EXTENSIONS OF THE CONCEPT OF TRADE INTENSITY

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The concept of "trade intensity" has been fruitfully used by several authors to study the composition of trade. While the use of the trade intensity index has been heretofore limited to a description of area composition of trade, the concept is extended in this note so that by using the various sector and area intensities defined one can achieve a much more complete description of trade composition.

Let

\[ X_{ijh} = \text{exports of area } i \text{ to area } j \text{ sector } h \]
\[ X_i = \sum_h X_{ijh} = i's \text{ exports to the world of sector } h \]
\[ X_{ij} = \sum_h X_{ijh} = i's \text{ exports of all sectors to } j \]
\[ X_i = \sum_j X_{ij} = i's \text{ exports to the world} \]
\[ M_{jh} = \sum_i X_{ijh} = \text{exports from the world to area } j \text{ sector } h \]
\[ M_j = \sum_h M_{jh} = \text{exports from the world to area } j \]
\[ M^h = \sum_j M_{jh} = \text{the world's exports of sector } h \]
\[ M = \sum_h M^h = \text{world exports} \]

From these, market shares are defined:

\[ \theta_{ijh} = \frac{X_{ijh}}{M^h} = i's \text{ share of } j's \text{ import market in sector } h \]
\[ \theta_i = \frac{X_i}{M_j} = i's \text{ share of } j's \text{ total import market} \] (1a)
\[ \theta_{ijh} = \frac{X_{ijh}}{M_j} = i's \text{ share of } j's \text{ total import market} \]
\[ \theta_{ih} = \frac{X_{ih}}{M^h-M_j} = i's \text{ share in the rest-of-the-world's imports of sector } h \]
\[ \theta_i = \frac{X_i}{M-M_i} = i's \text{ share in the rest-of-the-world's total imports} \]

We then define four concepts of trade intensity:

\[ \alpha_j = \frac{\theta_{ij}}{\theta_i} = i's \text{ area intensity of trade in } j \] (2a)
\[ \sigma^h = \frac{\theta_{ih}}{\theta_i} = i's \text{ sector intensity of trade in } h \] (2b)

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1 The concept was apparently first used by A.J. Brown [1948], and has since been applied by Balassa [1967], Kojima [1970], [1971], Yamazawa [1970], [1971], and Roemer [1975b].
The area intensity and sector intensity of trade (Equation 2a-b) measure the extent to which area \( i \) controls a particular area or sector market in relation to its strength (market share) in the world market. The \( j \)-area intensity of trade in sector \( h \) (Equation 2c) measures the extent to which area \( i \) controls the \( h \)-sector market in a particular area \( (j) \) compared to its control of the market world-wide. The concept which the authors referred to have called "trade intensity" is here given by \( \alpha_{ij} \).

It is clear that some appropriate averages of these intensities should be unity. These are given by:

\[
\sum_i \alpha_{ij} \frac{M_j}{M - M_i} = 1 \quad \text{for any } i
\]

\[
\sum_h \sigma_{ih} \frac{M^h_i}{M^h} = 1 \quad \text{for any } i
\]

\[
\sum_i \alpha_{ij} \frac{M_i^h}{M_i^h - M_i} = 1 \quad \text{for any } i, h
\]

\[
\sum_h \sigma_{ij} \frac{M^h_j}{M^h} = 1 \quad \text{for any } i, j.
\]

As an example, the third of these identities is derived:

\[
\sum_i \alpha_{ij} \frac{M_i^h}{M_i^h - M_i} = \sum_i \frac{\theta_{ij} M_i^h}{\theta_{ij} (M_i^h - M_i)} = \sum_i \frac{X_i^h}{X_i^h} = \frac{1}{X_i^h} \sum_i X_i^h = 1
\]

By using the four intensity concepts defined here, a much richer description of area/sector composition of trade can be provided than by consideration of the area intensity alone. The author has made use of the concepts in studying trade flows, and they have revealed interesting patterns in world trade. The most striking result is that countries tend to market their "weakest" sectors disproportionately in their "strongest" areas. More precisely, the rank correlation between the sequence \( \{\sigma_{ih}\}_h \) and the sequence \( \{\alpha_{ij}\}_h \) where \( i \) and \( j \) are fixed is significantly negative if \( j \) is an "area of strength" for exporter \( i \) (i.e., if \( \alpha_{ij} > 1 \)), and significantly positive when \( j \) is a weak area for \( i \) (i.e., when \( \alpha_{ij} < 1 \)).

The natural question which arises concerning the intensities as statistics is: how complete a description of world trade is given by knowledge of the arrays \( \{\alpha_{ij}\} \), \( \{\sigma_{ih}\} \), \( \{\alpha_{ij}\} \), \( \{\sigma_{ij}\} \)? The more information that can be captured about trade flows from the intensities, the richer is their descriptive power. More precisely, we ask: is the complete trade flow array \( \{X_{ij}\} \) retrievable from knowledge of the intensity arrays? In general, the answer is negative; however, if one makes the additional assumption that each country maintains a zero trade balance, then the entire array \( \{X_{ij}\} \) is retrievable—at least for the special case \( J = 3, H = 2 \) studied here. The proof of this result shall be outlined in this paper. During the process, the reader may observe the simple linear relationships which

1 Kojima and Balassa use the concept which I have defined in (2). Yamazawa, however, uses what he considers to be a "simpler" concept, in which he ignores the imports of the exporting country in computing its market share. That is, he defines \( \theta_{ij} \) as \( X_{ij}^h/M^h \), instead of the definition given in (1b). In this paper, it is shown that the correct definition (1b) provides no algebraic difficulties. (See Yamazawa [1970], p. 62, footnote 3.)

1 These results and their implications are demonstrated and discussed in Roemer [1975].
the intensity arrays obey.

We observe first that we need consider only three of the four intensity arrays in (2), since any one of the four intensities can be calculated from the other three. This follows from (5), which is immediately demonstrable from the definitions (1b) and (2):

\[ \alpha_j^h \cdot \alpha_i^h = \alpha_j^h \cdot \alpha_i \quad (5) \]

We shall henceforth assume a three country, two sector world \((J=3, H=2)\), which shall be sufficient to establish the irretrievability of the \(\{X_i^h\}\) from the intensity arrays. We proceed to establish:

**Proposition 1** Given the total volume of world exports \(M\), the area import vector \(\vec{\mu} = \langle M_1, M_2, \ldots, M_n \rangle\) is retrievable from the array \(\{\alpha_j\}\) as long as \(\alpha_{ij} \neq 1\) for all \(i, j\).

**Proof:**

From (2), we see:

\[ \alpha_j M_j = \frac{X_j(M-M_i)}{X_i} \]

and hence:

\[ \sum_{j \neq i} \alpha_j M_j = \frac{M - M_i}{X_i} \sum_{j \neq i} X_j = M - M_i = \sum_{j \neq i} M_j \quad (6) \]

since \( \sum_{j \neq i} X_j = X_i \).

From (6) it follows that:

\[ \sum_{j \neq i} (1 - \alpha_j) M_j = 0 \quad (7) \]

Let us define \(\alpha_{ii} = 0 = X_{ii} \forall i\).

We then modify (7) to:

\[ \sum_{j} (1 - \alpha_j) M_j = M_i. \quad (8) \]

Define the matrix \(\tilde{A} = (1-\alpha_j)\). Remembering the definition:

\[ \vec{\mu} = \langle M_1, M_2, \ldots, M_n \rangle \]

we write (8) in matrix form as:

\[ \tilde{A} \vec{\mu} = \vec{\mu} \quad (9) \]

Hence:

\[ (I - \tilde{A}) \vec{\mu} = 0. \quad (10) \]

We can retrieve \(\vec{\mu}\) from knowledge of the \(\{\alpha_j\}\), then, if and only if \(\dim \ker (I - \tilde{A}) = 1\). For if \(\dim \ker (I - \tilde{A}) = 1\), then \(\vec{\mu}\) is known to a scalar, and the scalar is determined by \(M\), which is given (i.e. \(M = \sum M_j\)). Now \(\dim \ker (I - \tilde{A}) = 1\) precisely when rank \(I - \tilde{A} = n - 1\), from

\[ n = \text{rank} (I - \tilde{A}) + \dim \ker (I - \tilde{A}) \quad (11) \]

which is the fundamental theorem of finite dimensional vector spaces.

We have:

\[ I - \tilde{A} = \begin{pmatrix} 0 & \alpha_{12} - 1 & \alpha_{13} - 1 \\ \alpha_{21} - 1 & 0 & \alpha_{23} - 1 \\ \alpha_{31} - 1 & \alpha_{32} - 1 & 0 \end{pmatrix} \quad (12) \]
and it is clear that rank \((I-\tilde{A})\geq 2\) as long as \(\alpha_{ij} \neq 1\) \(\forall i, j\) (since any 2\(\times\)2 minor will have a non-zero determinant). Hence rank \((I-\tilde{A})=2\), and the Proposition is shown.

**Proposition 2** Given the total value of world trade \(M\), then the values \(\{M_{jh}\}\) are all retrievable from the arrays \(\{\alpha_{ij}\}\) and \(\{\sigma_{jh}\}\) as long as \(\alpha_{ij} \neq 1\) \(\forall i,j\).

**Proof:**

From (2d), we compute:

\[
\sigma_j M_j^h = \frac{M_j}{X_j} X_j^h
\]

and hence

\[
\sum_h \sigma_j M_j^h = \sum_h \frac{M_j}{X_j} X_j^h = M_j = \sum_h M_j^h
\]

from which follows:

\[
\sum_h (1-\sigma_j) M_j^h = 0
\]

Define the matrices:

\[
\tilde{S}_j = (1-\sigma_j^h)
\]

where \(\tilde{S}_j\) is a \(J\times H\) matrix (with rows indexed by \(i\) and columns by \(h\)).

Define \(\mu_j = \langle M_j^1, M_j^2, \ldots, M_j^H \rangle\)

We rewrite (14) as:

\[
\tilde{S}_j \mu_j = 0
\]

Now we have

\[
H = \text{dim} \ker \tilde{S}_j + \text{rank} \tilde{S}_j
\]

Since \(H=2\) in our case and \(\text{dim} \ker \tilde{S}_j > 0\) by (17), it follows that \(\text{dim} \ker \tilde{S}_j = 1\) (as long as \(\tilde{S}_j\) is not identically zero). Hence from (17), \(\mu_j\) can be retrieved up to scalar \(\gamma_j\).

However, from **Proposition 1**, \(M_j\) can be uniquely retrieved, and

\[
M_j = \mu_j \cdot 1
\]

and hence the scalar \(\gamma_j\) can be computed from (19). The Proposition is shown.

We now outline the proof for the irretrievability of the \(X_j^h\) from the intensity data. From **Propositions 1 and 2**, given \(M\) we can retrieve the array \(\{M_j^h\}\). This allows us, of course, to replace all the import variables in equations (1b)-(2) with known quantities. Then, in equations (2), we can reduce all aggregated export variables to the disaggregated trade flows; i.e.

\[
X_i^h = \sum_{j,h} X_{ij}^h
\]

\[
X_i = \sum_j X_{ij}^h
\]

Equations (2) have now been transformed into linear equations in the unknowns \(\{X_j^h\}\), and we can investigate the invertibility of the system. More explicitly, the system of equations (2c) becomes the system:

\[
M_j^h \alpha_{jh} \sum_j X_{ij}^h - (M^h - M_j^h) X_j^h = 0 \quad \forall i, j, h
\]
\[ M_j^h \sigma_j \sum_h X_j^h - M_j X_j^h = 0 \quad \forall i, j, h \] (22)

Equations (2b) become:
\[ (M^h - M_i^h) \sigma^h \sum_j X_j^h - (M - M_i) \sum_j X_j^h = 0 \quad \forall i, h \] (23)

(We are treating the \( X_j^h \) as unknowns and the other parameters as coefficients.)

Now, as was remarked earlier, all the information in the system (2) must be contained in equations (2bcd), and hence we must only investigate the invertibility of equations (21)-(23). In the case of \( J=3, H=2 \) the array \( \{X_j^h\} \) consists of 12 elements (since \( X_j^h = 0 \) \( \forall i \)) and equations (21)-(23) can be described as a matrix equation
\[ \Omega \bar{X} = 0 \] (24)

where \( \bar{X} \) is the 12-vector \( <X_{12}^1, X_{12}^2, X_{13}^1, X_{13}^2, X_{21}^1, X_{21}^2, X_{23}^1, X_{23}^2, X_{31}^1, X_{31}^2, X_{32}^1, X_{32}^2> \) and \( \Omega \) is the appropriate matrix of coefficients; in particular, \( \Omega : \mathbb{R}^{12} \rightarrow \mathbb{R}^{30} \). Since it is known that
\[ \sum_{i,j,h} X_j^h = M \] (25)

we know that \( \bar{X} \) is uniquely retrievable from (24) if and only if \( \dim [\text{kernel } \Omega] = 1 \), or rank \( \Omega = 11 \). The laborious computation whose details shall be omitted here shows that rank \( \Omega = 9 \), and hence \( \bar{X} \) is not uniquely retrievable.

An outline of the rank computation is given below. Let us consider the system of equations defined by (21) as
\[ \Omega_1 \bar{X} = 0 \] (26)

where \( \Omega_1 : \mathbb{R}^{12} \rightarrow \mathbb{R}^{12} \);

the system defined by (22) as
\[ \Omega_2 \bar{X} = 0 \] (27)

where \( \Omega_2 : \mathbb{R}^{12} \rightarrow \mathbb{R}^{12} \);

the system defined by (23) as
\[ \Omega_3 \bar{X} = 0 \] (28)

where \( \Omega_3 : \mathbb{R}^{12} \rightarrow \mathbb{R}^{6} \)

That is, we have decomposed as:
\[ \Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \] (29)

An examination of \( \Omega_1 \) shows it to be of the form:
\[
\begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
and it is easily shown that a maximal linearly independent set of 6 rows can be chosen.

A consideration of $\Omega_3$ shows it to be also of rank 6, but a consideration of \( \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \) allows us to choose only 9 linearly independent rows.

A consideration of $\Omega_3$ shows it to be of rank 3, yet each row of $\Omega_3$ is dependent on the already chosen 9 rows of \( \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \). Hence, it is shown rank $\Omega=9$. (The reader will surely put up with this sketchy demonstration, whose alternative is pages of computations which can be supplied by the author.)

A final observation may be made. If we insist that each country maintain its trade in balance, then the \( \{X^h_j\} \) are retrievable uniquely, at least for the case $J=3$, $H=2$. To see this, we note that imposing balanced trade for each country means the addition of the following equations to the system determining $\bar{X}$:

$$\begin{align*}
\sum_{j,h} X_{1,jh}^h &= M_1 \\
\sum_{j,h} X_{2,jh}^h &= M_2 \\
\sum_{j,h} X_{3,jh}^h &= M_3
\end{align*}$$

Let us summarize these equations by:

$$\Omega_4 \bar{X} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$$

If we append $\Omega_4$ to to define the new system

$$\Omega' = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix}$$

where $\Omega' : \mathbb{R}^{12} \rightarrow \mathbb{R}^{33}$

then our new system (28) cum (30) becomes:

$$\Omega' \bar{X} = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix}$$

(where the first 30 components of the above image vector are zero).

Equation (33) defines an inhomogeneous system, from which $\bar{X}$ can be uniquely retrieved precisely if rank $\Omega' = 12$. It can be shown that rank $\Omega_4 = 3$, and in fact, all three rows of $\Omega_4$ are independent of the 9 rows already chosen as the maximal linearly independent set in $\Omega$; hence, rank $\Omega' = 12$, and $\bar{X}$ is retrievable.

In summary:

**Theorem:** In general, the trade flow array \( \{X^h_j\} \) is not uniquely retrievable from the intensity arrays \( \{a^h_j\}, \{\sigma^h_j\}, \{a^h\} \) and \( \{\sigma^h\} \) and the total volume of trade $M$. If, however, the added constraint that each country maintains zero trade balance is given, then the \( \{X^h_j\} \) are uniquely determined by the intensity data and total volume of trade,
in the case $J=3, H=2$.

A final remark: the author strongly conjectures that Propositions 1 and 2 are true in general, but conjectures with somewhat less assurance that all trade flows are in general retrievable under the assumption of balanced trade for each country.

REFERENCES

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