CAPITAL-OUTPUT RATIO IN A LIQUID CAPITAL GOODS MODEL

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§. 1

The model we consider in this paper is characterised by the following assumptions:

(1) Labor is the only one original means of production and wages are paid after the completion of production.

(2) The economy consists of n industrial sectors and each sector produces one kind of liquid capital goods respectively.

(3) Coefficients of production (namely capital and labor coefficients) are all fixed and the available technique is only one.

(4) The vector of labor coefficients is strictly positive and the matrix of non-negative capital coefficients is indecomposable.

(5) The wage rate and the notional rate of interest are all equal in all lines of industrial sectors.

We introduce the following notations.

l_i=the labor coefficient for the ith sector

$l=\{l_1, l_2, \ldots, l_n\}$=the column vector of labor coefficients

a_{ij}=the capital coefficient for the jth sector with respect to the ith liquid capital goods

$A=[a_{ij}]$=the indecomposable matrix with $n\times n$ capital coefficients

p_i=the price for one unit of the ith liquid capital goods which is assumed to be positive

$p=\{p_1, p_2, \ldots, p_n\}$=the column vector of prices

x_i=the output for the ith sector which is assumed to be positive

$x=\{x_1, x_2, \ldots, x_n\}$=the column vector of outputs

r=the notional rate of interest

w=the wage rate paid at the end of production.

Under these assumptions, we can present the equation

(1. 1)..... $p'=(1+r)\ p'A+wl'$

where a prime denotes the transposition of vectors (and matrix). By post-multiplying x to this equation, we can also have

(1. 2)..... $p'x=(1+r)\ p'Ax+wl'x.$

Let Y and K be the value of net national product and the value of liquid capital goods in the economy as a whole respectively. Under our liquid capital goods model, it is apparent that

(1. 3) .... $Y=wl'x+rp'Ax=p'[E-A]x$

(1. 4) .... $K=p'Ax$

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respectively, where E is the unit matrix of nth order.

Our problem in this paper is to examine the conditions for the capital-output raito in the economy as a whole (namely K/Y) to remain constant in spite of changes in either prices or outputs.

§. 2

We first prove the following

THEOREM 1:

For the capital-output ratio in the economy as a whole to remain constant for any output vector $x>0$, it is necessary and sufficient that the price vector p becomes the Frobenius characteristic vector of A from the left hand.

Proof of Necessity

Let us suppose that

$$\frac{p'Ax}{p'[E-A]x}=\frac{K}{Y}=\text{constant}\equiv\alpha.$$

Thus we have

$$p'\left[E-\left(\frac{1+\alpha}{\alpha}\right)A\right]x=0. \tag{2. 1}$$

For this equation to be true for any output vector $x>0$, we must have

$$p'\left[E-\left(\frac{1+\alpha}{\alpha}\right)A\right]=0. \tag{2. 2}$$

By assumption p is a positive vector, so that p must be the Frobenius characteristic vector of A which is associated to the Frobenius characteristic root from the left hand.

Q.E.D.

Proof of Sufficiency

Let us suppose that p is the Frobenius characteristic vector of A from the left-hand. Such a p we denote by p_*. Thus

$$p_*'A=\lambda_*p_*' \tag{2. 3}$$

where λ_* is the Frobenius characteristic root of A.

Then it follows

$$\frac{p_*'Ax}{p_*'[E-A]x}=\frac{\lambda_*p_*'x}{(1-\lambda_*)p_*'x}=\left(\frac{\lambda_*}{1-\lambda_*}\right)=\text{constant}$$

for any output vector $x>0$.

Q.E.D.

§. 3

We prove, next, the following

THEOREM 2:

For the capital-output ratio in the economy as a whole to remain constant for any price vector $p>0$, it is necessary and sufficient that the output vector x becomes the Frobenius characteristic vector of A from the right hand.

Proof of Necessity

Let us suppose that

$$\frac{p'Ax}{p'[E-A]x}=\frac{K}{Y}=\text{constant}=\alpha.$$

Thus we have

$$p'\left[E-\left(\frac{1+\alpha}{\alpha}\right)A\right]x=0. \tag{3. 1}$$

For this equation to be true for any price vector $p>0$, we must have

$$\left[E-\left(\frac{1+\alpha}{\alpha}\right)A\right]x=0. \tag{3. 2}$$

By assumption x is a positive vector, so that x must be the Frobenius characteristic vector of A which is associated to the Frobenius characteristic root from the right hand.

Q.E.D.

Proof of Sufficiency

Let us suppose that x is the Frobenius characteristic vector of A from the right hand. Such a x we denote by x_*. Thus

$$Ax_*=\lambda_* x_* \tag{3. 3}$$

where λ_* is the Frobenius characteristic root of A. Then it follows

$$\frac{p'Ax_*}{p'[E-A]x_*}=\frac{\lambda_* p'x_*}{(1-\lambda_*)p'x_*}=\left(\frac{\lambda_*}{1-\lambda_*}\right)=\text{constant}$$

for any price vector $p>0$.

Q.E.D.

§. 4

In another occasion [1], the author has proved that for the price vector p to be the Frobenius characteristic vector p_* of A from the left hand, it is necessary and sufficient that the vector of labor coefficients l becomes the Frobenius characteristic vector of A from the left hand and showed that this condition is equivalent in significance to say that the organitic composition of capital is the same in all lines of industrial sectors. Thus we may restate the theorem 1 as that for the capital-output ratio in the economy as a whole to remain constant for any output vector $x>0$, it is necessary and sufficient that the vector of labor coefficient l becomes the Frobenius characteristic vector of A from the left hand.

As to the theorem 2, it is well known that the Frobenius characteristic vector of A from the right hand (x_* in our model) is nothing but the "standard commodity" in the Sraffanian system [2; 3]. Thus in case if the organitic composition of capital is not the same in various industrial sectors, the standard commodity can make sure that the capital-output ratio remains constant even if the relative prices of capital goods are subject to changes in consequence of changes in the notional rate of interest.

REFERENCES

[1] K. Ara, "Parable and Realism in Capital Theory: A Generalisation of Prof. Samuelson's Theory of Surrogate Production Function", *The Economic Studies Quarterly* vol. XXVI, No. 1, April 1975 (in Japanese with English summary), pp. 1–13.

[2] P. Sraffa, *Production of Commodities by Means of Commodities*, 1960.

[3] E. Burmeister, "On a Theorem of Sraffa", *Economica*, Vol. XXXV, 1968, pp. 83-7.