BOUNDEDNESS OF THE CLOSED ECONOMY WITH SAMUELSON-LEONTIEF TECHNOLOGY*

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1. *Introduction*

The purpose of this brief note is to ascertain that the feasible production set of the closed economy with Samuelson-Leontief technology is bounded as well as closed, provided that the labour, the availability of which is finite, is indispensable in the production of each and every good. This is an intuitively appealing result and there seem to be some cases for providing its formal proof.[1] At the very least, it is an exercise of Occam's Razor.

In his first published rigorous proof of nonsubstitution theorem [1], Arrow assumes, together with the indispensability and finite supply of labour, that the feasible production set is compact.[2] Melvin [6] notices that "... his theorem can be strengthened in that he does not need to assume the compactness of S[feasible production set], this being implied by his earlier assumptions" [6, (306)]. Correct as he is, the proof is provided by Melvin only for the very special case of neoclassical production functions,[3] so that there is jarring discrepancy between his general observation and established theorem. Hence comes the present note. Melvin's theorem is generalised in what follows so as to make clear the essential properties of the generalised Leontief model which are responsible for the compact feasible production set.

2. *A Model of Closed Economy with Samuelson-Leontief Technology*

Suppose that there exist $m+1$ well-defined and homogeneous commodities, $G_0, G_1, \dots, G_m$, the 0-th of which, G_0, stands for the single primary resource, labour. Productive activities are performed by units of production, each one of which produces one and only one commodity. Under this assumption of non-joint production, we identify the unit producing the commodity k simply as k-sector ($k=1, 2, \dots, m$). Let T_k stand for the production

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1 For example, the feasible production set being compact is sufficient for the existence of Pareto-efficient economic states. See, [8]. Melvin [6, (305)] should also be consulted.
2 Arrow [1, (158, 164)]. See, also Mirrlees [7].
3 Melvin assumes that "the production functions are continuous concave functions defined for all non-negative values of their arguments, and that they are homogeneous of the first degree and have continuous first and second partial derivatives everywhere" [6, (308)]. Georgescu-Roegen's suggestion to the effect that "... the conclusion of Melvin's proposition requires only the continuity (everywhere) of the production functions" [4, (316)] hints at the present generalisation.

possibility set of the k-sector and assume:

(A_1) *T_k is a non-empty closed cone in R^{m+1} $(k=1, 2, \ldots, m)$.*[4,5]

That is to say, production is subject to continuity and constant returns to scale.

Following the usual sign convention, outputs will be represented by positive numbers, inputs by negative numbers. Then the assumption of non-joint production can simply be worded as follows. Letting $y^{(k)}=(y_0^{(k)}, y_1^{(k)}, \ldots, y_m^{(k)})$ be a generic element of T_k, we have:

(2. 1) $$y_j^{(k)} \leqq 0 \ (j=0, 1, \ldots, k-1, k+1, \ldots, m)$$

and

(2. 2) $$y_k^{(k)} \geqq 0.$$

In this model, labour is indispensable in the production of each and every commodity in the sense that:

(A_2) *$y^{(k)} \in T_k$ and $y_k^{(k)}>0$ imply $y_0^{(k)}<0$ $(k=1, 2, \ldots, m)$.*

Now it is time to introduce the concept of feasible production set. Let L, $0<L<+\infty$, denote the maximum available labour service and let:

(2. 3) $$T=\sum_{k=1}^{m} T_k$$

and

(2. 4) $$H=\{x \in R^{m+1} | x \geqq (-L, \overbrace{0, 0, \ldots, 0}^{m})\}.$$

We define the feasible production set T_* by:

(2. 5) $$T_*=T \cap H.$$

Any point $y \in T_*$ denotes the aggregative production of the closed economy which is technically admissible and satisfies the constraint enforced by labour service availability. We shall assume that technologies $\{T_1, T_2, \ldots, T_m\}$ are productive enough to the effect that:

(A_3) *There exists a $\bar{y}=(\bar{y}_0, \bar{y}_{)0(})\in T_*$ such that $\bar{y}_{)0(}>0$.*[6,7]

Our purpose will be served if we can prove that T_* is a compact subset of R^{m+1}.

3. *The Compactness of the Feasible Production Set*

At the outset, let us establish the following:

Lemma 3.1: *Let T_k' be defined by*:

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[4] So far as our present purpose is concerned, the convexity of T_k $(k=1, 2, \ldots, m)$ can be dispensed with.

[5] R^{m+1} stands for the Euclidean $(m+1)$-space. For any $x^1, x^2 \in R^{m+1}$, $x^1 \geqq x^2$ denotes $x_i^1 \geqq x_i^2$, $i=1, 2, \ldots, m+1$; $x^1 \geq x^2$ denotes $x^1 \geqq x^2$ and $x^1 \neq x^2$; and $x^1 > x^2$ denotes $x_i^1 > x_i^2$, $i=1, 2, \ldots, m+1$. For any $x \in R^{m+1}$, we let $\|x\|$ stand for the norm of x.

[6] Here $\bar{y}_{)0(}=(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$.

[7] It would be convenient to mention here some slips in Mirrlees' otherwise elegant contribution [7]. In our notation, he assumes that [7, $(A.\ 3)$]:

(*) *There exists a $\bar{y}=(\bar{y}_0, \bar{y}_{)0(})\in T$ such that $\bar{y}_{)0(}>0$ and that the set of all such $\bar{y}$ is bounded.*

T being cone, $\theta \bar{y}$ belongs to T for any $\theta>0$ and $\theta \bar{y}_{)0(}>0$. We have, however, $\|\theta \bar{y}\|=\theta\|\bar{y}\| \rightarrow \infty$ $(\theta \rightarrow \infty)$. Thus (*) never holds true in conjunction with the conic property of T. (Similar comment applies to [7, $(A'.\ 3)$].) This difficulty disappears if T is replaced, in our notation, by T_* in (*).

(3. 1) $T_k' = T_k \cap \{x = (x_0, x_1, \ldots, x_m) \in R^{m+1} | x_k = 1\}$ $(k = 1, 2, \ldots, m)$.

Then there exists an $\varepsilon_k > 0$ *for each* k *such that for any* $x^{(k)} \in T_k'$ *we have* $x_0^{(k)} \leqq -\varepsilon_k < 0$ $(k = 1, 2, \ldots, m)$.

Proof. We define:

(3. 2) $$\varepsilon_k = -\inf x_0^{(k)} \text{ over all } x^{(k)} \in T_k'.$$

By virtue of (A_2), we have $\varepsilon_k \geqq 0$. Suppose $\varepsilon_k = 0$. By definition (3. 2), there exists an infinite sequence $\{x^{(k)\mu}\}_{\mu=1}^{\infty}$ in T_k' such that $x^{(k)\mu} \to x^{(k)*}$ $(\mu \to \infty)$ with $x_0^{(k)*} = 0$. (A_1) and (A_3) assure us that T_k' is non-empty closed set in R^{m+1}, so that we must have $x^{(k)*} \in T_k' \subset T_k$, which contradicts (A_2) because $x_k^{(k)*} = 1$, $x_0^{(k)*} = 0$. This contradiction establishes the lemma. Q.E.D.

A generalised Melvin's theorem will now be given.

Theorem 3. 2: *T_* is a non-empty compact subset of R^{m+1}.*[8]

Proof.

Step 1 [Boundedness]: Suppose that there exists an infinite sequence $\{y^{\mu}\}_{\mu=1}^{\infty}$ in T_* such that $\|y^{\mu}\| \to \infty$ $(\mu \to \infty)$. Then there exists a sequence $\{y^{(k)\mu}\}_{\mu=1}^{\infty}$ in T_k for each and every k $(=1, 2, \ldots, m)$ such that

(3. 3) $$y^{\mu} = \sum_{k=1}^{m} y^{(k)\mu}$$

and that

(3. 4) $$y^{\mu} \geqq (-L, \overbrace{0, 0, \ldots, 0}^{m})$$

holds true for all $\mu = 1, 2, \ldots$ *ad inf.* Thus there exists a k $(1 \leqq k \leqq m)$ carrying the property:

(3. 5) $$y_k^{\mu} \to \infty \quad (\mu \to \infty).$$

Noticing (2.1), we must admit that, for this k:

(3. 6) $$y_k^{(k)\mu} \to \infty \quad (\mu \to \infty)$$

holds true. We may suppose without loss of generality that $y_k^{(k)\mu} > 0$ for all $\mu = 1, 2, \ldots$ *ad inf.*, so that

(3. 7) $$x^{(k)\mu} = y^{(k)\mu} / y_k^{(k)\mu} \quad (\mu = 1, 2, \ldots \textit{ad inf.})$$

is well-defined. For the sequence $\{x^{(k)\mu}\}_{\mu=1}^{\infty}$ in T_k' thus generated, we have:

(3. 8) $$-L / y_k^{(k)\mu} \leqq x_0^{(k)\mu} \leqq 0 \quad (\mu = 1, 2, \ldots \textit{ad inf.})$$

which yields $x_0^{(k)\mu} \to 0$ $(\mu \to \infty)$ by virtue of (3. 6), in contradiction to Lemma 3. 1.

Step 2 [Closedness]: Let Δ_k be defined as follows.

(3. 9) $$\Delta_k = \{y^{(k)} \in T_k | y^{(k)} + \sum_{s \neq k}^{m} y^{(s)} \in T_* \text{ for some } y^{(s)} \in T_s \ (s \neq k)\} \quad (k = 1, 2, \ldots, m).$$

In Step 1, we have in effect proved that:

(3. 10) *There exists a δ_k, $0 < \delta_k < +\infty$, such that $y_k^{(k)} \leqq \delta_k$ for any $y^{(k)} \in \Delta_k$ $(k = 1, 2, \ldots, m)$.*

We shall first show that Δ_k $(k = 1, 2, \ldots, m)$ is bounded utilising (3. 10). For any $y^{(k)} \in \Delta_k$, there exist some $y^{(s)} \in \Delta_s$ $(s \neq k)$ satisfying:

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[8] If each T_k $(k = 1, 2, \ldots, m)$ is a closed convex cone, then T_* is easily shown to be convex as well.

(3. 11) $$y_t^{(t)}+y_t^{(k)}+\sum_{s\neq t,k}^{m} y_t^{(s)} \geqq 0 \quad (t=1, 2, \dots, m)$$

which yields:

(3. 12) $$y_t^{(k)} \geqq -y_t^{(t)}-\sum_{s\neq t,k}^{m} y_t^{(s)} \geqq -\delta_t \quad (t=1, 2, \dots, k-1, k+1, \dots, m)$$

where uses are made of (2. 1) and (3. 10). Thus Δ_k $(k=1, 2, \dots, m)$ is bounded.

The closedness of T_* can now be verified. Taking an arbitrary accumulation point y^* of T_*, we let $\{y^\mu\}_{\mu=1}^\infty$ stand for an infinite sequence in T_* converging to y^*. By definition, there exists an infinite sequence $\{y^{(k)\mu}\}_{\mu=1}^\infty$ in Δ_k for each $k=1, 2, \dots, m$ such that:

(3. 13) $$y^\mu=\sum_{k=1}^{m} y^{(k)\mu} \quad (\mu=1, 2, \dots, ad\ inf.)$$

holds true. Δ_k being bounded, we may assume with full generality that $\{y^{(k)\mu}\}_{\mu=1}^\infty$ is originally convergent. Then we have:

(3. 14) $$y^*=\lim_{\mu\to\infty} y^\mu=\lim_{\mu\to\infty}\sum_{k=1}^{m} y^{(k)\mu}=\sum_{k=1}^{m}\lim_{\mu\to\infty} y^{(k)\mu}.$$

(A_1) entails $\lim_{\mu\to\infty} y^{(k)\mu}\in T_k$ $(k=1, 2, \dots, m)$, while $\sum_{k=1}^{m} y^{(k)\mu} \geqq (-L, \overbrace{0, 0, \dots, 0}^{m})$ for all $\mu=1, 2, \dots$ *ad inf.* yields $y^* \geqq (-L, \overbrace{0, 0, \dots, 0}^{m})$ by virtue of (3. 14). Thus we have $y^*\in T_*$, establishing the closedness of T_*. Q.E.D.

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