is, by definition of \hat{H} , $(\hat{x}^* - f^\circ(s), -\int_T \theta(s, t) \cdot \hat{\pi}(t) dt) \in \hat{H}(s)$ for such $s \in S$. Hence, by (6.10), $\bar{p} \cdot \hat{x}^* \geq \bar{p} \cdot f^\circ(s) + \gamma \cdot \int_T \theta(s, t) \cdot \hat{\pi}(t) dt$ for such $s \in S$. Letting \hat{p}^* converge to $\hat{p}(s)$, since \hat{x}^* converges to $\hat{f}(s)$ by Assumption 5.2 (ii) (iii), we have in the limit $\bar{p} \cdot \hat{f}(s) \geq \bar{p} \cdot f^\circ(s) + \gamma \cdot \int_T \theta(s, t) \cdot \hat{\pi}(t) dt$. Therefore, in any case, for consumers in S we have

$$(6.12) \quad \bar{p} \cdot \hat{f}(s) \geq \bar{p} \cdot f^\circ(s) + \gamma \cdot \int_T \theta(s, t) \cdot \hat{\pi}(t) dt \quad a.e. \text{ in } S.$$

Suppose that strict inequality holds for some firms and/or some consumers in inequalities (6.11) and (6.12). Then, integrating those inequalities over T and S respectively and adding them up, we have

$$\bar{p} \cdot \int_S (\hat{f} - f^\circ) ds > \bar{p} \cdot \int_T \hat{g} dt,$$

because $\int_S \left[\int_T \theta \cdot \hat{\pi} dt \right] ds = \int_T \left[\int_S \theta \cdot \hat{\pi} ds \right] dt = \int_T \hat{\pi} dt$ by Fubini's theorem. But, this is a contradiction to the feasibility of (\hat{f}, \hat{g}) , that is, $\int_S (\hat{f} - f^\circ) ds \leq \int_T \hat{g} dt$. Therefore, equality holds in (6.11) and (6.12).

Q.E.D.

Here, let us define a function $\tilde{\pi}: T \rightarrow R_+$ by

$$\tilde{\pi}(t) = \gamma \hat{\pi}(t) \text{ for each } t \in T.$$

Then, by Lemma 6.2 and (6.10), we have

$$(6.13) \quad \bar{p} \cdot \hat{f}(s) = \bar{p} \cdot f^\circ(s) + \int_T \theta(s, t) \cdot \tilde{\pi}(t) dt \leq \bar{p} \cdot x \text{ for all } x \in F(s) \text{ a.e. in } S \text{ and}$$

$$(6.14) \quad \bar{p} \cdot \hat{g}(t) = \tilde{\pi}(t) \geq \bar{p} \cdot y \text{ for all } y \in Y(t) \text{ a.e. in } T.$$

Suppose there exists a point $\hat{x}^* \in X(s)$ with $\bar{p} \cdot \hat{x}^* < \bar{p} \cdot \hat{f}(s)$ such that $\hat{x}^* \succ_s \hat{f}(s)$. Then we have $\hat{p}(s) \cdot \hat{x}^* > \hat{p}(s) \cdot \hat{f}(s)$. Since $\hat{p}(s) \cdot \hat{f}(s) = \hat{p}(s) \cdot f^\circ(s) + \int_T \theta \cdot \hat{\pi} dt$ by Assumption 5.1 (iv) and since we can assume $0 \leq \gamma \leq 1$ without loss of generality, we have

$$\hat{p}(s) \cdot \hat{x}^* > \hat{p}(s) \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

On the other hand, by (6.13), we have

$$\bar{p} \cdot \bar{x}^* < \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

Define $p^\lambda = \lambda \hat{p}(s) + (1 - \lambda) \bar{p}$, where $0 < \lambda < 1$. Then, for a sufficiently small λ , the following holds:

$$p^\lambda \cdot \hat{x}^* < p^\lambda f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

We must note here that $p^\lambda \in R_{++}^I$ since $\hat{p}(s) \in R_{++}^I$ by Assumption 5.2 (i) because $\hat{f}(s) \in D(\hat{p}(s), \hat{\pi}, s)$. Therefore, by Assumption 5.2 (i) and by Assumption 5.1 (iv), there exists $x^\lambda \in D(p^\lambda, \tilde{\pi}, s)$ such that $x^\lambda \succ_s \hat{x}^*$. By Assumption 5.1 (ii), $x^\lambda \succ_s \hat{f}(s)$. This simply implies that

$$\hat{p}(s) \cdot x^\lambda > \hat{p}(s) \cdot \hat{f}(s) \geq \hat{p}(s) \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

Moreover, since $x^\lambda \in F(s)$, by (6.13) we have

$$\bar{p} \cdot x^\lambda \geq \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

Therefore, we have $p^\lambda \cdot x^\lambda > p^\lambda f^o(s) + \int_T \theta \cdot \tilde{\pi} dt$, which contradicts that $x^\lambda \in D(p^\lambda, \tilde{\pi}, s)$. Hence, we can conclude that

$$(6.15) \quad \text{not } x \succ_s \hat{f}(s) \text{ for any } x \in X(s) \text{ with } \bar{p} \cdot x < \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

Furthermore, for all $x' \in X(s)$ with $\bar{p} \cdot x' = \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt$, Assumption 5.1 (i) insures that there is a sequence x^k converging to x' such that $\bar{p} \cdot x^k < \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt$ and $x^k \in X(s)$ for each $k = 1, 2, \dots$. Therefore, by Assumption 5.1 (iii) and (6.15), we have

$$(6.16) \quad \text{not } x' \succ_s \hat{f}(s) \text{ for any } x' \in X(s) \text{ with } \bar{p} \cdot x' = \bar{p} \cdot f^o(s) + \int_T \theta \cdot \tilde{\pi} dt.$$

To sum up, (6.13), (6.15), and (6.16) imply that

$$(6.17) \quad \hat{f}(s) \in D(\bar{p}, \tilde{\pi}, s) \quad a.e. \text{ in } S.$$

Hence, if we define a function $\tilde{p}: S \rightarrow R_+^I$ as $\tilde{p}(s) = \bar{p}$ a.e. in S , (6.14) and (6.17) imply that the contract configuration $(\hat{f}, \hat{g}; \tilde{p}, \tilde{\pi})$ is competitive. This completes the proof of the limit theorem.

REFERENCES

- [1] K.J. Arrow and G. Debreu, 'Existence of an equilibrium for a competitive economy,' *Econometrica*, 22 (1954), 265-290.
- [2] R.J. Aumann, 'Market with a continuum of traders,' *Econometrica*, 32 (1964), 39-50.
- [3] G. Debreu, *Theory of Value*, Wiley, New York, 1959.
- [4] G. Debreu and H. Scarf, 'A limit theorem of the core of an economy,' *International Economic Review*, 4 (1963), 235-246.
- [5] J.J. Gabszewicz and J-P. Vial, 'Oligopoly "a la Cournot" in a general equilibrium analysis,' *Journal of Economic Theory*, 4 (1972), 381-400.
- [6] W. Hildenbrand, 'The core of an economy with a measure space of economic agents,' *Review of Economic Studies*, 35 (1968), 443-452.
- [7] W. Hildenbrand, 'Measure spaces of economic agents,' *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley: University of California Press, (1972), 41-56.
- [8] W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey, 1974.
- [9] H. Scarf, 'An analysis of markets with a large number of participants,' *Recent Advances in Game Theory, Proceedings of the Princeton University Conference of October, 1961*, 127-155.
- [10] K. Vind, 'Edgeworth-allocations in an exchange economy with many traders,' *International Economic Review*, 5 (1964), 165-177.