ADDITIVITY OF INPUTS IN TRANSFORMATION FUNCTION

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§ 1

Let y and x_i $(i=1, 2, \ldots, n)$ be output and inputs respectively, and we say that the transformation function

$$y=F(x_1, x_2, \ldots, x_n)$$

is regular if the marginal contributions of inputs are all positive, namely

$$\frac{\partial y}{\partial x_i}>0 \qquad (i=1, 2, \ldots, n).$$

Our problem in this paper is to investigate the condition for various inputs to be integrated into a single variable.

§ 2

We can prove the following

THEOREM: *For the marginal rate of substitution between any two inputs to be all constant in the regular transformation function, namely*

$$\frac{\partial y}{\partial x_i}\Big/\frac{\partial y}{\partial x_j}=constant>0 \qquad (i, j=1, 2, \ldots, n),$$

it is necessary and sufficient that inputs in the function are additive in the sense that

$$y=F(a_1x_1+a_2x_2+\ldots+a_nx_n),$$

where a_i $(i=1, 2, \ldots, n)$ *are positive constants.*

PROOF OF NECESSITY: Let us put

$$z\equiv a_1x_1+a_2x_2+\ldots+a_nx_n.$$

It is clear that

$$\frac{\partial y}{\partial x_i}=\frac{dy}{dz}a_i \qquad (i=1, 2, \ldots, n),$$

so that we have

$$\frac{\partial y}{\partial x_i}\Big/\frac{\partial y}{\partial x_j}=\frac{a_i}{a_j}=\text{constant}>0 \qquad (i, j=1, 2, \ldots, n).$$

PROOF OF SUFFICIENCY: Let us put

$$\frac{\partial y}{\partial x_1}\Big/\frac{\partial y}{\partial x_2}\equiv a_{12}.$$

By assumption, a_{12} is a positive constant. Solving this, it follows that y is an arbitrary function of $a_1x_1+a_2x_2$ $(a_{12}\equiv a_1/a_2)$ with respect to x_1 and x_2, so that we have

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$$y=\phi(a_1x_1+a_2x_2, x_3 \ldots, x_n).$$

Let us put $z_1=a_1x_1+a_2x_2$. Then we have again

$$\frac{\partial y}{\partial x_3}=\frac{\partial y}{\partial x_1}a_{31}=\frac{\partial y}{\partial x_1}\frac{a_3}{a_1}=\frac{\partial y}{\partial z_1}a_3.$$

Solving this last equation, it follows that y is an arbitrary function of $z_1+a_3x_3$ with respect to z_1 and x_3, so that we have

$$y=\psi(a_1x_1+a_2x_2+a_3x_3, x_4, \ldots, x_n).$$

Repeating the same procedure properly, we can finally obtain the result desired, namely

$$y=F(a_1x_1+a_2x_2+\ldots+a_nx_n). \qquad \text{Q. E. D.}$$

§ 3

The applicability of our theorem to economic problems would be obvious. In the case of production function, it is almost straightforward. That the marginal rate of substitution between two kinds of input remains constant at all events means that they are really the same kind of input because the elasticity of substitution between them is infinite, so that we need not classify them as different inputs.

Next, let y and x_i $(i=1, 2, \ldots, n)$ be respectively the group welfare and the cardinal utilities of the members comprising the group. That the marginal rate of substitution between any two inputs is constant means, in this case, that the marginal significance of individual utilities to the group welfare is constant relatively, so that we may integrate the individual utilities into a single variable in such a way as indicated in our theorem.