PARABLE AND REALISM IN THE THEORY OF CAPITAL: A GENERALIZATION OF PROF. SAMUELSON'S THEORY OF SURROGATE PRODUCTION FUNCTION†

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1. *What the Present Study is All About*

Such neoclassical macroeconomic analyses as R. Solow [13] and J. Meade [7] presuppose for simplicity *well-behaved* production functions in which input factor variables are homogeneous labor and homogeneous capital goods of jelly-like malleable types, and go so far as to state that in competitive equilibrium the equalities between rate of profit and marginal productivity of capital and between real wage rate and marginal productivity of labor as well eventually prevail. However, it should be noted that Solow and Meade's statement is no more than the re-emergence of the point raised by Böhm-Bawerk and K. Wicksell of Austrian school along with J.B. Clark and P. Wicksteed.

Even if we are abstaining here from appraising the validity of the homogeneity assumption on labor, however, considering the technical nature of capital employed in the production process, we have every reason to launch doubt whether we are oversimplifying or not by studying within the confines of jelly-like, malleable and homogeneous capital goods. That is to say, the comparison of productivities of labor consequent upon its different capital-intensities has nothing in common with asking what if we increase labor input on the same acres of land. The basic reason is that it is impossible to conceive of a productivity of labor independently of its technically associated capital good. Therefore, unless we successfully measure heterogeneous capital goods in a single physical quantity, such concepts as capital-intensity of labor and productivity of capital in the whole economy have no significance. A natural clue to such measure for heterogeneous capital is provided in market valuation at their prevailing prices.

Nevertheless, in pursuing in this direction we must come to grips with a rather bothersome logical fact: If one persists in working with equilibrium market prices, the market pricing of capital goods is only possible with prior knowledge of the level of rate of profit or interest rate. To see the point, in equilibrium, market prices of fixed capital goods are equated to the added values of expected sequence of profits discounted at the current interest rate; the ruling level of interest rate must be known prior to these market valuations. The same is true with circulating capital goods. Thus, in a heterogeneous-capital world in

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† An earlier version of Sections 1 and 2 of this paper was presented in October 1974 at the Nagoya Congress of the Japanese Association of Theoretical Economics. The author is grateful to Professor Yoshio Kimura of Nagoya Municipal University for his comments and criticism on that occasion. Of course, the present article is titled after Samuelson [12].

arriving at a definite macroeconomic characterization of marginal productivity of capital, we conclude that some level of rate of profit or interest rate—which is what marginal productivity of capital has to explain—must be preassigned, and so there is evidently something of a vicious circle involved in the logic [10, 14].

Insofar as relative prices of capital goods are invariant for all levels of the rate of profit, we can evade the foregoing neoclassical macroeconomic difficulty—the jelly-like single capital supposition works well as a sufficient condition. But capital theory assuming single capital jelly is to be regarded as too enormous a simplification, then we are left with the important task of finding out under what conditions relative valuations of capital goods are independent of the changes in the rate of profit. That was much what Samuelson wanted to argue in his 1962 article "Parable and Realism in Capital Theory: The Surrogate Production Function" [12].

Samuelson begins with admittedly more general myriad heterogeneous capital model and brings out the following conditions which enable us to end up with his "surrogate production function"—essentially of the same nature as of the macro production function true of the homogeneous capital jelly:

(1) There exist n capital-good industries each producing only one kind of heterogeneous capital and a single consumption-good industry producing homogeneous consumption good;

(2) Each capital-good industry together with the consumption-good industry constitutes a *closed* production activity and each activity is characterized by fixed production coefficients;

(3) For simplicty, the wage is paid after production is carried out, and the same capital intensity of labor prevails throughout capital-good and consumption-good industries.

Now let l_1 and l_2 stand for the labor coefficient in the capital-good industry and in the consumption-good industry; a_{11} and a_{12} for the capital coefficient in the capital-good industry and in the consumption-good industry, respectively. Then in equilibrium

$$p_1=(\delta+r)a_{11}p_1+l_1w$$
$$p_2=(\delta+r)a_{12}p_1+l_2w$$

hold, where p_1=price of capital good, p_2=price of consumption good, δ=rate of depreciation, r=rate of profit and w=money wage rate. Taking the consumption good as the *numéraire* and remembering the aforementioned condition (3), these can be solved for the w-r relation

$$w=\frac{1}{l_2}\{1-(\delta+r)a_{11}\}$$

This formula depicts nothing other than Samuelson's "factor-price frontier" or what J.R. Hicks named a "wage curve" [5]. As is evident from its form, w is linearly related to r, and this linearlity is traced back to the assumed equality of capital intensities of labor between capital-good and consumption-good industries. Here we get the same number of straight-line r-w relations as of capital goods.

Let the outer envelope of n factor-price frontiers be given the special meaning as the "grand factor-price frontier." As we increase the number of capital goods, this grand factor-price frontier can be approximated to a smooth envelope. Samuelson's purpose in the above paper did lie in exhibiting that the smooth grand factor-price frontier thus reached coincides with the corresponding one in the neoclassical parable and that the foregoing conditions permit the aggregation of heterogeneous capital goods.

Unfortunately, Samuelson's analysis leans heavily on the extreme assumption that each production activity requires only one kind of capital good. For instance, an economist with input-output analytical inclination may feel uneasy even with *a priori* classification into the capital-good and consumption-good industries. The purpose of the present paper is twofold: First, removing Samuelson's singular assumption, we derive a neoclassical macro-production function in the more general Leontief's input-output context which admits of interdependences between industries. Second, we complicate the scenario a little by introducing the "neutral" technical progresses into our model and equip ourselves with their formulations.

2. *The Factor Price Frontier and the Factor Income Shares*

Let us write down the assumptions in full involved in the following analysis.

(1) Labor is the unique nonproducible primary factor and wages are paid out after the completion of production activity.

(2) There exist n industries, each producing one kind of heterogeneous circular capital.

(3) Production coefficients—capital and labor coefficients—are fixed.

(4) Labor-coefficient vectors are positive and capital-coefficient matrices are non-negative and indecomposable.[1]

(5) The same level of wage and the same level of profit rate prevail over all the industries in the economy.

Next, let us introduce our notational convention followed hereafter:

l_i=the labor coefficient in the i-th industry,

$l\equiv\{l_i\}$=the vector of labor coefficients,

a_{ij}=the capital input coefficient of the i-th industry's product to produce one unit of the j-th industry's output,

$A\equiv[a_{ij}]$=the matrix of capital coefficients,

p_i=the market price of the i-th industry's product,

$p\equiv\{p_i\}$=the vector of market prices,

r=the rate of profit,

w=the money wage rate.

By the preceding assumptions we then have

(2.1) $$p'=(1+r)p'A+wl',$$

where the prime indicates the vector transposition—the transpose l' is a row vector for instance. Further, adopting C. von Weizsäcker's [16, p. 19] definition of the real wage rate which takes a standard wage commodity basket $b\equiv\{b_i\}$, we proceed to assume that b remains invariant at any levels of relative prices and wage rate. Here, we take b as the *numéraire* for which

(2.2) $$p'b=\sum p_ib_i=1.$$

[1] Admittedly the assumed indecomposability of the capital-coefficient matrix may be too restrictive. However, after admitting of the decomposability our analysis remains valid only if the left-hand side Frobenius vector of the capital-coefficient matrix is positive.

Let the Frobenius root of A denoted by λ^* and write it in the form

$$\lambda_* = \frac{1}{1+r_*}.$$

Here, our production technique is assumed to be "productive" in the sense that λ_* is less than unity. Then by a well-known theorem [17, p. 130], we are guaranteed to have

(2.3) $$[E-(1+r)A]^{-1} > 0 \text{ for } r_* > r \geq 0,$$

and this is an increasing function of r. Now if we introduce the new notation $\hat{w}$ to represent w corresponding to the numéraire b, the new variable $\hat{w}$ may safely be said as denoting the real wage rate. Anyhow, from the above equation we have

(2.4) $$\hat{w} = \frac{1}{l'[E-(1+r)A]^{-1}b} \equiv \Phi(r).$$

Evidently, this relationship gives the factor-price frontier for the given technology; and Φ is a monotone decreasing function of r.

Let the capital-intensity of labor in the j-th industry be given by

$$\mu_j = \frac{\sum p_i a_{ij}}{l_j} \qquad (j=1,\ 2,\ \ldots,\ n),$$

and let M stand for the diagonal matrix with μ_j as the j-th diagonal element. By easy manipulation

(2.5) $$l'M = p'A.$$

Further, if all the industries throughout the economy are characterized by the same capital intensity of labor, then by stipulating $\mu_j = \mu$, M emerges as

(2.6) $$M = \mu E,$$

and reduces to a scalar matrix. We are now in a position to state and prove the following theorem.

THEOREM (1)

For the capital intensities of labor to be the same between any industries throughout the economy it is both necessary and sufficient that l be the left-hand Frobenius vector of A.

PROOF OF NECESSITY

For this, it is enough to prove that if $M=\mu E$ then l' is the Frobenius vector of A. Let us start with $M=\mu E$. Then we have $\mu l' = p'A$ from (2.5). Combining this with (2.1)

(2.7) $$\begin{aligned} p' &= (1+r)\mu l' + wl' \\ &= \{(1+r)\mu + w\} l', \end{aligned}$$

which tells us that p' stands in direct proportion to l. Using $\mu l' = p'A$ again, we find

(2.8) $$l'A = \left\{\frac{\mu}{(1+r)\mu + w}\right\} l' \equiv \lambda l',$$

so that

(2.9) $$l'[\lambda E - A] = 0.$$

Remembering that A is a non-negative indecomposable matrix and l is a positive vector by construction, Frobenius theorem guarantees that l' is the Frobenius vector of A.

PROOF OF SUFFICIENCY

It suffices to show that if l' is the Frobenius vector of A, then $M=\mu E$ holds. Suppose l' be the Frobenius vector of A, we have

$$l'A = \lambda_* l'$$

by definition. So with the equation

$$l'(1+r)^tA^t(1+r)^t\lambda_*^t l' \equiv \left(\frac{1+r}{1+r_*}\right)^t l'$$

in mind, which constitutes a part of the expanded Neumann series [27, p. 130] of the type

$$[E-(1+r)A]^{-1}=\sum_0^\infty(1+r)^tA^t \qquad \text{for } r_*>r\geq 0,$$

we can restate the equation

$$p'=wl'[E-(1+r)A]^{-1} \qquad \text{for } r_*>r\geq 0,$$

solved for p' from (2.1) as

$$p'=\left\{\frac{1+r_*}{r_*-r}\right\}wl' \qquad \text{for } r_*>r\geq 0. \tag{2.10}$$

Multiply both sides by A from the right hand. Since $l'A=\lambda_*l'$, the right hand side reduces to

$$\left\{\frac{1+r_*}{r_*-r}\right\}wl'A=\left\{\frac{1+r_*}{r_*-r}\right\}w\lambda_*l'\equiv\mu l',$$

which establishes $M=\mu E$. Q.E.D.

Let us call our attention to the equation

$$p'=wl'[E-(1+r)A]^{-1} \qquad \text{for } r_*>r\geq 0,$$

obtained from (2.1). For a moment, let the price-vector p be evaded from the condition of $p'b=1$. Clearly, for a given w, p' increases along with an increase in r. So for the price vectors $p_{(1)}'$ and $p_{(2)}'$ each corresponding to any two different levels of interest rate r_1 and r_2 ($r_1>r_2$) if we have the relation

$$p_{(1)}'=\nu p_{(2)}' \qquad (\nu>1),$$

we have every reason to contend that the relative prices of capital goods remain invariant irrespective of the changes in the level of r. Further, if in the above circumstances l' is always characterized as the Frobenius vector of A, we can safely conclude that for the relative prices of capital goods to remain unchanged it is necessary that l' be the Frobenius vector of A. Equipped with these preparatory considerations, we present the following theorem.

THEOREM (2)

For the relative prices of capital goods to remain unchanged irrespective of any variation in the rate of profit it is both necessary and sufficient that the capital intensities of labor be the same all over the industries.

PROOF OF SUFFICIENCY[2]

Evident from the proportionality between p and l.

PROOF OF NECESSITY

Write $(1+r_i)\equiv\rho_i$. Of course, $\rho_1>\rho_2$. So if for two different price systems

$$p_{(i)}'=wl'[E-\rho_iA]^{-1} \qquad (i=1,\ 2)$$

we stipulate $p_{(1)}'=\nu p_{(2)}'$, then we find

$$l'[E-\rho_1A]^{-1}=\nu l'[E-\rho_2A]^{-1}$$

$$\therefore\quad l'[E-\rho_1A]^{-1}[E-\rho_2A]=\nu l'$$

$$\therefore\quad l'[E-\rho_1A]^{-1}-\rho_1\frac{\rho_2}{\rho_1}l'[\textstyle\sum_0^\infty\rho_1^tA^t]=\nu l'-\frac{\rho_2}{\rho_1}l'$$

$$\therefore\quad \left\{\frac{\rho_1-\rho_2}{\rho_1}\right\}l'[E-\rho_1A]^{-1}=\left\{\frac{\nu\rho_1-\rho_2}{\rho_1}\right\}l'$$

[2] The sufficiency has already been established by E. Burmeister [3].

$$\left\{\frac{\rho_1-\rho_2}{\nu\rho_1-\rho_2}\right\}l'=l'[E-\rho_1 A]=l'-\rho_1 l'A$$

$$\therefore\quad l'A=\left\{\frac{\nu-1}{\nu\rho_1-\rho_2}\right\}l'\equiv\lambda l'.$$

Since l is a positive vector and A is an indecomposable non-negative matrix by assumption, Frobenius theorem guarantees that l' be the Frobenius vector of A. Q.E.D.

We cannot put too much emphasis on the vital relevance of this THEOREM (2) to such economic analyses as the "transformation problem between values and prices" in Marxian economics and the "price rigidity" problem in the modern macroeconomic analysis. In what follows we investigate the characteristics of the factor price frontier for the economy wherein the requisites of this theorem are met.

For this, writing

x_i=the output produced by the i-th industry,

$x\equiv\{x_i\}$=the output vector,

we have the following accounting identities in circular-capital-good economy.

$p'[E-A]x\equiv Y$=national income,

$p'Ax\equiv K$=amount of capital in value terms,

$l'x\equiv L$=amount of labor employment.

Using these and post-multiplying (2.1) by x we get

(2.11) $Y=rK+\hat{w}L$

or alternatively dividing both sides by L

(2.12) $y=rk+\hat{w}$,

where $p'b=1$ and y and k denote net value productivity per capital and the capital intensity of labor, respectively, prevailing throughout the economy.

With these preparations in mind, let us concentrate on (2.10), which holds valid whenever all the industries in the economy are operating at the uniform capital intensity level of labor. Multiplication of both sides from the right hand by the standard wage commodity vector b and a little manipulation yield

$$\hat{w}=\frac{1}{l'b}\left\{\frac{r_*-r}{1+r_*}\right\}. \qquad (2.13)$$

Needless to say, these configurations of real wage and interest rate give rise to the factor price frontier and when the uniform capital intensity of labor rules over all the industries this frontier is summarized in a linear equation. Figure 1 portrays these configurations. Obviously, the vertical intercept gives the labor productivity associated with $r=0$, and is not affected by the level of profit rate provided that $p'b=1$. No less evident is

$$\frac{y}{k}=\frac{p'[E-A]x}{p'Ax}=\left(\frac{1-\lambda_*}{\lambda_*}\right)=r_*$$

from $p'A=\lambda_* p'$. Thus, in Figure 1 the slope of the straight-line frontier corresponding to the angle θ therein easily reads

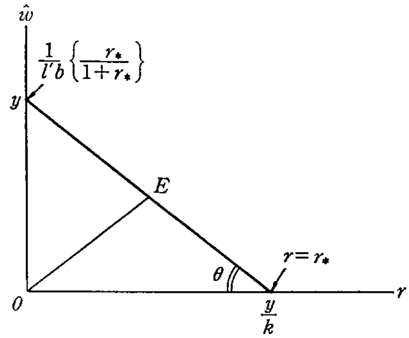
$$-\frac{d\hat{w}}{dr}=k. \qquad (2.14)$$

Calculate the elasticity at a point like E on the factor price frontier. Obviously, the elasticity can be conceived of in conjunction with the correspondent relative shares of income (1). To see this, write down the calculated result in full

$$-\frac{d\hat{w}}{dr}\Big/\frac{\hat{w}}{r}=\frac{rK}{\hat{w}L}. \tag{2.15}$$

Of course, this elasticity becomes zero at $r=0$ and infinity at $\hat{w}=0$. Anyway, this noticeable property that the elasticity of the factor price frontier corresponds to some specified factor income shares will prove essential for our subsequent analysis of neutral technical progress.

FIG. 1



3. *Alternative Techniques and the Surrogate Production Function*

So far we have confined our analysis to the economy wherein a single technique is available to each industry. Now it is time to admit of the substitutability alleged by the neo-classical capital theory and consider the more general economy in which each industry is equipped with a number of alternative techniques.

In this connection, we are familiar with the theorem of Samuelson [12]-Morishima [8]-Levhari [6] which asserts that at the intersection where two factor price frontiers, each associated with one alternative technique, meet both price systems of capital goods are identical. Of course, this theorem holds true irrespective of the difference in capital intensities of labor between industries. For expository reason, however, we hereafter consider an extreme case where any point on the grand factor price frontier—compounded as the outer envelope of the distinct factor price frontiers—lies on the composite linear factor price frontier characterized by the uniform-capital-intensity-of-labor production structure. Figure 2 shows the case in question.

In Figure 2, suffixes are numbered in accordance with the numbering of production techniques, and y is the aggregative labor productivity and k the aggregative capital intensity of labor. Of course, by the uniformity assumption on the capital intensity of labor, the factor price frontier each associated with one production technique reduces to a straight line. Together with the constancy of the relative prices of capital goods along every composite linear factor price frontier with the uniform capital intensity of labor as the foregoing THEOREM (2) asserts, by the equality of prices of capital goods produced by the alternative techniques at the intersections like s_1 and s_2 in the figure which is assured by the theorem of Samuelson-Morishima-Levhari, we are led to the conclusion that we come up with the constant prices of capital goods along the whole portion of the bold grand factor price frontier.

FIG. 2

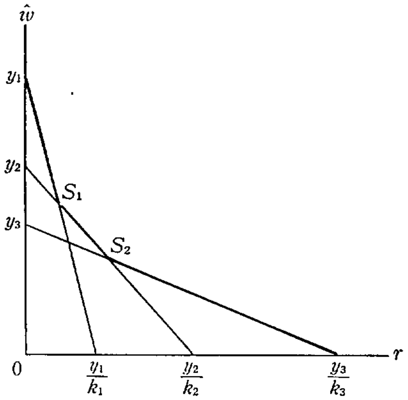
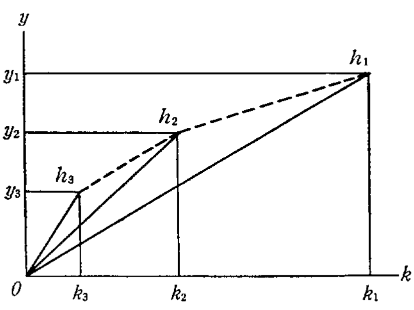


FIG. 3



If we array the configurations of y's and k's in Figure 2 in accordance to their correspondent production technique, we obtain something like Figure 3. To see this, note $y_1>y_2>y_3$, $k_1>k_2>k_3$, $\frac{y_1}{k_1}>\frac{y_2}{k_2}>\frac{y_3}{k_3}$. Apparently, the slope of the line connecting h_1 and h_2 is less than that joining h_2 and h_3. So write these relationships between y and k in the form

(3.1) $$y=H(k)$$

and assume for simplicity that we can conceive of this function as continuous by increasing sufficiently the density of techniques. Then we can read $H'>0$ and $H''<0$, nothing but a well-known property of neoclassical production functions. Multiply both sides of (3.1) by the labor employment L, and we get

(3.2) $$Y=H(k)L\equiv F(K,\ L).$$

This relation is named the "surrogate production function" in Samuelson's tradition, and exhibits quite similar properties to those of the aggregative neoclassical production function which takes the homogeneity of capital for granted. First, as is easily seen from (3.2), this surrogate production function features constant returns to scale in the sense that Y is a linear homogeneous function of K and L. Second, Y increases at a decreasing rate with an increase in K or L, that is the surrogate production function suffers the rule of decreasing returns to factor input. Above all, note should be taken of the relationship between the marginal productivity and the rate of return on the factor input.

Total differentiation of (2.12) yields

$$dy = r\,dk + \{k\,dr + d\hat{w}\}. \tag{3.3}$$

Due to (2.14) which holds when the uniform capital intensity of labor prevails, the second term of the right-hand side of (3.3) vanishes and

$$\frac{dy}{dk} = r \tag{3.4}$$

is derived. In this way we can establish the equality between the partial derivative of Y with respect to K and the rate of profit, that is

$$\frac{\partial Y}{\partial K} = r \tag{3.5}$$

and similarly

$$\frac{\partial Y}{\partial L} = \hat{w} \tag{3.6}$$

as well. These are nothing but the well-known relationships neoclassical marginal productivity theory dictates.

Thus we have departed from the realistic world which allows for heterogeneity of capital goods to settle down in the parable realm with jelly-like homogeneous capital, and this is only possible as long as we stick to the uniformity assumption on the capital intensities of labor in all the industries. This restriction applies to Samuelson's analysis as well [1]. To sum up: It is only within the confines of the uniform capital intensity of labor simultaneously applicable to all the industries that we are justified in going on to assume constant relative prices of admittedly more general heterogeneous capital goods.

4. *The Analysis of Neutral Technical Progresses*

Next, equipped with the foregoing analysis of the properties of the factor price frontier for the given spectrum of techniques, we introduce into our model such technical progresses as to cause the upward-shift of the factor price frontier.

Let us begin with the neutrality analysis of technical progresses. As is well-known, the "neutrality" is conceived of as the neutrality of factor income shares: To be more precise, under some suitable criterion such types of technical progresses as cause no change on their corresponding factor income shares are said to be neutral. What are suitable criteria for neutrality? We can cite Harrod's and Hicks' criteria as the most popular ones. Harrod's criterion compares the distributions of income both before and after the technical progress in question at the arbitrarily chosen level of profit rate, and if at any chosen level of profit rate there is no change in factor income shares then that technical progress is concluded to be Harrod-neutral. On the contrary ,Hicks' criterion compares two states before and after the occurrence of technical progress at the arbitrarily chosen factor price ratio of the wage rate to the profit rate, and if at any chosen facotr price ratio there is no alteration in factor income shares then the technical progress in consideration is said to be Hicks-neutral [15].

Let us extremely simplify our model so that there is only one choice of production technique open at any point of time and the capital intensities of labor are the same in all industries. Then, writing the capital-coefficient matrix at the initial point of time as $A(0)$, the labor-coefficient vector as $l(0)$, the afore-mentioned THEOREM (1) maintains that $l(0)$ be the left-hand side Frobenius vector of $A(0)$. First, let us go further to specify the type of technical progress by the constancy of the capital-coefficient matrix along with the proportionate annual decrease in the labor-coefficient vector. Symbolically

$$\begin{cases} A(t)=A(0) \\ l(t)=l(0)e^{-\xi t} \end{cases} \tag{4.1}$$

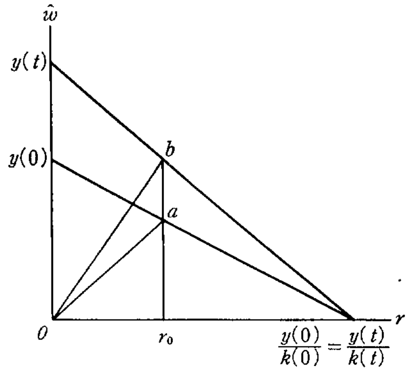
where t=the point of time, e=the base for natural logarithms, $\xi=a$ positive parameter.[3] As is easily checked, in our present context where $l'(0)$ is the Frobenius vector of $A(0)$, $l'(t)$ is characterized as the Frobenius vector of $A(t)$ as well.

Figure 4 portrays the case in consideration in terms of the factor price frontier. As the figure shows, the technical progress shifts the factor price frontier upwards with the horizontal intercept fixed. The horizontal intercept is fixed because the matrix of capital coefficients—consequently its Frobenius characteristic root—remains invariant as is evident from Figure 1. Contrarily, the vertical intercept depicting the productivity of labor shifts upward at the rate of the dwindling factor of the vector of labor coefficients—that is ξ. For this, similarly note

$$l'(t)b=l'(0)be^{-\xi t}$$

and Figure 1. Now compare the two points a and b at the arbitrarily chosen level of profit rate r_o. Evidently, the equal elasticity applies both to the point a and to the point b, and moreover this equality holds at any level of the profit rate. Since the elasticity calculated at the point on the factor price frontier is known to give the corresponding factor income shares, the noted equality of elasticities leads us to the conclusion that Figure 4 shows such situations as the technical progress is of Harrod-neutral type.

FIG. 4



[3] We do not hesitate to cite L. Pasinetti's article [9] as a pioneering effort to discuss the technical progress of the similar type. Of course, he does not discuss in such context as to stipulate that l' be the Frobenius vector of A.

Next, consider the case where both the capital-coefficient matrix and the labor-coefficient vector are decreasing at the same rate, that is symbolically

$$(4.2)\qquad \begin{cases} A(t)=A(0)e^{-\xi t} \\ l(t)=l(0)e^{-\xi t} \end{cases}$$

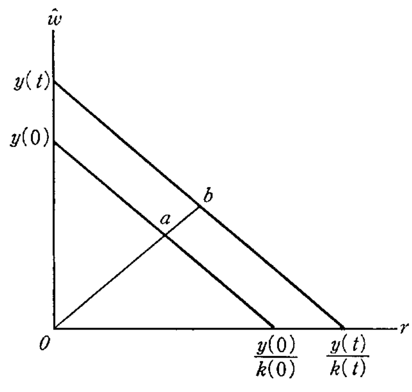
Again refer to Figure 1. Since each element of the capital-coefficient matrix is supposed to decrease at the rate ξ, so does its Frobenius characteristic root $\frac{1}{1+r_*}$ at the rate ξ—consequently r_* increases. Besides, since the labor-coefficient vector decreases at the rate ξ as well, the vertical intercept of Figure 1 which gives the labor productivity

$$\frac{1}{l'b}\left\{\frac{r_*}{1+r_*}\right\}$$

shifts upward in accordance with the increase in r_*. Thus we end up with Figure 5 depicting the parallel shift of the factor price frontier. Arbitrarily selecting the two points a and b on some ray from the origin we know the elasticities at a and b are the same and so are their corresponding factor income shares. This equality holds for any ray from the origin. So we can safely conclude that the technical progress depicted in Figure 5 is of Hicks-neutral type.[4]

So far restricted our analysis within the supposed confine of single available production technique, but we can extend the above results to the case of multiple alternative techniqes without vital alterations. To show this, we again resort to the supposition that the grand factor price frontier formed by each alternative production technique be composed of the spectrum of the uniform capital-intensity-of-labor techniques. For instance, suppose

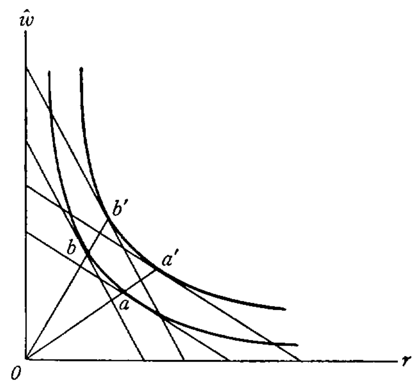
FIG. 5



the technical progress to be of Hicks-neutral type. Figure 6 well illustrates this possibility. Writing the capital-coefficient matrix and the labor-coefficient vector associated with the point a in the figure as $A_\alpha(0)$ and $l_\alpha(0)$, $l_\alpha(0)$ is the Frobenius vector of $A_\alpha(0)$ from the left hand side. Further, let $A_\alpha(t)$ and $l_\alpha(t)$ stand for the capital-coefficient matrix and the labor-coefficient vector corresponding to the point a', and then $A_\alpha(t)=A_\alpha(0)$ and $l_\alpha(t)=$

[4] For representative purposes the case where $A(t)=A(0)e^{-\xi t}$ and $l(t)=l(0)$ would complete our present analysis.

FIG. 6



$l_\alpha(0)e^{-\xi t}$ hold. On the other hand, reverting our attention to the point b and denoting the associated capital-coefficient matrix and labor-coefficient vector by $A_\beta(0)$ and $l_\beta(0)$, $l_\beta(0)$ is similarly the left-hand side Frobenius vector of $A_\beta(0)$. Further, writing the capital-coefficient matrix and the labor-coefficient vector corresponding to the point b' as $A_\beta(t)$ and $l_\beta(t)$, we can check $A_\beta(t)=A_\beta(0)$ and $l_\beta(t)=l_\beta(0)e^{-\xi t}$ as well. Thus we can establish the same relationship for any ray from the origin and are certified to conclude that Figure 6 well illustrates Hicks-neutral technical progress along all the portions of the grand factor price frontier just as Figure 5 did in the single technique case.

Repeating quite the same procedure, we can illustrate the case of Harrod-neutral technical progress, but we reasonably refrain from this repetition.

5. *The Surrogate Production Function of the Cobb-Douglasian Type*

Now following the tradition of the neoclassical aggregative growth theory, let us ask what the consequences will be if the technical progress is of both Harrod-neutral and Hicks-neutral type at the same time in our present context. Figure 7 shows such a possibility.

Assumed Harrod-neutrality on the technical progress requires the identical factor income shares both at the point a and at the point b in Figure 7. In addition, Hicks-neutrality assumption also requires the identical factor income shares both at the point a and at the point c. Repetition of this reasoning leads to the requirement that the factor income shares be constant at any point on the grand factor price frontier.

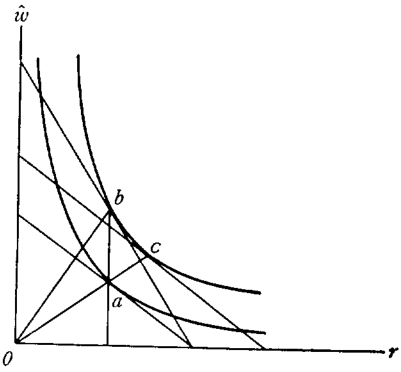
Let us write down the grand factor price frontier at the point of time t formally as

(5.1) $$\hat{w}=\Phi(r, t).$$

Here, t is introduced to visualize the effect of the technical progress through the passage of time so that Φ is reasonably thought of as increasing along with the passage of time t. Remembering that Figure 7 is depicted in such a way as to ensure the constancy of factor income shares applicable to any point on the frontier, that is

(5.2) $$-\frac{\partial \hat{w}}{\partial r}\Big/\frac{\hat{w}}{r}=\text{const}\equiv\alpha,$$

FIG. 7



(5.1) can be transformed into

(5.3) $$\hat{w}=\xi(t)r^{-\alpha},$$

and $\xi(t)$ is an increasing function of t. Thus, (5.3) incorporates the technical progress of Harrod neutral type and of Hicks neutral type at the same time.

Equipped with these preparations, we are in a position to propose the following theorem.

THEOREM (3)

Let the grand factor price frontier be composed of spectrum of uniform capital-intensity-of-labor techniques. Then, for the technical progress to be Harrod-neutral and Hicks-neutral at the same time, it is both necessary and sufficient that the surrogate production function be of the Cobb-Douglasian type, that is formally

(5.4) $$y=\phi(t)k^{\alpha/(1+\alpha)},$$

where ϕ is an increasing function of t and α is a positive parameter.

PROOF OF NECESSITY

To establish the necessity it suffices to derive (5.4) from the presupposed (5.3). Since any point on the grand factor price frontier corresponds to some technique with assumed uniform capital intensity of labor in all industries, the foregoing (2.13) is valid for any such technique. Write down this relationhips in full for the θth technique as

(5.5) $$\hat{w}=\frac{1}{l'_\theta b}\left\{\frac{r_\theta-r}{1+r_\theta}\right\},$$

which with the aid of the additional notation

$$y_\theta=\frac{1}{l'_\theta b}\left\{\frac{r_\theta}{1+r_\theta}\right\}$$

and

$$k_\theta=\frac{1}{l'_\theta b}\left\{\frac{1}{1+r_\theta}\right\},$$

reduces to

(5.6) $$\hat{w}=y_\theta-rk_\theta.$$

By the linearity of the factor price frontier we are assured to have

(5.7) $$-\frac{\partial \hat{w}}{\partial r}=k_\theta.$$

Now suppose (5.3) holds. Partial differentiation of $\hat{w}$ with respect to r and (5.6) yield

$$\hat{w}=\alpha^{-\alpha/(1+\alpha)}\zeta(t)^{1/(1+\alpha)}k_\theta{}^{\alpha/(1+\alpha)}.$$

Again resort to (5.6), and

(5.8) $$y_\theta=\alpha^{(1-\alpha)/(1+\alpha)}\zeta(t)^{1/(1+\alpha)}k_\theta{}^{\alpha/(1+\alpha)}\equiv\phi(t)k_\theta{}^{\alpha/(1+\alpha)}$$

results after a little manipulation. This holds for any θ-th technique by supposition, so that the desired validity of (5.4) has been established. Q.E.D.

PROOF OF SUFFICIENCY

For the sufficiency, conversely derive (5.2) from (5.4). Let us start with (5.4). Since (5.4) denotes the envelope of the composite discriminate factor price frontiers, (5.8) evidently holds. As informed by (3.4), the partial derivative of y_θ with respect to k_θ is equivalent to the rate of profit. Carrying out this differentiation we get

(5.9) $$r=\left(\frac{\alpha}{1+\alpha}\right)\phi(t)k_\theta{}^{-1/(1+\alpha)}.$$

With the aid of (5.9) the combination of (5.6) and (5.8) and the appropriate manipulation lead to

(5.10) $$\hat{w}=\phi(t)^{1+\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\left(\frac{1}{1+\alpha}\right)r^{-\alpha}\equiv\zeta(t)r^{-\alpha}$$

as was desired. Q.E.D.

We are familiar with the neoclassical macro growth theory characteristic of the presupposed homogeneous single capital good which asserts that for the technical progress to be Harrod-neutral as well as Hicks-neutral the production function be of the Cobb-Douglasian type. Thus we have arrived at quite the same conclusion admitting of the heterogeneity of multiple capital goods. Here we have brought forward our analysis far enough to be in a position to conclude that there is no essential difference involved between our present realistic heterogeneous-capital-good model and the neoclassical parable of homogeneous-capital-good macro-model. Of course, the crux of this conclusion leans heavily on the supposed uniformity of capital intensity of labor in all the industries. Admittedly the application of the identical capital intensity of labor to any point on the grand factor price frontier is too restrictive a presupposition. Indeed, the modern "capital controversies" center on the uneven case with different capital intensities of labor between industries.

6. *Concluding Remarks*

We have to leave to other efforts [4, 14] the discussion of such problems as of "reswitching" of techniques which the neoclassical capital theory encounters once it departs from the parable world and steps into the more realistic world characterized by the industries with uneven capital intensities of labor. However, let us refer briefly to the so-called "invariable measure of value" hinted by D. Ricardo.

Ricardo admits what we call the uneven case with different capital intensities of labor in industries as the most ferquently observable economic reality. As the foregoing THEOREM (2) dictates, in the real observed economy even if they are not accompanied by the changes in the output vector x, changes in the rate of profit inevitably give rise to changes in the relative valuation of capital goods, which in turn affects the value of the national income. Here arises an urgent need to seek for the invariable measure of value applicable to characterize the distribution of income between wages and profits irrespective of any change in the relative prices. Unfortunately, Ricardo ended his life before arriving at a satisfactory device. Today, 150 years later the procedure for solution has been provided by P. Sraffa [14].

To follow Sraffa's procedure, note the relationship (2.4) between w and r

$$\hat{w}=\frac{1}{l'[E-(1+r)A]^{-1}b}\equiv\Phi(r).$$

We have shown that if l is the Frobenius vector of A from the left hand side then this relation can be expressed in a linear equation. Of course, this holds true irrespective of the composition of the standard-wage-good vector b. Note Ricardo does not necessarily characterize l as the left-hand side Frobenius vector of A. However, if b is successfully characterized as the Frobenius vector from the right hand side, then we are guaranteed to end up in the linear $\hat{w}$-r relationship unaffected by any change in the relative prices just as we did in Figure 1. Write such b as b_*, and then b_* is nothing but what Ricardo hinted at and Sraffa discovered. Of course, Frobenius theorem guarantees the uniqueness of b_*. Sraffa named b_* as the "standard commodity" in terms of which he measured wages and national income to analyse the distributive relationship independently of the value relationship.

Admittedly little probability is attributed to the case where the laborers' actually purchased wage-commodity vector stands in direct proportion to b_*. If we stick to the measure of wages in terms of b_* nevertheless, $\hat{w}$ no longer carries the meaning of the real wage rate and represents the wage rate measured against some purely aritificial unit commodities of value at best. Thus, to afford Sraffa's theory of standard commodities the relevance to the real economy, we are obliged to ask into the benefits of adopting the measure b_*, presumably in different composition from the actual vector of wage commodities, for understanding the real economic workings [3].

Probably the similar difficulty arises when we ask into the relevance of the analysis in terms of the left-hand side Frobenius vector l_* of A with which the actual l does not coincide up to the scalar multiplication. We cannot but postpone the detailed study in this direction to another opportunity. However, note that we have focussed so far on the price system resorting to the "left-hand side" Frobenius vector of A, and that the untouched "right-hand side" Frobenius vector of A can be inferred to play the same essential role in the corresponding analysis of the quantity system. The duality between the price system and the quantity system offers a basic viewpoint to the modern economic analyses, and the present capital theory is no exception. In the present study we have concentrated on the price system leaving unattempted the analysis of the quantity system in the context of the multiple capital goods and its relation to the price system [2]. These problems awaits to be analysed in detail in the future.

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